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AN EXTENSION OF A CONVEXITY THEOREM OF THE GENERALIZED NUMERICAL RANGE ASSOCIATED WITH $SO(2n+1)$

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Abstract. For any $C, A_1, A_2, A_3 \in \mathfrak{so}(2n+1)$, let $W_C(A_1, A_2, A_3)$ be the following subset of $\mathbb{R}^3$:

$$\{(\operatorname{tr} CO^T A_1 O, \operatorname{tr} CO^T A_2 O, \operatorname{tr} CO^T A_3 O) : O \in SO(2n+1)\}.$$

We show that if $n \geq 2$, then $W_C(A_1, A_2, A_3)$ is always convex. When $n = 1$, it is an ellipsoid, probably degenerate. The convexity result is best possible in the sense that if we have $W_C(A_1, \dots, A_p)$ defined similarly, then there are examples which fail to be convex when $p \geq 4$ and $n \geq 1$.

The set is also symmetric about the origin for all $n \geq 1$, and contains the origin when $n \geq 2$. Equivalent statements of this result are given. The convexity result for $\mathfrak{so}(2n+1)$ is similar to Au-Yeung and Tsing's extension of Westwick's convexity result for $\mathfrak{u}(n)$.

1. Introduction

Very recently the notion of numerical range has been generalized in the context of compact connected Lie groups [13]. See [13] and [6] for historical remarks. Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ which is equipped with a G-invariant inner product $\langle \cdot, \cdot \rangle$. For $A_1, \dots, A_p, C \in \mathfrak{g}$, the C-numerical range of $(A_1, \dots, A_p)$ is defined to be the following subset of $\mathbb{R}^p$:

$$W_C(A_1, \dots, A_p) = \{(\langle A_1, Z\rangle, \dots, \langle A_p, Z\rangle) \ : \ Z \in O(C)\},$$

where $O(C) = \{\operatorname{Ad}(g)C : g \in G\}$, the adjoint orbit of C.

Theorem 1 ([13]). *Let G be a compact connected Lie group. For $A_1, A_2, C \in \mathfrak{g}$, $W_C(A_1, A_2)$ is a compact convex set in $\mathbb{R}^2$.*

Corollary 2. 1. [15] *If $G = U(n)$ or $SU(n)$, then*

$$W_C(A_1, A_2) = \{(\operatorname{tr} A_1 U^* C U, \operatorname{tr} A_2 U^* C U) : U \in G\}$$

is convex, where A_1, A_2 and C are Hermitian matrices.

2. *The set $W_C(A_1, A_2) = \{(\operatorname{tr} A_1 O^T C O, \operatorname{tr} A_2 O^T C O) : O \in SO(n)\}$ is convex, where $A_1, A_2,$ and C are real skew symmetric matrices.*

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Theorem 1 is best possible in the sense that $W_C(A_1,\dots,A_p)$ fails to be convex when $p > 2$, e.g., convexity fails to hold if $p > 3$ or $n = 2$ when $G = U(n)$ [2], and if $p = 3$ and $n = 1$ when $G = SO(2n+1)$ [14]. Nevertheless, there are equivalent statements for the convexity of $W_C(A_1,\dots,A_p)$ [13]. In order to state the result, we need to introduce some notations. Let $\mathfrak{t}$ be the Lie algebra of a maximal torus T of the compact connected Lie group G. We denote by W the Weyl group of G and by $\hat{K}$ the convex hull of the set K. Since every adjoint orbit intersects $\mathfrak{t}$ in a finite non-empty set, we can assume [13] that $C \in \mathfrak{t}$ for $W_C(A_1,\dots,A_p)$ or even $C \in \mathcal{C}$, where $\mathcal{C}$ is the (closed) fundamental Weyl chamber.

Theorem 3 ([13]). *Let $A_1,\dots,A_p$ be elements in $\mathfrak{g}$ and let $C \in \mathfrak{t}$. The following statements are equivalent:*

1. *$W_C(A_1,\dots,A_p)$ is convex.*
2. *If $(r_1,\dots,r_p) \in \mathbb{R}^p \setminus W_C(A_1,\dots,A_p)$, then there exist $\alpha_1,\dots,\alpha_p \in \mathbb{R}$ such that $\min_{g\in G} \langle \sum_{i=1}^p \alpha_i A_i, \mathrm{Ad}(g)C\rangle > \sum_{i=1}^p \alpha_i r_i$.*
3. *If $(r_1,\dots,r_p) \in \mathbb{R}^p \setminus W_C(A_1,\dots,A_p)$, then there exist $\alpha_1,\dots,\alpha_p \in \mathbb{R}$ such that $\min_{\omega\in W} \langle \omega \cdot \tilde{A}, C\rangle > \sum_{i=1}^p \alpha_i r_i$, where $\tilde{A} \in O(A) \cap \mathcal{C}$ and $A = \sum_{i=1}^p \alpha_i A_i$.*
4. *$W_B(A_1,\dots,A_p) \subset W_C(A_1,\dots,A_p)$ if $B \in \mathfrak{t}$ and $B \in \widehat{W(C)}$.*

