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THE S-ELEMENTARY WAVELETS ARE PATH-CONNECTED

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ABSTRACT. A construction of wavelet sets containing certain subsets of $\mathbb{R}$ is given. The construction is then modified to yield a continuous dependence on the underlying subset, which is used to prove the path-connectedness of the s-elementary wavelets. A generalization to $\mathbb{R}^n$ is also considered.

0. INTRODUCTION

A function $f \in L^2(\mathbb{R})$ is a *dyadic orthogonal wavelet* (or simply a *wavelet* if no confusion can arise) if $\{2^{n/2}f(2^nx+l)\}_{l,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$. Alternatively, if we define $D : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $D(f)(t) = \sqrt{2}f(2t)$ and $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $T(f)(t) = f(t-1)$, then by definition f is a wavelet if and only if f is a complete wandering vector for the unitary system $\mathcal{U} = \{D^nT^l\}_{l,n\in\mathbb{Z}}$, written $f \in \mathcal{W}(\mathcal{U})$. In operator theory, the set $\mathcal{W}(\mathcal{V})$ has been studied mostly in the case that $\mathcal{V}$ is a singly generated infinite group of unitaries [H]. In our case, however, $\mathcal{U}$ is not even a semigroup, and much less is known about the structure of $\mathcal{W}(\mathcal{U})$. For example, while it is known that $\mathcal{W}(\mathcal{U})$ is not closed but still has dense span, it is not known whether $\mathcal{W}(\mathcal{U})$ is connected, i.e. pathwise in the $L^2(\mathbb{R})$ metric. However, Dai and Larson [DL] showed that $\mathcal{W}(\mathcal{U})$ does not have any trivial components.

Now, if we define $\hat{D} = \mathcal{F}D\mathcal{F}^{-1} = D^{-1}$ and $\hat{T} = \mathcal{F}T\mathcal{F}^{-1} = M_{e^{-is}}$, where $\mathcal{F}$ is the Fourier transform and M_g is multiplication by g, then f is a wavelet if and only if $\hat{f} \in \mathcal{W}(\hat{\mathcal{U}}) = \mathcal{W}(\{\hat{D}^n\hat{T}^l\}_{l,n\in Z})$. One example of such a function is $\frac{1}{\sqrt{2\pi}}1_E$, where $E = [-2\pi, -\pi) \cup [\pi, 2\pi)$. It is not hard to see that when W is measurable, $\frac{1}{\sqrt{2\pi}}1_W$ is the Fourier transform of a wavelet if and only if (modulo null sets)

$$\bigcup_{i=-\infty}^{\infty} 2^iW = \mathbb{R}, \qquad \bigcup_{i=-\infty}^{\infty} (W + 2\pi i) = \mathbb{R}$$

with both unions disjoint. Following Dai and Larson, we call such a set W a *wavelet set* and the inverse Fourier transform of $\frac{1}{\sqrt{2\pi}}1_W$ an *s-elementary wavelet.*

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The importance of translations by 2π and dilations by 2 is such that we are led to define maps $\tilde{\tau}:\mathbb{R}\to[-2\pi,-\pi)\cup[\pi,2\pi)$ and $\tilde{d}:\mathbb{R}_0=(\mathbb{R}\backslash\{0\})\to[-2\pi,-\pi)\cup[\pi,2\pi)$ by

$$\tilde{\tau}(x)=x+2\pi m(x),\qquad \tilde{d}(x)=2^{n(x)}x,$$

where m and n are the unique integers which map x into $[-2\pi,-\pi)\cup[\pi,2\pi)$ under translation and dilation respectively. In this notation, W is a wavelet set if and only if $\tilde{d}|_W$ and $\tilde{\tau}|_W$ are bijections (modulo null sets). If $\tilde{d}|_W$ a nd $\tilde{\tau}|_W$ are injections, modulo null sets, then we say that W is a *sub-wavelet set*. It should be noted that not every sub-wavelet set is a subset of a wavelet set; for example, it is not hard to see that $[2\pi,4\pi)$ is such a set.

We say that measurable sets A and B are *2-dilation congruent* if there is a measurable partition $\{A_n\}_{n=-\infty}^{\infty}$ of A such that $B=\bigcup_{n=-\infty}^{\infty}2^nA_n$ disjointly. Similarly, A and B are 2π-*translation congruent* if there is a measurable partition $\{A_n\}_{n=-\infty}^{\infty}$ of A such that $B=\bigcup_{n=-\infty}^{\infty}2\pi n+A_n$ disjointly.

In addition, we let λ denote Lebesgue measure on $\mathbb{R}$ and μ denote the measure defined by $\mu(A)=\int 1_A(x)\frac{d\lambda}{|x|}$. Also, for any measurable subset A of $\mathbb{R}$ we let $\mathcal{M}(A)$ denote the collection of all measurable subsets B of A such that $\lambda(B)<\infty$ and $\mu(B)<\infty$. Then,

$$d:(\mathcal{M}(\mathbb{R}_0),\mu)\to(\mathcal{M}([-2\pi,-\pi)\cup[\pi,2\pi)),\mu)$$

and

$$\tau:(\mathcal{M}(\mathbb{R}),\lambda)\to(\mathcal{M}([-2\pi,-\pi)\cup[\pi,2\pi)),\lambda)$$

defined by $A^d=d(A)=\tilde{d}(A)$ and $A^\tau=\tau(A)=\tilde{\tau}(A)$ are locally measure preserving injections (by this we mean that for any x [resp. $x\neq 0$], there is a neighborhood U of x such that $\tau|_{\mathcal{M}(U),\lambda}$ [resp. $d|_{\mathcal{M}(U),\mu}$] is a measure preserving injection). The maps d and τ will be studied in more detail in section 2.

