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A LIOUVILLE-TYPE THEOREM ON HALFSPACES FOR THE KOHN LAPLACIAN

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ABSTRACT. Let $\Delta_{\mathbb{H}^n}$ be the Kohn Laplacian on the Heisenberg group $\mathbb{H}^n$ and let Ω be a halfspace of $\mathbb{H}^n$ whose boundary is parallel to the center of $\mathbb{H}^n$. In this paper we prove that if u is a non-negative $\Delta_{\mathbb{H}^n}$-superharmonic function such that

$$u \in L^1(\Omega),$$

then $u \equiv 0$ in Ω.

1. INTRODUCTION

Let $\Delta_{\mathbb{H}^n}$ be the Kohn Laplacian on the Heisenberg group $\mathbb{H}^n$. In this note we prove the following result.

Theorem 1.1. *Let Ω be a halfspace of $\mathbb{H}^n$ whose boundary is parallel to the center of $\mathbb{H}^n$. If u is a non-negative $\Delta_{\mathbb{H}^n}$-superharmonic function such that $u \in L^1(\Omega)$, then $u \equiv 0$ in Ω.*

From this theorem the next corollary follows.

Corollary 1.2. *If $u \in L^1(\Omega) \cap C^2(\Omega)$, $u \geq 0$ and $\Delta_{\mathbb{H}^n} u \leq 0$, then $u \equiv 0$ in Ω.*

While we refer to the next section for the notation and definitions used in the theorem above, here we would like to briefly comment on the result and the technique used for its proof.

Theorem 1.1 cannot be improved in the "scale" of the Heisenberg group $\mathbb{H}^n$. As a matter of fact, for every $p > 1$ and $n > \frac{1}{p-1}$, there exist $\Delta_{\mathbb{H}^n}$-superharmonic functions on $\mathbb{H}^n$ that belong to $L^p(\Omega)$ and are strictly positive everywhere (see Remark 3.5).

Our technique strongly relies on the use of a mean value operator suitably adapted to the geometry of Ω, and it is not applicable if the boundary of Ω is not parallel to the center of $\mathbb{H}^n$. Nevertheless, this technique can easily be used to prove an analogous result for the classical Laplace operator Δ on $\mathbb{R}^n$.

Theorem 1.3. *Let Ω be a halfspace of $\mathbb{R}^n$. If u is a non-negative Δ-superharmonic function such that $u \in L^1(\Omega)$, then $u \equiv 0$ in Ω.*

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We emphasize that our proof does not require any knowledge of the Poisson kernel.

Liouville theorems on halfspaces of the Heisenberg group for semilinear $\Delta_{\mathbb{H}^n}$-inequalities have recently been proved by Birindelli-Capuzzo Dolcetta-Cutrì [BCDC] by different techniques. It should be mentioned that our result cannot be deduced from those in [BCDC]. Liouville theorems for functions with polynomial growth have been proved by Korányi-Stanton [KS] for the complex Kohn Laplacian and by Geller [Ge] for homogeneous left-invariant hypoelliptic operators on homogeneous groups.

This paper is organized as follows. In Section 2 we fix the notation and the main definitions, and we recall some known results. Section 3 is devoted to the proof of Theorem 1.1 and its corollary. The proof of Theorem 1.3 closely follows the lines of that of Theorem 1.1; therefore it is omitted.

2. Some known results on the Heisenberg group and the Kohn Laplacian

The Heisenberg group $\mathbb{H}^n$, whose points will be denoted by $\xi = (z,t) = (x,y,t)$, is the Lie group $(\mathbb{R}^{2n+1}, \circ)$, with composition law defined by

$$\xi \circ \xi' = (z + z', t + t' + 2(\langle x', y\rangle - \langle x, y'\rangle)),$$

where $\langle\ ,\ \rangle$ denotes the inner product in $\mathbb{R}^n$. The t-axis, i.e. the set $\{(0,0,t) | t \in \mathbb{R}\}$, is called the center of $\mathbb{H}^n$. A halfspace of $\mathbb{H}^n$ is merely a halfspace of $\mathbb{R}^{2n+1}$. The Kohn Laplacian on $\mathbb{H}^n$ is the operator