While Westwick's convexity result [15] implies that the statements of Theorem 3 ($G = U(n)$) are valid if $p = 2$, Au-Yeung and Tsing [4] proved that the statements of Theorem 3 are true when $p = 3$ and $n > 2$ ($G = U(n)$). This obviously extends Westwick's result when $n > 2$. See [11] for further generalization. When $n = 2$, it can be shown that $W_C(A_1, A_2, A_3)$ is an ellipsoid and hence is not convex in general.

In this paper, we deal with another special case, namely, $SO(2n+1)$ when $p = 3$ and $n \geq 2$.

When $G = SO(2n+1)$, let $\mathfrak{t}$ be the space of skew symmetric matrices of the form

$$\begin{pmatrix} 0 & c_1 \\ -c_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & c_n \\ -c_n & 0 \end{pmatrix} \oplus 0. \tag{1}$$

Every $C \in \mathfrak{so}(2n+1)$ has the canonical form (1), i.e., there exist $O \in SO(2n+1)$ such that $O^T C O$ is in $\mathfrak{t}$. We can identify $\mathfrak{t}$ with $\mathbb{R}^n$ by sending (1) to $(c_1,\dots,c_n)$. So the (closed) fundamental Weyl chamber $\mathcal{C}$ can be identified with $\mathbb{R}^n_+ = \{(x_1,\dots,x_n) \in \mathbb{R}^n, x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$. The Weyl group W operates on $\mathfrak{t}$, and its action is given by $(c_1,\dots,c_n) \mapsto (\pm c_{\theta(1)},\dots,\pm c_{\theta(n)})$, where $\theta \in \Sigma_n$, and for any choice of sign.

Let C and B be skew symmetric matrices with canonical forms corresponding to c and b respectively, according to (1). Then $B \in \widehat{W(C)}$ amounts to the condition [12] that b is in the convex hull of the vectors $(\pm c_{\theta(1)},\dots,\pm c_{\theta(n)})$, where $\theta \in \Sigma_n$, and for any choice of sign. This is equivalent to saying that $\sum_{i=1}^k |b_i| \leq \sum_{i=1}^k |c_i|$, $k = 1,\dots,n$, after rearranging the entries of b and c in descending order of absolute values. We will denote the relation by $y \prec_w x$. If b and c are both nonnegative vectors, then the inequalities become the usual weak majorization [7] for $\mathbb{R}^n_+$.

With respect to $W_C(A_1,\dots,A_p)$, C can be assumed in the form (1), where $c \in \mathbb{R}^n_+$. We can even assume that $C \in \mathcal{C}$. The reason is that

$$W_C(A_1,\dots,A_p) = W_{P^TCP}(A_1,\dots,A_p) \qquad \text{for any } P \in SO(2n+1). \tag{2}$$

Corollary 4 ([13])**.** *Let $C, A_1, \dots, A_p \in \mathfrak{so}(2n+1)$ and let C be in the canonical form* (1) *corresponding to $c = (c_1, \dots, c_n) \in \mathbb{R}^n_+$. The following statements are equivalent:*

1. *$W_C(A_1, \dots, A_p)$ is convex.*
2. *If $(r_1, \dots, r_p) \in \mathbb{R}^p \setminus W_C(A_1, \dots, A_p)$, then there exist $\alpha_1, \dots, \alpha_p \in \mathbb{R}$ such that*
$$\min_{O \in SO(2n+1)} \operatorname{tr} CO^T (\sum_{i=1}^p \alpha_i A_i) O > \sum_{i=1}^p \alpha_i r_i.$$
3. *If $(r_1, \dots, r_p) \in \mathbb{R}^p \setminus W_C(A_1, \dots, A_p)$, then there exist $\alpha_1, \dots, \alpha_p \in \mathbb{R}$ such that*
$$\min_{\sigma \in \Sigma_n} -2 \sum_{i=1}^n c_i a_{\sigma(i)} > \sum_{i=1}^p \alpha_i r_i,$$
where the canonical form of $\sum_{i=1}^p \alpha_i A_i$ corresponds to $a = (a_1, \dots, a_n) \in \mathbb{R}^n_+$, according to (1)*.*
4. *$W_B(A_1, \dots, A_p) \subset W_C(A_1, \dots, A_p)$ if $b \prec_w c$, where $b \in \mathbb{R}^n_+$ and the canonical form of B corresponds to $b = (b_1, \dots, b_n)$, according to* (1)*.*

2. Convexity and some equivalent statements

Mirsky [9] obtained the following result, which was redicovered by Chong [5]. Also see [7].