Dai and Larson showed that given two wavelet sets of a certain type, it is possible to interpolate between the corresponding s-elementary wavelets. In this manner, they were able to reconstruct Meyer's wavelet. So, it is natural to ask whether it is possible to connect any two given s-elementary wavelets by a path of wavelets. In this paper we show that in fact, the collection of all s-elementary wavelets forms a path-connected subset of $L^2(\mathbb{R})$ by showing that the collection of all wavelet sets is path-connected in the symmetric difference metric. In particular, there is a path connecting Shannon's wavelet to Journe's wavelet, so the collection of all MRA wavelets does not form a connected component of the collection of all wavelets. This answers a question Larson posed in a seminar at Texas A&M University in the summer of 1994.

In addition to the paper [DL], there have been two recent developments in the theory of s-elementary wavelets. Hernandez, Wang and Weiss have considered the smoothing of s-elementary wavelets in [HWW1] and [HWW2], while Fang and Wang [FW] have considered properties of wavelet sets as subsets of $\mathbb{R}$.

1. Subsets of wavelet sets

For motivation, we present a special case of the criterion in [DLS] for a set to be contained in a wavelet set. The construction given will be modified in section 2 to give continuity.

Theorem 1.1. *Let A be a sub-wavelet set. Suppose*

(1) *there is an $\epsilon > 0$ such that $A^\tau \subset [-2\pi + \epsilon, -\pi) \cup [\pi, 2\pi - \epsilon)$ and*
(2) *$S = ([-2\pi, -\pi) \cup [\pi, 2\pi)) \setminus A^d$ has non-empty interior.*

Then, there is a wavelet set $W \supset A$.

For the proof of Theorem 1.1 we need two lemmas.

Lemma 1.2. *Let E be a subset of $[-2\pi, -\pi) \cup [\pi, 2\pi)$ which has non-empty interior. Let F be any subset of $[-2\pi, -\pi) \cup [\pi, 2\pi)$. Then there is a set G such that the following two conditions hold.*

(i) *G is 2-dilation congruent to a subset of E and*
(ii) *G is 2π-translation congruent to F.*

Proof. Let x, ϵ be such that $B_\epsilon(x) \subset E$. Let n be an integer such that $2^n\epsilon > 6\pi$. Then, there is an integer m such that $([-2\pi, -\pi) \cup [\pi, 2\pi)) + 2\pi m$ is contained in $2^n B_\epsilon(x)$. Let $G = F + 2\pi m$ and see that G satisfies conditions (i) and (ii). $\square$

Lemma 1.3. *Let $\epsilon > 0$. The sets $[-2\pi, -\pi) \cup [\pi, 2\pi)$ and $[-\epsilon, -\epsilon/2) \cup [\epsilon/2, \epsilon)$ are 2-dilation congruent. As a consequence, for any set E a subset of $[-2\pi, -\pi) \cup [\pi, 2\pi)$, there is a unique $G \subset [-\epsilon, -\epsilon/2) \cup [\epsilon/2, \epsilon)$ such that G is 2-dilation congruent to E.*

Proof. Since $[-2\pi, -\pi) \cup [\pi, 2\pi)$ and $[-\epsilon, -\epsilon/2) \cup [\epsilon/2, \epsilon)$ are sub-wavelet sets, by Lemma 4.2 in [DL], it suffices to show

$$\bigcup_{i=-\infty}^{\infty} 2^i([-2\pi, -\pi) \cup [\pi, 2\pi)) = \bigcup_{i=-\infty}^{\infty} 2^i([-\epsilon, -\epsilon/2) \cup [\epsilon/2, \epsilon)).$$

It is easy to see that

$$\bigcup_{i=-\infty}^{\infty} 2^i([-2\pi, -\pi) \cup [\pi, 2\pi)) = \mathbb{R} = \bigcup_{i=-\infty}^{\infty} 2^i([-\epsilon, -\epsilon/2) \cup [\epsilon/2, \epsilon)).$$

The second statement of the lemma is now immediate. $\square$

Proof of Theorem 1.1. Write S as the disjoint union of sets $\{E_i\}_{i=1}^{\infty}$, each of which has non-empty interior. Let $F_1 = ([-2\pi + \epsilon, -\pi) \cup [\pi, 2\pi - \epsilon)) \setminus A^\tau$. By Lemma 1.2, there is a set G_1 which is 2π-translation congruent to F_1 and 2-dilation congruent to a subset of E_1.

Use Lemma 1.3 with $E = E_1 \setminus (G_1 \cup A)^d$ to get G_2 contained in $[-\epsilon, -\epsilon/2) \cup [\epsilon/2, \epsilon)$ which is 2-dilation congruent to E.

Note that by construction, $A \cup G_1 \cup G_2$ is a sub-wavelet set which satisfies condition (1) of Theorem 1.1 with $\epsilon' = \epsilon/2$ and condition (2) with $S = \bigcup_{i=2}^{\infty} E_i$.

Let $F_2 = ([-2\pi + \epsilon/2, -\pi) \cup [\pi, 2\pi - \epsilon/2)) \setminus (A \cup G_1 \cup G_2)^\tau$. By Lemma 1.2, there is a set G_3 which is 2π-translation congruent to F_2 and 2-dilation congruent to a subset of E_2.

Use Lemma 1.3 with $E = E_2 \setminus (G_3 \cup G_2 \cup G_1 \cup A)^d$ to get G_4 contained in $[-\epsilon/2, -\epsilon/4) \cup [\epsilon/4, \epsilon/2)$ which is 2-dilation congruent to E.