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n} ({X_j}^2 + {Y_j}^2),$$

where for all $j \in \{1, \dots, n\}$

$$X_j = \partial_{x_j} + 2y_j\partial_t\ ,\quad Y_j = \partial_{y_j} - 2x_j\partial_t.$$

The operator $\Delta_{\mathbb{H}^n}$ has positive semidefinite characteristic form, and it is not elliptic at any point of $\mathbb{H}^n$. A natural group of dilations on $\mathbb{H}^n$ is given by

$$\delta_\lambda(\xi) = (\lambda z, \lambda^2 t)\ ,\quad \lambda > 0.$$

The Jacobian determinant of δ_λ is λ^Q, where

$$Q = 2n + 2$$

is called the homogeneous dimension of $\mathbb{H}^n$. The operator $\Delta_{\mathbb{H}^n}$ is invariant with respect to left translations of $\mathbb{H}^n$ and homogeneous of degree two with respect to the dilations δ_λ. More precisely, if we set

$$\tau_\xi(\xi') = \xi \circ \xi',$$

we have

$$\Delta_{\mathbb{H}^n}(u \circ \tau_\xi) = (\Delta_{\mathbb{H}^n} u) \circ \tau_\xi\ ,\quad \Delta_{\mathbb{H}^n}(u \circ \delta_\lambda) = \lambda^2 (\Delta_{\mathbb{H}^n} u) \circ \delta_\lambda.$$

A remarkable analogy between the Kohn Laplacian and the classical Laplace operator is that a fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole at zero is given by

$$\Gamma(\xi) = \frac{c_Q}{d(\xi)^{Q-2}},$$

where c_Q is a suitable positive constant and

$$d(\xi) = (|z|^4 + t^2)^{1/4} \tag{2.1}$$

(see [F]). Moreover, if we define $d(\xi, \xi') = d(\xi'^{-1} \circ \xi)$, then d is a distance on $\mathbb{H}^n$; see [C] for a complete proof of this statement. If we denote by $B_d(\xi, r)$ the d-ball of center ξ and radius r, then, due to the left translation invariance of the distance d, we have

$$\tau_\xi(B_d(0, r)) = B_d(\xi, r).$$

Moreover, since d is homogeneous of degree 1 with respect to the dilation δ_λ,

$$\delta_\lambda(B_d(0, r)) = B_d(0, \lambda r)$$

and

$$|B_d(\xi, r)| = r^Q |B_d(0, 1)|.$$

Here $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^{2n+1}$. We also recall that the Lebesgue measure is a Haar measure on $\mathbb{H}^n$. If $u = u(d)$ is a smooth radial function (i.e. u only depends on the function d in (2.1)) then, as it has been noticed by Garofalo and Lanconelli in [GL],

$$\Delta_{\mathbb{H}^n} u = \psi\Big(u''(d) + \frac{Q-1}{d} u'(d)\Big), \tag{2.2}$$

where

$$\psi(\xi) = \frac{|z|^2}{d(\xi)^2}.$$

In [GL] the following mean value formula was also shown (see also Gaveau [G]): if Ω is an open subset of $\mathbb{H}^n$, $u \in C^2(\Omega)$ and $\overline{B_d(\xi, r)} \subseteq \Omega$, then

$$u(\xi) = (M_r u)(\xi) - \frac{Q}{r^Q} \int_0^r \varrho^{Q-1} \int_{B_d(\xi, \varrho)} \Big(\Gamma(\xi, \xi') - \frac{c_Q}{\varrho^{Q-2}}\Big) \Delta_{\mathbb{H}^n} u(\xi') d\xi' d\varrho, \tag{2.3}$$

where M_r is the mean value operator defined by

$$(M_r u)(\xi) = \frac{1}{\alpha_Q r^Q} \int_{B_d(\xi, r)} \psi(\xi, \xi') u(\xi') d\xi'. \tag{2.4}$$

Here we have set $\psi(\xi, \xi') = \psi(\xi'^{-1} \circ \xi)$ and $\Gamma(\xi, \xi') = \Gamma(\xi'^{-1} \circ \xi)$. Moreover,

$$\alpha_Q = \int_{B_d(0,1)} \psi(\xi) d\xi.$$

For more details on the Heisenberg group one can see, for example, [FS].