Lemma 5. *Let $x, y \in \mathbb{R}^n$. Then $y \prec_w x$ if and only if $y \le P_1 \cdots P_k x$ for some pinching matrices $P_1, \dots, P_k$. Hence, if $x, y \in \mathbb{R}^n_+$, then $y \prec_w x$ if and only if $y = \Gamma P_1 \cdots P_k x$ for some pinching matrices $P_1, \dots, P_k$ and $\Gamma = \operatorname{diag}(\gamma_1, \dots, \gamma_n)$ with $0 \le \gamma_i \le 1$, $i = 1, \dots, n$.*

Lemma 6. *Let $C \neq 0$ and $A_1, A_2, A_3 \in \mathfrak{so}(3)$, where*
$$A_1 = \begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & y_1 & y_2 \\ -y_1 & 0 & y_3 \\ -y_2 & -y_3 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & z_1 & z_2 \\ -z_1 & 0 & z_3 \\ -z_2 & -z_3 & 0 \end{pmatrix}.$$
Set
$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}.$$
Then $W_C(A_1, A_2, A_3) = \{(\operatorname{tr} CO^T A_1 O, \operatorname{tr} CO^T A_2 O, \operatorname{tr} CO^T A_3 O) : O \in SO(3)\} \subset \mathbb{R}^3$ is

1. *a nondegenerate ellipsoid centered at the origin, if* $\operatorname{rank} A = 3$*;*
2. *an elliptical disk centered at the origin, if* $\operatorname{rank} A = 2$*;*
3. *a line segment centered at the origin, if* $\operatorname{rank} A = 1$*;*
4. *the origin if* $\operatorname{rank} A = 0$.

Proof. By using the technique in [14], it can be deduced that the ellipsoid $\mathcal{E}$ in Lemma 6 is given in the following way: Let $A = O_1 \Lambda O_2$ be a singular value decomposition of A, where $O_1 \in SO(3)$. Let rS^2 be the 2-sphere centered at the origin and with radius $r = \sqrt{\frac{1}{2} C^T C}$. The ellipsoid $\mathcal{E}$ is $O_1(\mathcal{E}')$ where $\mathcal{E}' = \Lambda O_2(rS^2) = \Lambda(rS^2)$ is the ellipsoid in the standard position, centered at the origin and principal axes x_1, x_2 and x_3, respectively. The principal axes lengths are λ_1, λ_2 and λ_3, respectively, where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$. ☐

The following is the main result of this paper.

Theorem 7. *Let* $C, A_1, A_2, A_3 \in \mathfrak{so}(2n+1)$ *and let* C *be in the canonical form* (1) *corresponding to* $c = (c_1, \dots, c_n) \in \mathbb{R}^n_+$. *The following statements hold.*

1. $W_C(A_1, A_2, A_3)$ *is convex.*
2. *If* $(r_1, r_2, r_3) \in \mathbb{R}^3 \setminus W_C(A_1, A_2, A_3)$, *then there exist* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ *such that*
$$\min_{O\in SO(2n+1)} \operatorname{tr} CO^T(\sum_{i=1}^n \alpha_i A_i)O > \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3.$$
3. *If* $(r_1, r_2, r_3) \in \mathbb{R}^3 \setminus W_C(A_1, A_2, A_3)$, *then there exist* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ *such that*
$$\min_{\sigma\in\Sigma_n} -2\sum_{i=1}^n c_i a_{\sigma(i)} > \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3,$$
where the canonical form of $\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$ *corresponds to* $a = (a_1, \dots, a_n) \in \mathbb{R}^n_+$, *according to* (1).
4. $W_B(A_1, A_2, A_3) \subset W_C(A_1, A_2, A_3)$ *if* $b \prec_w c$, *and* $B \in \mathfrak{so}(2n+1)$ *has canonical form corresponding to* $b = (b_1, \dots, b_n) \in \mathbb{R}^n_+$ *according to* (1).

Proof. We are going to use the continuity argument [4] to furnish the proof. Due to Corollary 4, it suffices to show that the last statement of Theorem 7 is valid. Because of Lemma 5, it is sufficent to establish that $W_B(A_1, A_2, A_3) \subset W_C(A_1, A_2, A_3)$ when

1. $b = \Gamma c$, where $\Gamma = \operatorname{diag}(1, \dots, 1, \alpha, 1, \dots, 1)$, $0 \le \alpha < 1$; and
2. $b = Pc$, where P is a pinching matrix.