Continuing in this fashion, it is easy to see that $W = A \cup (\bigcup_{i=1}^{\infty} G_i)$ is a wavelet set containing A. $\square$

Note that $A = [2\pi, 4\pi)$ is not a wavelet set since $A^d = [\pi, 2\pi)$. (In fact, since every wavelet set must have Lebesgue measure 2π, A is not even contained in a wavelet set.) However, by Theorem 1.1 there are wavelet sets W_n containing

$[2\pi+1/n, 4\pi-1/n)$ which necessarily converge to $[2\pi, 4\pi)$. Since it is generally easier to prove connectedness of closed sets, this partially explains why the next section is technical.

2. Paths of wavelet sets

It is shown that the wavelet sets are path-connected under the metric $d_\lambda(A,B) = \lambda(A\triangle B)$, where λ is Lebesgue measure on $\mathbb{R}$, which is equivalent to the $L^2(\mathbb{R})$ metric restricted to the s-elementary wavelets. The idea is to first connect any two wavelet sets by a path of sub-wavelet sets, then for each sub-wavelet set in the path, we will construct a containing wavelet set which depends continuously on the underlying sub-wavelet set. In order to do so, we will have to construct continuous analogues of Lemma 1.2 and Lemma 1.3.

Given a measure m, d_m will denote the metric $d_m(A,B) = m(A\triangle B)$. We note here that neither the identity map

$$I : (\mathcal{M}(\mathbb{R}), d_\lambda) \to (\mathcal{M}(\mathbb{R}), d_\mu)$$

nor its inverse is continuous. We do, however, have the following lemma.

Lemma 2.1. *Let V be a measurable subset of $\mathbb{R}_0$ such that $\lambda(V), \mu(V) < \infty$. Then $I : (\mathcal{M}(V), d_\lambda) \to (\mathcal{M}(V), d_\mu)$ is a homeomorphism.*

Proof. The statement follows from the absolute continuity of the integral, but we give a proof from the first principles to familiarize the reader with the type of argument used below.

For the continuity of I, it suffices to show that for every $\epsilon > 0$, there is a $\delta > 0$ such that whenever $\lambda(U) < \delta, \mu(U) < \epsilon$. So, fix $\epsilon > 0$ and note that for all $c > 0$, the set which maximizes $\{\mu(S) : \lambda(S) \le c\}$ is of the form $(-a, a)\cap V$ for some positive a. Now, since $\mu(V) < \infty$ and μ is non-atomic, there is some $b > 0$ such that $\mu((-b,b)\cap V) < \epsilon$. Let $\delta = \lambda((-b,b)\cap V)$. Then, whenever $\lambda(U) < \delta$, we have $\mu(U) \le \mu((-b,b)\cap V) < \epsilon$ as desired.

For the continuity of I^{-1}, note that for all $d > 0$, the set which maximizes $\{\lambda(T) : \mu(T) \le d\}$ is of the form $((-\infty,-N]\cup[N,\infty))\cap V$ for some non-negative N. Pick N such that $\lambda(((-\infty,-N]\cup[N,\infty))\cap V) < \epsilon$ and let $\delta = \mu(((-\infty,-N]\cup[N,\infty))\cap V)$. Then, whenever $\mu(U) < \delta$, we have $\lambda(U) < \epsilon$ as desired. □

An immediate consequence of Lemma 2.1 is the following lemma.

Lemma 2.2. *Let W be a sub-wavelet set. Then, the functions $d|_W$ and $\tau|_W$: $(\mathcal{M}(W), d_\lambda) \to (\mathcal{M}([-2\pi,-\pi)\cup[\pi,2\pi)), d_\lambda)$ are continuous with continuous inverses.*

Proof. The function $\tau|_W$ is an isometry. On the other hand, $d|_W$ is an isometry from $(\mathcal{M}(W), d_\mu) \to (\mathcal{M}([-2\pi,-\pi)\cup[\pi,2\pi)), d_\mu)$; hence, it is a homeomorphism. So, the lemma as stated follows from Lemma 2.1. □

The following two lemmas are continuous analogues of Lemma 1.2 and 1.3.

Lemma 2.3. *Let $A \subset ([-2\pi,-\pi)\cup[\pi,2\pi)), s > 0$, and let $N = N(A,s)$ be the unique subset of $([-s,-1/2s)\cup[1/2s,s))$ which is 2-dilation congruent to A. Then N depends (jointly) continuously on A and s with respect to the metric d_λ.*

Proof. The uniqueness follows from Lemma 1.3.

Write $F_r = [-r, -\frac{1}{2}r)\cup[\frac{1}{2}r, r)$. Then, $F_r^d = [-2\pi,-\pi)\cup[\pi,2\pi)$. □

Claim. For $s, t > 0$,

(1) $N(A \cap (F_s \cap F_t)^d, s) = N(A \cap (F_s \cap F_t)^d, t)$ and
(2) $N(A \cap (F_s \setminus F_t)^d, s) \subset F_s \setminus F_t$.

Proof of Claim. (1) $N(A \cap (F_s \cap F_t)^d, s)$ is the unique subset of F_s which is 2-dilation congruent to $A \cap (F_s \cap F_t)^d$. Since $N(A \cap (F_s \cap F_t)^d, t)$ is also 2-dilation congruent to $A \cap (F_s \cap F_t)^d$, it suffices to show that $N(A \cap (F_s \cap F_t)^d, t)$ is a subset of F_s. This follows since $N((F_s \cap F_t)^d, t) = F_s \cap F_t \subset F_s$.

(2) It suffices to show that $N((F_s \setminus F_t)^d, s) = F_s \setminus F_t$. To see this, note that $F_s \setminus F_t$ is 2-dilation congruent to $(F_s \setminus F_t)^d$ and is a subset of F_s. Therefore, since $N((F_s \setminus F_t)^d, s)$ is the unique set satisfying the conditions in the previous sentence, it follows that $N((F_s \setminus F_t)^d, s) = F_s \setminus F_t$.