Definition 2.1. A function $u : \Omega \to]-\infty, +\infty]$ is $\Delta_{\mathbb{H}^n}$-superharmonic if

(*i*) u is lower semicontinuous,

(*ii*) $u \in L^1_{loc}(\Omega)$, and

(*iii*) $u(\xi) \geq (M_r u)(\xi)$ for every d-ball $B_d(\xi, r)$ with closure contained in Ω; here M_r is the mean value operator (2.4).

Remark 2.2. From (2.3) we immediately get that if $u \in C^2(\Omega)$ and $\Delta_{\mathbb{H}^n} u \leq 0$ in Ω, then u is $\Delta_{\mathbb{H}^n}$-superharmonic in Ω.

Definition 2.3. A mapping $\varrho : \mathbb{H}^n \to \mathbb{H}^n$ is a rotation around the t-axis if there exists a complex unitary transformation $F : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\varrho(z,t) = (F(z), t).$$

It is easy to verify that the superarmonicity property is invariant with respect to left translations and rotations around the t-axis. Moreover it is well known that the operator $\Delta_{\mathbb{H}^n}$ has the same invariance properties.

The following proposition can be proved with simple geometric arguments.

Proposition 2.4. *Let Ω be a halfspace of $\mathbb{H}^n$, and define*

$$\Omega_t = \{(x,y,t) \in \mathbb{R}^{2n+1} \mid t > 0\}, \qquad \Omega_1 = \{(x,y,t) \in \mathbb{R}^{2n+1} \mid x_1 > 0\}.$$

Then, one of the two following cases always occurs:

(i) There exists $\xi_0 \in \mathbb{H}^n$ such that either $\Omega = \tau_{\xi_0}(\Omega_t)$ or $\Omega = \tau_{\xi_0}(-\Omega_t)$.

(ii) There exist $\xi_0 \in \mathbb{H}^n$ and a rotation ϱ around the t-axis such that

$$\Omega = \varrho\tau_{\xi_0}(\Omega_1).$$

The first case occurs when $\partial\Omega$ is transverse to the t-axis, the second when they are parallel.

3. Proof of Theorem 1.1 and its corollary

In light of the results of section 2 we only need to prove Theorem 1.1 when $\Omega = \{x_1 > 0\}$. Hence, from now on, we will assume $\Omega = \Omega_1$. Let us introduce the following notation for the point $\xi \in \mathbb{H}^n$: $\xi = (z,t) = (x_1; \hat{z}, t)$, where $\hat{z} = (x_2, \ldots, x_n, y)$. For every $\xi \in \Omega$ we set

$$r(\xi) = \frac{x_1}{2}\ ,\ B_\xi = B_d(\xi, r(\xi))$$

and

$$A_\xi = \{\eta \in \Omega \mid d(\xi,\eta) < r(\eta)\}.$$

We explicitly remark that $\chi_{B_\xi}(\eta) = \chi_{A_\eta}(\xi)$, where χ_{B_ξ} and χ_{A_η} denote, respectively, the characteristic functions of the sets B_ξ and A_η. We also define

$$K : \Omega \times \Omega \to \mathbb{R}, \quad K(\xi,\xi') = \frac{\psi(\xi,\xi')}{\alpha_Q r(\xi)^Q}\chi_{B_\xi}(\xi')$$

and, for $u \in L^1_{loc}(\Omega)$,

$$Tu : \Omega \to \mathbb{R}, \quad (Tu)(\xi) = (M_{r(\xi)}u)(\xi) = \int_\Omega K(\xi,\xi')u(\xi')d\xi'.$$

Finally we set

$$\eta_0 = (1;0,0) \quad \text{and} \quad \beta = \int_\Omega K(\eta,\eta_0)d\eta.$$

If u is a $\Delta_{\mathbb{H}^n}$-harmonic function, i.e. if $\Delta_{\mathbb{H}^n} u = 0$, from (2.3) it straightforwardly follows that $Tu = u$. In particular $T(1) = 1$, which means

$$\int_{\Omega} K(\xi, \xi') d\xi' = 1 \quad \forall \xi \in \Omega. \tag{3.1}$$

It is not obvious that the integration of $K(\xi, \xi')$ with respect to the variable ξ also yields a constant value. This fact will be crucial in the proof of Theorem 1.1.