Due to (2), we only have to deal with the following two cases:

Case 1. $b = \Gamma c$, where $b, c \in \mathbb{R}^n_+$, and

$$\Gamma = \operatorname{diag}(\alpha, 1, \dots, 1), \quad 0 \le \alpha < 1. \tag{3}$$

Hence we can assume that C and B are of the form (1) corresponding to c and b respectively, where $0 \le b_1 < c_1$ and $c_i = b_i$ for $i = 2, \dots, n$. Suppose that $x = (x_1, x_2, x_3) \in W_B(A_1, A_2, A_3)$, i.e., there exists $E \in SO(2n+1)$ such that

$$x = (\operatorname{tr} BE^T A_1 E, \operatorname{tr} BE^T A_2 E, \operatorname{tr} BE^T A_3 E).$$

Let $e_1, e_2, \dots, e_{2n+1}$ be the columns of E. Then

$$x_i = -2(b_1 e_1^T A_i e_2 + \sum_{j=2}^n b_j e_{2j-1}^T A_i e_{2j}), \quad i = 1, 2, 3.$$

The point $\gamma = -2b_1(e_1^T A_1 e_2, e_1^T A_2 e_2, e_1^T A_3 e_2)$ belongs to $W_{B'}(A_1', A_2', A_3')$, which is an ellipsoid (probably degenerate) centered at the origin, by Lemma 6, where $A_i' = (E^T A_i E)[1, 2, 2n+1|1, 2, 2n+1]$, $i = 1, 2, 3$, are elements of $\mathfrak{so}(3)$ and $A[\alpha|\beta]$ denotes the submatrix of A lying in the rows given by the sequence α and in the columns given by the sequence β and

$$B' = \begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix} \oplus 0.$$

If we denote by $\mathcal{E}_{b_1,E}$ the ellipsoid obtained by translating $W_{B'}(A_1', A_2', A_3')$ by

$$-2(\sum_{j=2}^n b_j e_{2j-1}^T A_1 e_{2j}, \sum_{j=2}^n b_j e_{2j-1}^T A_2 e_{2j}, \sum_{j=2}^n b_j e_{2j-1}^T A_3 e_{2j}),$$

then $x \in \mathcal{E}_{b_1,E} \subset W_B(A_1, A_2, A_3)$.

Similarly we have the ellipsoid $\mathcal{E}_{c_1,E} \subset W_C(A_1, A_2, A_3)$. Since $0 \le b_1 < c_1$, $b_j = c_j$, $j = 2, \ldots, n$, the ellipsoid $W_{B'}(A_1', A_2', A_3')$ lies within the interior of the ellipsoid $W_{C'}(A_1', A_2', A_3')$, where

$$C' = \begin{pmatrix} 0 & c_1 \\ -c_1 & 0 \end{pmatrix} \oplus 0.$$

Notice that $\mathcal{E}_{c_1,E}$ is the ellipsoid obtained by translating $W_{C'}(A_1', A_2', A_3')$ by

$$-2(\sum_{j=2}^{n} c_j e_{2j-1}^T A_1 e_{2j}, \sum_{j=2}^{n} c_j e_{2j-1}^T A_2 e_{2j}, \sum_{j=2}^{n} c_j e_{2j-1}^T A_3 e_{2j}),$$

which is identical to

$$-2(\sum_{j=2}^{n} b_j e_{2j-1}^T A_1 e_{2j}, \sum_{j=2}^{n} b_j e_{2j-1}^T A_2 e_{2j}, \sum_{j=2}^{n} b_j e_{2j-1}^T A_3 e_{2j}).$$

So $\mathcal{E}_{b_1,E}$ lies within the interior of $\mathcal{E}_{c_1,E}$. If $W_{C'}(A_1', A_2', A_3')$ degenerates, so do $\mathcal{E}_{c_1,E}$ and $\mathcal{E}_{b_1,E}$. In this case, $x \in \mathcal{E}_{b_1,E} \subset \mathcal{E}_{c_1,E} \subset W_C(A_1, A_2, A_3)$. Thus we assume that $W_{C'}(A_1', A_2', A_3')$ does not degenerate.

Now pick two orthonormal vectors z_1 and z_2 in $\mathbb{R}^{2n+1}$ such that $\operatorname{span}\{z_1, z_2\}$ is an invariant subspace of A_1. Since $n \ge 2$, there exists $z_{2n+1} \in \mathbb{R}^{2n+1}$ such that z_{2n+1} is perpendicular to the four vectors z_1, z_2, $A_2 z_1$ and $A_2 z_2$. Then extend $\{z_1, z_2, z_{2n+1}\}$ to an orthonormal basis $\{z_1, z_2, z_3, \ldots, z_{2n}, z_{2n+1}\}$ of $\mathbb{R}^{2n+1}$ so that the matrix $Z = \begin{pmatrix} z_1 & z_2 & \ldots & z_{2n+1} \end{pmatrix}$ is an element of $SO(2n+1)$. Then the two matrices $(Z^T A_1 Z)[1, 2, 2n+1|1, 2, 2n+1]$, $(Z^T A_2 Z)[1, 2, 2n+1|1, 2, 2n+1]$ are of the following form:

$$\begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

According to Lemma 6, the ellipsoid $\mathcal{E}_{c_1,Z}$ degenerates. Let Z', $E' \in \mathfrak{so}(2n+1)$ such that $Z = e^{Z'}$ and $E = e^{E'}$. Then consider a continuous path $F(t)$ in $SO(2n+1)$ defined by $F(t) = e^{tZ' + (1-t)E'}$, where $0 \le t \le 1$. Notice that $\mathcal{E}_{c_1,F(t)} \subset W_C(A_1, A_2, A_3)$ for all $t \in [0, 1]$. Now x lies within the interior of $\mathcal{E}_{c_1,F(0)} \equiv \mathcal{E}_{c_1,E}$. Moreover, $\mathcal{E}_{c_1,F(1)} \equiv \mathcal{E}_{c_1,Z}$ degenerates and is then convex. So by the continuity argument, as in [4] there exists $t \in [0, 1]$ such that $x \in \mathcal{E}_{c_1,F(t)}$. Thus $x \in W_C(A_1, A_2, A_3)$.

Case 2. $b = Pc$, where $b, c \in \mathbb{R}_+^n$, and

$$P = \alpha I + (1-\alpha)Q, \quad 0 < \alpha < 1, \quad \text{and} \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-2}. \tag{4}$$

We can also assume that $c_1 > c_2$ and $b_1 \ge b_2$. In other words, we have $b = (b_1, b_2, \ldots, b_n)$, $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}_+^n$ such that

$$b_1 + b_2 = c_1 + c_2, \quad 0 \le b_1 - b_2 < c_1 - c_2, \quad b_i = c_i, \quad i = 3, \ldots, n.$$

Because of (2), we can assume that C and B are in the form (1) corresponding to c and b respectively. Suppose that $x = (x_1, x_2, x_3) \in W_B(A_1, A_2, A_3)$, i.e., there exists $E \in SO(2n+1)$ such that

$$x = (\operatorname{tr} BE^T A_1 E, \operatorname{tr} BE^T A_2 E, \operatorname{tr} BE^T A_3 E).$$

Let $e_1, e_2, \ldots, e_{2n+1}$ be the columns of E. Then for $i = 1, 2, 3$,

$$\begin{aligned} x_i &= -2b_1 e_1^T A_i e_2 - 2b_2 e_3^T A_i e_4 - 2\sum_{j=3}^{n} b_j e_{2j-1}^T A_i e_{2j} \\ &= -(b_1+b_2)(e_1^T A_i e_2 + e_3^T A_i e_4) \\ &\quad -(b_1-b_2)(e_1^T A_i e_2 - e_3^T A_i e_4) - 2\sum_{j=3}^{n} b_j e_{2j-1}^T A_i e_{2j}. \end{aligned}$$

Let f_1, f_2, f_3 and $f_4 \in \mathbb{R}^{2n+1}$ be the vectors defined by the relation:

$$\begin{aligned} f_1 + if_2 &= e^{-i\phi}\cos\theta(e_1 + ie_2) + e^{i\phi}\sin\theta(e_3 + ie_4), \\ f_3 + if_4 &= -e^{-i\phi}\sin\theta(e_1 + ie_2) + e^{i\phi}\cos\theta(e_3 + ie_4). \end{aligned}$$

The matrix which sends (e_1, e_2, e_3, e_4) to (f_1, f_2, f_3, f_4) is an element of $SO(4)$:

$$\begin{pmatrix} \cos\phi\cos\theta & -\sin\phi\cos\theta & -\cos\phi\sin\theta & \sin\phi\sin\theta \\ \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\phi\sin\theta & -\cos\phi\sin\theta \\ \cos\phi\sin\theta & \sin\phi\sin\theta & \cos\phi\cos\theta & \sin\phi\cos\theta \\ -\sin\phi\sin\theta & \cos\phi\sin\theta & -\sin\phi\cos\theta & \cos\phi\cos\theta \end{pmatrix}.$$