Returning to the proof of Lemma 2.3, write $U' = U \cap (F_s \cap F_t)^d$ for $U = A, B$. Then, $A' \cup (A \cap (F_s \setminus F_t)^d) = A$. So, using the definition of N and property (2) above, we have

$$\begin{aligned} N(A, s) =& N(A' \cup (A \cap (F_s \setminus F_t)^d), s) \\ =& N(A', s) \cup N(A \cap (F_s \setminus F_t)^d, s) \end{aligned}$$

and

$$N(B, t) = N(B', t) \cup N(B \cap (F_t \setminus F_s)^d, t).$$

So, since $N(A \cap (F_s \setminus F_t)^d, s) \cap N(B \cap (F_t \setminus F_s)^d, t) = \emptyset$, we have (writing $T = N(A, s) \triangle N(B, t)$)

$$\begin{aligned} T &= ((N(A', s) \cup N(A \cap (F_s \setminus F_t)^d, s)) \triangle (N(B', t) \cup N(B \cap (F_t \setminus F_s)^d, t)) \\ &\subset (N(A', s) \triangle N(B', t)) \cup N(A \cap (F_s \setminus F_t)^d, s) \cup N(B \cap (F_t \setminus F_s)^d, t) \\ &\subset (F_s \triangle F_t) \cup (N(A', s) \triangle N(B', t)) \\ &= (F_s \triangle F_t) \cup (N(A', s) \triangle N(B', s)). \end{aligned}$$

Therefore, $N(A, s) \triangle N(B, t) \to \emptyset$ as $A \to B$ and $s \to t$ since $F_s \triangle F_t \to \emptyset$ as $s \to t$ (by definition of F_r) and $N(A', s) \triangle N(B', s) \to \emptyset$ as $A \to B$ by Lemma 2.2 with $W = F_s$. Therefore, the map $(A, s) \to N(A, s)$ is continuous. □

Lemma 2.4. *Let A be a subset of $[-2\pi, -\pi) \cup [\pi, 2\pi)$, and let (x_n) be a sequence of real numbers which decreases to -2π and is bounded above by $-\pi$. In addition, let $s > 0$ and let $k = k(s)$ be the smallest integer such that $x_{k(s)} < -2\pi + s$. Then, there is a function $M = M(A, s)$ such that*

(i) *M is 2π-translation congruent to A*
(ii) *M is 2-dilation congruent to a subset of $[x_{k(s)+1}, -2\pi + s)$*
(iii) *M is a (jointly) continuous function of A and s with respect to the metric $d\lambda$, and*
(iv) *if $A_1 \cap A_2 = \emptyset$ and $s_1, s_2 > 0$, then $M(A_1, s_1)^d \cap M(A_2, s_2)^d = \emptyset$.*

Proof. We give an explicit construction of M. For each n, choose the smallest integer m_n such that $\lambda(2^{m_n}[x_{n+1}, x_n)) \geq 6\pi$ and the smallest integer r_n such that $([-2\pi, -\pi) \cup [\pi, 2\pi)) + 2\pi r_n$ is a subset of $2^{m_n}[x_{n+1}, x_n)$. Note that m_n, r_n and x_n depend neither on A nor s.

Let $M_1 = M_1(A, s) = (A + 2\pi r_{k(s)-1}) \cap 2^{m_{k(s)-1}}[x_{k(s)}, -2\pi + s)$. Let $B = B(A, s) = A \setminus M_1^\tau$, and let $M_2 = M_2(A, s) = B + 2\pi r_{k(s)}$. Then, we claim that $M = M_1 \cup M_2$ satisfies (i) through (iv) of Lemma 2.4.

Conditions (i) and (ii) are immediate. Indeed, $B + 2\pi r_{k(s)} \subset ([-2\pi, -\pi) \cup [\pi, 2\pi)) + 2\pi r_{k(s)} \subset 2^{m_{k(s)}}[x_{k(s)+1}, x_{k(s)})$. Therefore, (ii) follows from the fact $[x_{k(s)+1}, x_{k(s)}) \cap [x_{k(s)}, -2\pi + s) = \emptyset$.

To prove (iii), let $A_n \to A$ and $t_n \to t$. There are two cases to consider. First, suppose that $k(t_n) \to k(t)$ (since k is integer valued, this says $k(t_n) = k(t)$ for large n). It suffices to show that $M_1(A_n, t_n) \to M_1(A, t)$ and $M_2(A_n, t_n) \to M_2(A, t)$. To see this, we have

$$\begin{aligned} M_1(A_n, t_n) =&(A_n + 2\pi r_{k(t_n)-1}) \cap 2^{m_{k(t_n)-1}}[x_{k(t_n)}, -2\pi + t_n) \\ \to&(A + 2\pi r_{k(t)-1}) \cap 2^{m_{k(t)-1}}[x_{k(t)}, -2\pi + t) \\ =&M_1(A, t). \end{aligned}$$

So, $B(A_n, t_n) = A_n \setminus M_1(A_n, t_n)^\tau \to A \setminus M_1(A, t)^\tau = B(A, t)$. Thus, $M_2(A_n, t_n) = B(A_n, t_n) + 2\pi r_{k(t_n)} \to B(A, t) + 2\pi r_{k(t)} = M_2(A, t)$, as desired.