Lemma 3.1. *We have*

$$\int_{\Omega} K(\xi, \xi_0) d\xi = \beta$$

for every $\xi_0 \in \Omega$.

Proof. We fix $\xi_0 = (x_0^1; \hat{z_0}, t_0) \in \Omega$ and we set $\overline{\xi_0} = \xi_0 \circ (-x_0^1; 0, 0) = \tau_{\xi_0}(\delta_{x_0^1}\eta_0)^{-1}$. For sake of brevity, we also set $\tau = \tau_{\overline{\xi_0}}$ and $\delta = \delta_{x_0^1}$. Then $\xi_0 = \tau\delta\eta_0$. Moreover $A_{\xi_0} = \tau\delta A_{\eta_0}$. Indeed, if $\eta \in \Omega$ we have

$$\begin{aligned} d(\tau\delta\eta, \xi_0) &= d(\xi_0^{-1} \circ \overline{\xi_0} \circ \delta\eta) = d((\delta\eta_0)^{-1} \circ \delta\eta) \\ &= d(\delta(\eta_0^{-1} \circ \eta)) = x_0^1 d(\eta, \eta_0) \end{aligned}$$

and

$$r(\tau\delta\eta) = r(\overline{\xi_0} \circ \delta\eta) = \frac{(\overline{\xi_0} \circ \delta\eta)_1}{2} = \frac{(\delta\eta)_1}{2} = x_0^1 \frac{\eta_1}{2} = x_0^1 r(\eta).$$

It is worthwhile noticing that $\delta(\Omega) \subseteq \Omega$ and $\tau(\Omega) \subseteq \Omega$. Hence

$$\eta \in A_{\eta_0} \iff d(\eta, \eta_0) < r(\eta) \iff d(\tau\delta\eta, \xi_0) < r(\tau\delta\eta) \iff \tau\delta\eta \in A_{\xi_0}.$$

Therefore we have

$$\begin{aligned} \beta = \int_{\Omega} K(\eta, \eta_0) d\eta &= \int_{A_{\eta_0}} \frac{\psi(\eta, \eta_0)}{\alpha_Q r(\eta)^Q} d\eta = (x_0^1)^{-Q} \int_{\delta A_{\eta_0}} \frac{\psi(\delta^{-1}\zeta, \eta_0)}{\alpha_Q r(\delta^{-1}\zeta)^Q} d\zeta \\ &= (x_0^1)^{-Q} \int_{\delta A_{\eta_0}} \frac{\psi(\zeta, \delta\eta_0)}{\alpha_Q ((x_0^1)^{-1} r(\zeta))^Q} d\zeta = \int_{\delta A_{\eta_0}} \frac{\psi(\zeta, \delta\eta_0)}{\alpha_Q r(\zeta)^Q} d\zeta \end{aligned}$$

(ψ is homogeneous of degree zero with respect to the dilation δ)

$$\begin{aligned} &= \int_{\tau\delta A_{\eta_0}} \frac{\psi(\overline{\xi_0}^{-1} \circ \xi, \delta\eta_0)}{\alpha_Q r(\overline{\xi_0}^{-1} \circ \xi)^Q} d\xi \\ &= \int_{A_{\xi_0}} \frac{\psi(\xi, \overline{\xi_0} \circ \delta\eta_0)}{\alpha_Q r(\xi)^Q} d\xi = \int_{A_{\xi_0}} \frac{\psi(\xi, \xi_0)}{\alpha_Q r(\xi)^Q} d\xi = \int_{\Omega} K(\xi, \xi_0) d\xi \end{aligned}$$

($r(\overline{\xi_0}^{-1} \circ \xi) = r(\xi)$ for $\overline{x}_0^1 = 0$). ☐

Remark 3.2. The technique used in the proof of the previous lemma does not work when $\Omega = \Omega_t$ since Ω_t is not invariant with respect to the left translations "parallel" to the hyperplane $\{t = 0\}$.

The next corollary easily follows from Lemma 3.1 and the identity (3.1) by standard arguments.