So $f_1, f_2, f_3, f_4 \in \mathbb{R}^{2n+1}$ are orthonormal vectors and the matrix

$$F = (f_1\ f_2\ f_3\ f_4\ e_5\ \cdots\ e_{2n+1})$$

is an element of $SO(2n+1)$. It is clear that $(e_1 + ie_2)^* A_j (e_1 + ie_2) = 2ie_1^T A_j e_2$ since A_j is real skew symmetric. By direct computation, we have

$$f_1^T A_j f_2 + f_3^T A_j f_4 = e_1^T A_j e_2 + e_3^T A_j e_4, \quad j = 1, 2, 3,$$

and

$$f_1^T A_j f_2 - f_3^T A_j f_4 = p_j \cos 2\theta + \sin 2\theta(q_j \sin 2\phi + s_j \cos 2\phi), \qquad j = 1, 2, 3,$$

where

(5)

$$p_j = e_1^T A_j e_2 - e_3^T A_j e_4, \quad q_j = e_1^T A_j e_3 - e_2^T A_j e_4, \quad s_j = -e_2^T A_j e_3 + e_1^T A_j e_4.$$

If we set $y_i = \operatorname{tr} BF^T A_i F$, then for $i = 1, 2, 3$,

$$\begin{aligned} y_i &= -(b_1+b_2)(f_1^T A_i f_2 + f_3^T A_i e_4) \\ &\quad - (b_1-b_2)(f_1^T A_i f_2 - f_3^T A_i e_4) - 2\sum_{j=3}^{n} b_j e_{2j-1}^T A_i e_{2j} \\ &= -(b_1+b_2)(e_1^T A_i e_2 + e_3^T A_i e_4) \\ &\quad - (b_1-b_2)[p_j \cos 2\theta + \sin 2\theta(q_j \sin 2\phi + s_j \cos 2\phi)] - 2\sum_{j=3}^{n} b_j e_{2j-1}^T A_i e_{2j}. \end{aligned}$$

As θ and ϕ vary in $\mathbb{R}$, the locus of the point (y_1, y_2, y_3) in $\mathbb{R}^3$ is an ellipsoid (compare [4]) which is denoted by $\mathcal{E}_{b_1,b_2,E}$. Indeed it is the translation of the ellipsoid (centered at the origin; (θ, ϕ) and $(\pi + \theta, \phi)$ give opposite points on the ellipsoid) generated by the middle term. The ellipsoid $\mathcal{E}_{b_1,b_2,E}$ contains the point $x = (x_1, x_2, x_3)$, since x corresponds to $\theta = 0$ and $\phi = \pi/2$. Similarly we have $\mathcal{E}_{c_1,c_2,E}$. Since $c_1 + c_2 = b_1 + b_2$ and $0 \le b_1 - b_2 < c_1 - c_2$, we conclude that $\mathcal{E}_{b_1,b_2,E}$ lies within the interior of $\mathcal{E}_{c_1,c_2,E}$. Let $\pm i\lambda_1, \dots, \pm i\lambda_n, 0$ be the eigenvalues of the skew symmetric matrix A_1, where λ's are nonnegative. Let

$$Z = \begin{pmatrix} z_1 & z_2 & \dots & z_{2n+1} \end{pmatrix} \in SO(2n+1)$$

be such that

$$Z^T A_1 Z = \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix} \oplus 0.$$

In particular, we have

$$\begin{aligned} A_1 z_1 &= \lambda_1 z_2, & A_1 z_2 &= -\lambda_1 z_1, \\ A_1 z_3 &= \lambda_2 z_4, & A_1 z_4 &= -\lambda_2 z_3, \end{aligned}$$

and thus

$$z_2^T A_1 z_3 = z_1^T A_1 z_4 = 0, \qquad z_1^T A_1 z_3 = z_2^T A_1 z_4.$$

If $\lambda_1 = \lambda_2$, then $z_1^T A_1 z_2 - z_3^T A_1 z_4 = 0$. The ellipsoid $\mathcal{E}_{c_1,c_2,Z}$ degenerates and lies within the yz plane. By applying the continuity argument in case 1 to the matrices E and Z, we have $x \in W_C(A_1, A_2, A_3)$. So we can assume that $\lambda_1, \dots, \lambda_n$ are distinct and $\lambda_1 > \lambda_2$. Let

$$A_1' = (Z^T A_1 Z)[1, 2, 2n+1|1, 2, 2n+1] = \begin{pmatrix} 0 & \lambda_1 & 0 \\ -\lambda_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = P^T A_1 P \in \mathfrak{so}(3),$$

where $P = \begin{pmatrix} z_1 & z_2 & z_{2n+1} \end{pmatrix}$.