Now suppose that $k(t_n) \nrightarrow k(t)$. By definition of k, this can only occur when $-2\pi + t = x_m$ for some m. In addition, if t_n increases to t, then $k(t_n) \to k(t)$, so we can assume that t_n decreases to t. Note that then $k(t_n) \to k(t) - 1$, so we may assume that $k(t_n) = k(t) - 1$. Then,

$$\begin{aligned} M_1(A_n, t_n) =&(A_n + 2\pi r_{k(t_n)-1}) \cap 2^{m_{k(t_n)-1}}[x_{k(t_n)}, -2\pi + t_n) \\ =&(A_n + 2\pi r_{k(t)-2}) \cap 2^{m_{k(t)-2}}[x_{k(t)-1}, -2\pi + t_n) \\ \to&(A + 2\pi r_{k(t)-2}) \cap 2^{m_{k(t)-2}}[x_{k(t)-1}, -2\pi + t) = \emptyset. \end{aligned}$$

(Since $k(t)$ is the smallest natural number such that $x_{k(t)} < -2\pi + t$, we have $x_{k(t)-1} \geq -2\pi + t$. Therefore, $[x_{k(t)-1}, -2\pi + t) = \emptyset$ modulo a null set.) So, $M(A_n, t_n) = M_2(A_n, t_n) = (A_n \setminus M_1^\tau) + 2\pi r_{k(t_n)} \to A + 2\pi r_{k(t)-1}$. On the other hand,

$$\begin{aligned} M_1(A, t) =&(A + 2\pi r_{k(t)-1}) \cap 2^{m_{k(t)-1}}[x_{k(t)}, -2\pi + t) \\ =&A + 2\pi r_{k(t)-1}. \end{aligned}$$

Indeed, since $-2\pi + t = x_m$ for some m, $([-2\pi, -\pi) \cup [\pi, 2\pi)) + 2\pi r_{k(t)-1} \subset [x_{k(t)}, -2\pi + t)$. Therefore, $M(A, t) = M_1(A, t) = A + 2\pi r_{k(t)-1}$. So, we have $M(A_n, t_n) \to M(A, t)$ and (iii) is proved.

To prove (iv), note that by construction we can write $M(A_1, s_1) = \bigcup_{n=0}^\infty 2^{m_n} G_n$ and $M(A_2, s_2) = \bigcup_{n=0}^\infty 2^{m_n} H_n$ with $G_n, H_n \subset [x_{n+1}, x_n)$ and $x_0 = -\pi$. Then, $M(A_1, s_1)^d = \bigcup_{n=0}^\infty G_n$ and $M(A_2, s_2)^d = \bigcup_{n=0}^\infty H_n$, so $M(A_1, s_1)^d \cap M(A_2, s_2)^d = \bigcup_{n=0}^\infty (H_n \cap G_n)$. Therefore, it suffices to show that $G_n \cap H_n = \emptyset$ for all n. But, this follows from the fact that $2^{m_n} G_n - 2\pi r_n \subset A_1$, $2^{m_n} H_n - 2\pi r_n \subset A_2$ and $A_1 \cap A_2 = \emptyset$. □

We are now ready for the main result of this paper.

Theorem 2.5. *The wavelet sets are path-connected in the symmetric difference metric.*

Proof. It is shown that every wavelet set W is path-connected to $[-2\pi, -\pi) \cup [\pi, 2\pi)$. The first step is to construct a path of subsets of wavelet sets, then for each set in the path, we will find a wavelet superset which depends continuously on the subset. So, for $0 \leq t \leq \pi$ let R_t be the subset of W which is 2π-translation congruent to $[-\pi - t, -\pi) \cup [\pi, \pi + t)$, i.e. $R_t^\tau = [-\pi - t, -\pi) \cup [\pi, \pi + t)$. Let P_t

be the subset of W 2-dilation congruent to $[-2\pi, -2\pi + t) \cup [2\pi - t, 2\pi)$ and let $Q_t = [-2\pi + t, -\pi - t) \cup [\pi + t, 2\pi - t)$. Then the path of sets defined by

$$S_t = \begin{cases} [(Q_t \cup R_t) \setminus (R_t^d \cap Q_t)] \setminus P_t & \text{if } 0 \leq t \leq \pi/2, \\ R_t \setminus P_{\pi - t}, & \text{if } \pi/2 \leq t \leq \pi \end{cases}$$

is a path of subsets of wavelet sets which connect E to W and which are of the type mentioned in Proposition 1.1 for $0 < t < \pi$. Note that the path is continuous by the continuity of dilations, translations and set operations with respect to d_m.

The second step is a recursive construction of sets M_i, continuous functions of t, such that $S_t \cup (\bigcup_{i=0}^{\infty} M_i)$ is a wavelet set for each t. Fix x_n and the function k as in Lemma 2.4. Also, define $C_i = C_i(t)$ to be $[-2\pi + t/2^i, -2\pi + t/2^{i-1}) \cup [2\pi - t/2^i, 2\pi - t/2^{i-1})$ for $i = 1, 2, 3, \ldots$ and C_0 to be $[-2\pi + t, -\pi) \cup [\pi, 2\pi - t)$. For notational convenience in what follows, we denote

$$t' = \begin{cases} t & \text{if } 0 \leq t \leq \pi/2, \\ \pi - t & \text{if } \pi/2 \leq t \leq \pi. \end{cases}$$

Finally, recall the definitions of the functions N and M from Lemma 2.3 and 2.4 respectively.

Notice that we have the following relations.

(1) $C_0(t') \supset S_t^d$,

(2) $C_0(t') \supset S_t^\tau$.

Now, let $A_0 = A_0(t) = C_0(t') \setminus S_t^d$, $s_0 = s_0(t) = t'$ and define $M_0 = N(A_0, s_0)$, which is a continuous function of t by Lemma 2.3. Note that

(1) $[x_{k(t')+1}, -\pi) \cup [\pi, -x_{k(t')}) \supset (M_0 \cup S_t)^d \supset C_0(t')$, and

(2) $C_0(t') \cup C_1(t') \supset (M_0 \cup S_t)^\tau$.