Corollary 3.3. *T is a bounded linear operator in $L^p(\Omega)$ for every $p \in [1,+\infty]$. More precisely:*

$$\| Tu \|_\infty \le \| u \|_\infty \quad \forall u \in L^\infty(\Omega), \tag{3.2}$$

$$\| Tu \|_p^p \le \beta \| u \|_p^p \quad \forall u \in L^p(\Omega),\ \forall p \in [1,+\infty[. \tag{3.3}$$

Moreover,

$$\int_\Omega Tu \ = \beta \int_\Omega u \quad \forall u \in L^1(\Omega). \tag{3.4}$$

Lemma 3.4. *In the notation of Lemma 3.1, $\beta > 1$.*

Proof. Let us consider the function

$$v = d^{-Q} \circ \tau_{\eta_0}.$$

Then $v \in C^\infty(\Omega) \cap L^2(\Omega)$. Moreover,

$$\Delta_{\mathbb{H}^n} v = (2Q\psi d^{-Q-2}) \circ \tau_{\eta_0} > 0 \quad \text{in} \ \ \Omega,$$

since, by (2.2),

$$\Delta_{\mathbb{H}^n}(d^{-Q}) = \psi\Big(Q(Q+1)d^{-Q-2} + \frac{Q-1}{d}(-Q)d^{-Q-1}\Big).$$

On the other hand, by the mean value formula (2.3),

$$\begin{aligned} Tv(\xi) - v(\xi) &= (M_{r(\xi)}v)(\xi) - v(\xi) \\ &= \frac{Q}{r(\xi)^Q} \int_0^{r(\xi)} \varrho^{Q-1}\Big(\int_{B_d(\xi,\varrho)} (\Gamma(\xi,\cdot) - \frac{c_Q}{\varrho^{Q-2}})\Delta_{\mathbb{H}^n} v\Big) d\varrho \end{aligned}$$

for every $\xi \in \Omega$. As a consequence, if we choose a compact subset K of Ω, with $|K| > 0$, there exists $\varepsilon > 0$ such that $Tv \ge v + \varepsilon$ in K. Hence, from (3.3),

$$\begin{aligned} \beta \| v \|_{L^2(\Omega)}^2 \ge \| Tv \|_{L^2(\Omega)}^2 &= \int_K |Tv|^2 + \int_{\Omega\setminus K} |Tv|^2 \\ &\ge \int_K |v+\varepsilon|^2 + \int_{\Omega\setminus K} |v|^2 \ge \varepsilon^2|K| + \int_\Omega |v|^2 \ > \| v \|_{L^2(\Omega)}^2, \end{aligned}$$

and so $\beta > 1$. □

Proof of Theorem 1.1. Since u is a $\Delta_{\mathbb{H}^n}$-superharmonic function we have $u \ge Tu$. Then

$$\int_\Omega u \ \ge \int_\Omega Tu \ = \ (\text{from (3.4)}) \ = \beta \int_\Omega u.$$

On the other hand, $u \in L^1(\Omega)$ by hypothesis and $\beta > 1$ by Lemma 3.4. Therefore, if $u \ge 0$, it must be $u = 0$ a.e. in Ω. Recalling that u is lower semicontinuous, we get $u \equiv 0$. □

Proof of Corollary 1.2. The proof immediately follows from Theorem 1.1 and Remark 2.2. □

Remark 3.5. Theorem 1.1 no longer holds on every $\mathbb{H}^n$ if we replace the hypothesis $u \in L^1(\Omega)$ with $u \in L^p(\Omega)$. Indeed, if $p \in]1, +\infty]$ and $n > \frac{1}{p-1}$, the fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole outside Ω is a strictly positive harmonic function on Ω which belongs to $L^p(\Omega)$.

Note. We do not know if Theorem 1.1 holds for halfspaces that are not parallel to the center of $\mathbb{H}^n$. However the following interesting borderline example has been pointed out to us by the referee: the function $-\partial_t(d(\xi)^{2-Q}) = nt(|z|^4 + t^2)^{-1-\frac{n}{2}}$ is $\Delta_{\mathbb{H}^n}$-harmonic away from the origin, and in the halfspace $\{t > 1\}$ it is positive and belongs to L^p for all $p > 1$ as well as to weak L^1.

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