It is well known that the adjoint orbits, $\{OXO^T : O \in SO(3)\}$ for all $X \in \mathfrak{so}(3)$, can be identified (using $\operatorname{tr} X^T Y$ as the inner product on $\mathfrak{so}(3)$) with the 2-spheres in $\mathbb{R}^3$ centered at 0 and with radius $\sqrt{\frac{1}{2} \operatorname{tr} X^T X}$. So there exists $O \in SO(3)$ such that $\begin{pmatrix} z_1' & z_2' & z_{2n+1}' \end{pmatrix} = \begin{pmatrix} z_1 & z_2 & z_{2n+1} \end{pmatrix} O$ and $z_1'^T A_1' z_2' = z_3^T A_1' z_4 = \lambda_2$, since $\lambda_1 > \lambda_2 \geq 0$. In other words,

$$Z'^T A_1' Z' = \begin{pmatrix} 0 & \lambda_2 & 0 & 0 & \dots & 0 & u \\ -\lambda_2 & 0 & 0 & 0 & \dots & 0 & v \\ 0 & 0 & 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & -\lambda_2 & 0 & 0 & \dots & 0 \\ & & \dots & & & & \\ -u & -v & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \qquad u^2 + v^2 + \lambda_2^2 = \lambda_1^2,$$

where $Z' = \begin{pmatrix} z_1' & z_2' & z_3 & z_4 & \dots & z_{2n} & z_{2n+1}' \end{pmatrix}$. It follows that

$$z_2'^T A_1' z_3 = z_1'^T A_1 z_4 = z_1'^T A_1' z_3 = z_2'^T A_1 z_4 = 0, \quad z_1'^T A_1' z_2' = z_3^T A_1 z_4.$$

So the corresponding p_1, q_1 and s_1 in (5) are all zeros. Thus the ellipsoid $\mathcal{E}_{c_1,c_2,Z'}$ degenerates and lies within the yz plane. Applying the continuity argument again, we are done. $\square$

Corollary 8. *Let* $C, A_1, A_2, A_3 \in \mathfrak{so}(2n+1)$ *and let* $\alpha \in \{\operatorname{tr} CO^T A_3 O : O \in SO(2n+1)\} \subset \mathbb{R}$. *Then in* $\mathbb{R}^2$ *the subset*

$$\{(\operatorname{tr} CO^T A_1 O, \operatorname{tr} CO^T A_2 O) : O \in SO(2n+1), \operatorname{tr} CO^T A_3 O = \alpha\}$$

is either empty or convex.

Remark. We remark that the above results are valid if $G = O(2n+1)$ ($\mathfrak{o}(k) \equiv \mathfrak{so}(k)$ for all k) whenever they are true for $G = SO(2n+1)$, since $G = SO(2n+1)$ and $O(2n+1)$ give the same $W_C(A_1, \ldots, A_p)$.

Due to case 1 in the above proof, we conclude that $W_C(A_1, A_2, A_3)$ always contains the origin when $n \geq 2$, since $0 \prec_w c$ for $c \in \mathbb{R}^n_+$. One can deduce the same conclusion by the convexity of $W_C(A_1, A_2, A_3)$ when $n \geq 2$. The following symmetry result of $W_C(A_1, A_2, A_3)$ follows directly from the remark on the Weyl group.

Proposition 9. *Let $G = SO(2n+1)$ or $O(2n+1)$ and $C, A_1, \ldots, A_p \in \mathfrak{so}(2n+1)$. Then $W_C(A_1, \ldots, A_p)$ is symmetric about the origin, i.e., if $x \in W_C(A_1, \ldots, A_p)$, so is $-x$.*

Theorem 7 is best possible in the sense that there are examples of $W_C(A_1, \ldots, A_p)$ which fail to be convex when $p > 3$ and $n \geq 1$. When $n = 1$, Lemma 6 is also valid for $W_C(A_1, \ldots, A_p)$, except that the matrix A is $p \times 3$ and is formed from $A_1, \ldots, A_p$, i.e., $W_C(A_1, \ldots, A_p)$ is a 2-ellipsoid in $\mathbb{R}^p$. So it is not always convex. When $n \geq 2$, it suffices to consider the case $p = 4$. The following matrices are $k \times k$ skew symmetric matrices, where $k \geq 4$, even or odd.

Example.

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus 0,$$

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus 0, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus 0,$$

$$A_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus 0, \qquad A_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix} \oplus 0.$$

The points $(-2, -2, 0, 0)$ and $(2, -2, 0, 0)$ are obviously in $W_C(A_1, A_2, A_3, A_4)$. However, $W_C(A_1, A_2, A_3, A_4)$ does not contain the midpoint $(0, -2, 0, 0)$. Indeed, otherwise there would exist $O \in SO(k)$ such that

$$C' = (O^T C O)[1,2,3,4|1,2,3,4] = \begin{pmatrix} 0 & \frac{1}{2} & \alpha & \beta \\ -\frac{1}{2} & 0 & -\alpha & -\beta \\ -\alpha & \alpha & 0 & \frac{1}{2} \\ -\beta & \beta & -\frac{1}{2} & 0 \end{pmatrix}.$$

Since the rank of C is 2, the rank of the submatrix C' is not greater than 2. Now the first two rows of C' are obviously linearly independent. The third row has to be a linear combination of the first two rows, i.e.,

$$(-\alpha, \alpha, 0, \frac{1}{2}) = \kappa_1(0, \frac{1}{2}, \alpha, \beta) + \kappa_2(-\frac{1}{2}, 0, -\alpha, -\beta).$$

So $\kappa_1 = \kappa_2 = 2\alpha$ from the first two coordinates of both sides of the equality. But then the fourth coordinate will yield $\frac{1}{2} = 0$. This is absurd.