Then let $B_0 = B_0(t) = C_0(t') \setminus (M_0 \cup S_t)^\tau$, $r_0 = t'$ and define $M_1 = M(B_0, r_0)$, which is a continuous function of t by Lemma 2.4. Note that

(1) $[x_{k(t')+1}, -\pi) \cup [\pi, -x_{k(t')+1}) \supset (M_0 \cup M_1 \cup S_t)^d \supset C_0(t')$, and

(2) $C_0(t') \cup C_1(t') \supset (M_0 \cup M_1 \cup S_t)^\tau \supset C_0(t')$.

For the recursive definition, suppose we have $M_0, \ldots, M_{2p-1}$ defined such that each is a continuous function of t and for all r between 0 and $p - 1$ the following conditions hold.

(1a) $[x_{k(t'/2^r)+1}, -\pi) \cup [\pi, -x_{k(t'/2^r)+1}) \supset (\bigcup_{i=0}^{2r} M_i \cup S_t)^d \supset \bigcup_{i=0}^{r} C_i(t')$, and

(1b) $\bigcup_{i=0}^{r+1} C_i a(t') \supset (\bigcup_{i=0}^{2r} M_i \cup S_t)^\tau \supset \bigcup_{i=0}^{r-1} C_i(t')$;

(2a) $[x_{k(t'/2^r)+1}, -\pi) \cup [\pi, -x_{k(t'/2^r)+1}) \supset (\bigcup_{i=0}^{2r+1} M_i \cup S_t)^d \supset \bigcup_{i=0}^{r} C_i(t')$, and

(2b) $\bigcup_{i=0}^{r+1} C_i(t') \supset (\bigcup_{i=0}^{2r+1} M_i \bigcup S_t)^\tau \supset \bigcup_{i=0}^{r} C_i(t')$.

Then, we need to define M_{2p}, M_{2p+1} having the same properties. Let $A_p = C_p(t') \setminus (\bigcup_{i=1}^{2p-1} M_i \cup S_t)^d$, $s_p = t'/2^p$ and define $M_{2p} = N(A_p, s_p)$, which is a continuous function of t. Let $B_p = C_p(t') \setminus (\bigcup_{i=1}^{2p} M_i \cup S_t)^\tau$, $r_p = t'/2^p$, and define $M_{2p+1} = M(B_p, r_p)$, which is a continuous function of t. Note that M_{2p+1}^d has empty intersection with M_i^d for $i \leq 2p$ by Lemma 2.4 when i is odd and by construction when i is even. Thus, we have defined wavelet sets $W_t = \bigcup_{i=0}^{\infty} M_i \cup S_t$.

In order to see that this is a continuous construction, note that

$$m(\bigcup_{i=2r+1}^{\infty} M_i) \leq t'/2^r$$

by condition (2) and the definition of C_i. This estimate together with the continuity of the pieces gives the continuity of W_t. $\square$

Corollary 2.6. *The s-elementary wavelets form a path-connected subset of $L^2(\mathbb{R})$.*

Proof. This follows immediately from Theorem 2.5 and the fact that d_λ is equivalent to the $L^2(\mathbb{R})$ metric restricted to the s-elementary wavelets. $\square$

3. Generalizations to $\mathbb{R}^n$

We begin with the definition of an n-dimensional wavelet set. Let $D : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear transform whose inverse has norm less than 1, and let $\{T_i : i = -\infty ... \infty\}$ be the translations by multiples of 2π in every coordinate direction, i.e. $T_i(x_1, ..., x_n) = (x_1 + 2\pi k_{1,i}, ..., x_n + 2\pi k_{n,i})$. Then, we say a measurable set W is an *n-dimensional wavelet set* if $(2\pi)^{-n/2} 1_W$ is the Fourier transform of a wavelet f, i.e. $\{(\det(D))^{m/2} f(D^m x + (k_1, ..., k_n) : m, k_1, ..., k_n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathbb{R}^n$. We call this inverse Fourier transform an *n-dimensional s-elementary wavelet,* where D is understood.

As was shown in [DLS], a set W is an n-dimensional wavelet set if and only if

$$\bigcup_{i=-\infty}^{\infty} D^i(W) = \mathbb{R}^n \quad \text{and} \quad \bigcup_{i=-\infty}^{\infty} T_i(W) = \mathbb{R}^n,$$

with both unions disjoint. The proof of the existence of such sets is given in [DLS], and the idea is similar to the proof of Theorem 1.1.

The proof that the n-dimensional wavelet sets are path-connected is largely a matter of developing the correct analogues to the 1-dimensional case. Therefore, we will omit some details when convenient. (The full proof appears in the author's dissertation [S].) For sets $A, B \subset \mathbb{R}^n$ with $\mathrm{conv}(A) \subset \mathrm{conv}(B)$, we define the annulus $\mathcal{A}(A, B)$ to be $\mathrm{conv}(B) \setminus \mathrm{conv}(A)$. As suggested by the name, we will be mostly concerned with the case that A and B are ellipsoids.

Let S^n denote the unit sphere in $\mathbb{R}^n$ and consider $\mathcal{A}(S^n, D(S^n))$. (Note that since $\|D^{-1}\| < 1$, $\mathcal{A}(S^n, D(S^n))$ is not empty.) Now, let $(w_1, \ldots, w_n)$ be orthonormal axes of $D(S^n)$; in other words, there exist constants $(a_1, \ldots, a_n)$ such that $a_i > 1$ and $D(S^n) = \{(x_1, \ldots, x_n) : \sum_{i=1}^n \frac{x_i^2}{a_i^2} = 1\}$, where $(x_1, \ldots, x_n)$ is the coordinatization of a point in $\mathbb{R}^n$ with respect to the orthonormal basis $(w_1, \ldots, w_n)$. Define $f_i : [1, 2] \to [1, a_i]$ by $f_i(s) = (2 - s) + (1 - s)a_i$, and for $1 \le s \le 2$, let P_s be the ellipsoid defined by $P_s = \{(x_1 \ldots, x_n) : \sum_{i=1}^n \frac{x_i^2}{f_i(s)^2} = 1\}$.

We extend P_s to the positive axis as follows. Let $s > 0$ and let n be the unique integer such that $2^n t \in (1, 2]$. Define $P_s = D^{-n}(P_{2^n s})$. Finally, we define $P_0 = \{0\}$.

As in the one-dimensional case, we say that $E, F \subset \mathbb{R}^n$ are D-congruent if there is a partition $\{E_i\}_{i=-\infty}^{\infty}$ of E such that

$$\bigcup_{i=-\infty}^{\infty} D^i(E_i) = F.$$

Similarly, we say $E, F \subset \mathbb{R}^n$ are T-congruent if there is a partition $\{E_i\}_{i=-\infty}^{\infty}$ of E such that

$$\bigcup_{i=-\infty}^{\infty} T_i(E_i) = F.$$

Also, given wavelet sets W and $E \subset W$, we define E^τ to be the subset of $[-\pi,\pi]^n$ which is T-congruent to E and E^d to be the subset of $\mathcal{A}(S^n, D(S^n))$ which is D-congruent to E. One can check that since D and D^{-1} are bounded, and all wavelet sets have finite Lebesgue measure, the functions $E \to E^d$ and $E \to E^\tau$ are continuous with continuous inverses when restricted to subsets of a given wavelet set.

We now give the n-dimensional analogues to Lemmas 2.3 and 2.4.

Lemma 3.1. *Let $A \subset \mathcal{A}(P_\pi, P_{2\pi}), s > 0$. Then there is a unique subset $N(A,s)$ of $\mathcal{A}(P_{s/2}, P_s)$ which is D congruent to A. Furthermore, N is a jointly continuous function of A and s.*

Proof. The uniqueness follows from the fact that $\mathcal{A}(P_{s/2}, P_s)$ is a D-dilation generator of $\mathbb{R}^n$. (By this, we mean that $\bigcup_{i=-\infty}^{\infty} D^i(\mathcal{A}(P_{s/2}, P_s)) = \mathbb{R}^n$, with the union disjoint.) We will, however, need the explicit formula for N, so we give it here. Let $n = n(s)$ be the unique integer such that $\pi \le 2^n s < 2\pi$. Then,

$$N(A,s) = D^{-(n+1)}(\mathcal{A}(P_{2^n s}, P_{2\pi}) \cap A) \cup D^{-n}(\mathcal{A}(P_\pi, P_{2^n s}) \cap A).$$

To prove continuity, let $A_m \to A$ and $s_m \to s$. Then, there are two cases to consider. First, suppose $n(s_m) \to n(s)$. Then, we may assume that $n(s_m) = n(s) = n$. So, the continuity follows from the continuity of unions, intersections, P_s and D.

The second case is that $n(s_m) \nrightarrow n(s)$. By passing to a subsequence if necessary, we may assume $\pi \le 2^{n(s_m)} s < 2\pi$ and $2^{n(s_m)} s \to 2\pi$. Then, for all m, $2^{n(s_m)} s = 2\pi$, $n(s) = n(s_m) - 1$ and $2^{n(s)} = \pi$. We write $n = n(s)$. So,

$$\begin{aligned} N(A_m, s_m) =& D^{-n-2}(\mathcal{A}(P_{2^{n+1} s_m}, P_{2\pi}) \cap A_m) \cup D^{-n-1}(\mathcal{A}(P_\pi, P_{2^n s_m}) \cap A_m) \\ \to& \emptyset \cup D^{-(n+1)}(\mathcal{A}(P_\pi, P_{2\pi}) \cap A) \\ =& N(A,s), \end{aligned}$$

as desired. ∎

Lemma 3.2. *Let A be a subset of $[-\pi,\pi)^n$ and let $\{x_m\}_{m=1}^\infty$ be a sequence of real numbers which decreases to π and is bounded above by 2π. In addition, let $\pi > s > 0$, and let $k = k(s)$ be the smallest integer such that $x_{k(s)} < \pi + s$. Then, there is a function $M = M(A,s)$ such that*

(i) *M is T-congruent to A,*
(ii) *M is D-congruent to a subset of $\mathcal{A}(P_{x_{k(s)+1}}, P_{\pi+s})$,*
(iii) *M is a jointly continuous function of A and s, and*
(iv) *if $A_1 \cap A_2 = \emptyset$ and $s_1, s_2 > 0$, then $M(A_1,s_1)^d \cap M(A_2,s_2)^d = \emptyset$.*

Proof. We define M explicitly as in the proof of Lemma 2.4. For each natural number j, let m_j be a large enough integer such that there is an r_j with

$$T_{r_j}([-\pi,\pi)^n) \subset D^{m_j}(\mathcal{A}(P_{x_{j+1}}, P_{x_j})).$$

To define M, let $M_1 = M_1(A,s) = T_{r_{k(s)-1}}(A) \cap D^{m_{k(s)-1}}(\mathcal{A}(P_{x_{k(s)}}, P_{\pi+s}))$. Let E be the subset of A which is T-congruent to $M_1(A,s)$. Define $B = B(A,s) = A \backslash E$. Let $M_2 = M_2(A,s) = T_{r_{k(s)}}(B)$. Then $M = M_1 \cup M_2$ satisfies conditions (i) through (iv) of Lemma 3.2.