3. Some remarks and questions

Question 1. If $G = SO(2n)$, is there a convexity result when $p = 3$ and $n \geq k$ for some k? If not, it would be interesting to have nonconvex examples when $n \geq 2$. Obviously, if $n = 1$, $W_C(A_1, A_2, A_3)$ is a singleton set and hence is convex. If there is such convexity theorem for $SO(2n)$ when $p = 3$, then according to the example in the previous section (k can be even), it will be best possible in the same sense.

If $x, y \in \mathbb{R}^n$, we denote by $y \ll x$ the relation defined by the inequalities

$$\begin{aligned} \sum_{i=1}^{k} |y_i| &\leq \sum_{i=1}^{k} |x_i|, \quad k = 1, \ldots, n, \\ \sum_{i=1}^{n-1} |y_i| - |y_n| &\leq \sum_{i=1}^{n-1} |x_i| - |x_n|, \end{aligned}$$

and in addition, if the total number of negative terms of the sequences x and y is odd,

$$\sum_{i=1}^{n} |y_i| \leq \sum_{i=1}^{n-1} |x_i| - |x_n|,$$

after rearranging the sequences x and y in decreasing order with respect to the absolute values.

The geometry behind the partial ordering $\ll$ is like that behind $\prec$ and $\prec_w$, namely, $y \ll x$ if and only if y is in the convex hull of the vectors $(\pm c_{\theta(1)}, \ldots, \pm c_{\theta(n)})$, where $\theta \in \Sigma_n$ and the number of negative signs is even [12].

Lemma 5 provides a characterization of the partial relation $\prec_w$ which is useful to the proof of our main result. See [13] for the statements equivalent to the convexity of $W_C(A_1, A_2, A_3)$ when $G = SO(2n)$. The partial order $\ll$ is involved.

Question 2. Is there a characterization of $\ll$ similar to those of $\prec$ ($x, y \in \mathbb{R}^n$, [8]) and $\prec_w$ ($x, y \in \mathbb{R}^n_+$, Lemma 5) ?

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References

[1] M. F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), 523-615. MR **80k:**14006

[2] Y. H. Au-Yeung and Y. T. Poon, A remark on the convexity and positive definitness concerning Hermitian matrices, Southeast Asian Bull. Math. **3** (1979), 85-92. MR **81c:**15026

[3] Y. H. Au-Yeung and N. K. Tsing, Some theorems on the generalized numerical ranges, Linear and Multilinear Algebra, **15** (1984), 3–11. MR **85c:**15039

[4] Y. K. Au-Yeung and N. K. Tsing, An extension of the Hausdorff-Toeplitz theorem on the numerical range, Proc. Amer. Math. Soc. **89**, (1983), 215-218. MR **85f:**15021

[5] K. M. Chong, An induction theorem for rearrangements, Canad. J. Math. **28** (1976), 154-160. MR **52:**14200

[6] K. E. Gustafson and D. K. M. Rao, *Numerical Range: the field of values of linear operators and matrices*, Springer, New York, 1997. CMP 97:03

[7] A. W. Marshall and I. Olkin, *Inequalities: Theory of majorization and its applications*, New York: Academic Press, 1979. MR **81b:**00002

[8] R. F. Muirhead, Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters, Proc. Edinburgh Math. Soc. **21** (1903), 144-157.

[9] L. Mirsky, On a convex set of matrices, Arch. Math. **10** (1959), 88-92. MR **21:**5643

[10] Y. T. Poon, Another proof of a result of Westwick, Linear and Multilinear Algebra, **9** (1980), 35-37. MR **81h:**15015

[11] Y. T. Poon, Generalized numerical ranges, joint positive definiteness and multiple eigenvalues, Proc. Amer. Math. Soc. **125** (1997), 1625–1634. MR **97g:**15030

[12] T.Y. Tam, Kostant's convexity theorem and the compact classical groups, Linear and Multilinear Algebra **43** (1997), 87–113.

[13] T.Y. Tam, Generalized numerical ranges, numerical radii, and Lie groups, manuscript, 1996.

[14] T.Y. Tam, Plotting the generalized numerical range associated with the compact connected Lie groups, manuscript, 1996.

[15] R. Westwick, A theorem on numerical range, Linear and Multilinear Algebra **2** (1975), 311-315. MR **51:**11132

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