The proof that M satisfies the above conditions is identical to the proof of Lemma 2.4 with T playing the role of translation, D playing the role of dilation, and P_{x_n} playing the role of x_n, so we omit the details. ∎

Theorem 3.3. *The collection of all n-dimensional wavelet sets is path-connected in the symmetric difference metric.*

Proof. Let W_1, W_2 be wavelet sets. The first step is to construct a path of sub-wavelet sets which connects W_1 to W_2. Let $0 < t_0 \leq \pi/2$ be small enough so that $P_{t_0} \subset [-\pi, \pi]^n$. (This is possible since $\|D^{-1}\| < 1$.) Let $B_t = [-t, t]^n$. For $0 \leq t \leq \pi$, we have the following definitions.

O_t is the subset of W_1 such that $O_t^\tau = B_\pi \setminus B_t$,
Q_t is the subset of W_2 such that $Q_t^\tau = B_t$,
R_t is the subset of W_1 such that $R_t^d = \mathcal{A}(P_\pi, P_{\pi+\min(t,t_0)})$,
X_t is the subset of W_2 such that $X_t^d = \mathcal{A}(P_\pi, P_{\pi+\min(t,t_0)})$,
U_t is the subset of W_1 such that $U_t^d = O_t^d \cap Q_t^d$, and
V_t is the subset of $O_t \cup Q_t$ such that $V_t^\tau = \mathcal{A}(\{0\}, P_{\min(t,t_0)})$.

Then, we have that S_t defined by

$$S_t = \begin{cases} (O_t \cup Q_t) \setminus (R_t \cup X_t \cup U_t \cup V_t) & \text{if } 0 \leq t \leq \pi/2, \\ (O_t \cup Q_t) \setminus (R_{\pi-t} \cup X_{\pi-t} \cup U_t \cup V_{\pi-t}) & \text{if } \pi/2 \leq t \leq \pi \end{cases}$$

is a path of sub-wavelet sets connecting W_1 to W_2 by the continuity of $E \to E^d$ and $E \to E^\tau$, Lemma 3.1 and Lemma 3.2.

Now, we will construct sets $\{M_i\}_{i=0}^\infty$ which are continuous functions of t such that $S_t \cup_{i=0}^\infty M_i(t)$ is a wavelet set for each t.

Fix $\{x_n\}_{n=1}^\infty$ and the function k as in Lemma 3.2. Define $C_i = C_i(t)$ to be $\mathcal{A}(P_{\pi+t/2^i}, P_{\pi+t/2^{i-1}})$ for i a natural number and C_0 to be $\mathcal{A}(P_{\pi+t}, P_{2\pi})$. Define t' by

$$t' = \begin{cases} t & \text{if } t \leq t_0, \\ t_0 & \text{if } t_0 \leq t \leq \pi - t_0, \\ \pi - t & \text{if } \pi - t_0 \leq t \leq \pi. \end{cases}$$

Finally, recall the definitions of N and M in Lemma 3.1 and Lemma 3.2.

Now, let $A_0 = A_0(t) = C_0(t') \setminus S_t^d$, $s_0 = s_0(t) = t'$ and define $M_0 = M_0(t) = N(A_0, s_0)$, which is a continuous function of t by Lemma 3.1. Note

(1) $\mathcal{A}(P_{\pi+t'}, P_{2\pi}) \supset (M_0 \cup S_t)^d = C_0(t')$, and
(2) $[-\pi, \pi]^n \setminus \mathcal{A}(0, P_{t'/2}) \supset (M_0 \cup S_t)^\tau$.

Then let $B_0 = B_0(t) = ([-\pi, \pi]^n \setminus \mathcal{A}(0, P_{t'/2})) \setminus (M_0 \cup S_t)^\tau$, $r_0 = t'$ and define $M_1 = M(B_0, r_0)$, which is a continuous function of t by Lemma 3.2. Note that

(1) $\mathcal{A}(P_{k(t')+1}, P_{2\pi}) \supset (M_0 \cup M_1 \cup S_t)^d \supset C_0(t')$, and
(2) $(M_0 \cup M_1 \cup S_t)^\tau = [-\pi, \pi]^n \setminus \mathcal{A}(0, P_{t'/2})$.

Now, let $A_1 = C_1(t') \setminus (M_0 \cup M_1 \cup S_t)^d$ and $s_1 = t'/2$. Define $M_2 = N(A_1, s_1)$. Then,

(1) $\mathcal{A}(P_{k(t'/2)+1}, P_{2\pi}) \supset (\bigcup_{i=0}^2 M_i \cup S_t)^d \supset C_0(t') \cup C_1(t')$, and
(2) $[-\pi, \pi]^n \setminus \mathcal{A}(0, P_{t'/4}) \supset (\bigcup_{i=0}^2 M_i \cup S_t)^\tau \supset [-\pi, \pi]^n \setminus \mathcal{A}(0, P_{t'/2})$.

Let $B_1 = ([-\pi, \pi]^n \setminus \mathcal{A}(0, P_{t'/4})) \setminus (\bigcup_{i=0}^2 M_i \cup S_t)^\tau$ and $r_1 = t'/2$. Define $M_3 = M(B_1, r_1)$. Note that

(1) $\mathcal{A}(P_{k(t'/2)+1}, P_{2\pi}) \supset (\bigcup_{i=0}^3 M_i \cup S_t)^d \supset C_0(t') \cup C_1(t')$, and
(2) $(M_0 \cup M_1 \cup S_t)^\tau = [-\pi, \pi]^n \setminus \mathcal{A}(0, P_{t'/4})$.

Note that $M_3^d \cap M_i^d = \emptyset$ for $i < 3$ by construction when i is even, and by Lemma 3.2 (v) when i is odd.

Continuing in this fashion, one can check as in the one-dimensional case that the resulting sets $L_t = \bigcup_{i=0}^{\infty} M_i \cup S_t$ forms a path of wavelet sets connecting W_1 to W_2.

□

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