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A COMBINATORIAL FORMULA OF LEIBNIZ TYPE WITH APPLICATION TO GEGENBAUER'S POLYNOMIALS

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ABSTRACT. We establish a combinatorial formula of Leibniz type, which is an identity for a certain differential polynomial. The formula leads to new quadratic relations between Gegenbauer's orthogonal polynomials.

1. Introduction and results

In the course of his studies on the fourth Painlevé equation in several variables [3], the second author came across the following interesting identity:

$$\sum_{i=0}^{n} \frac{1}{\alpha+i}\binom{n}{i}(D^i f^{-\alpha-i})(D^{n-i} f^{\alpha+i}) = 0 \qquad (n = 1, 2, 3, \ldots) \tag{1.1}$$

where α is a parameter, f is any function in one variable x and D is the differentiation with respect to x. He checked by the computer algebra system REDUCE that (1.1) actually holds for $n = 1, 2, \ldots, 10$. This experiment naturally convinced him that (1.1) should hold for any positive integer n. Afterwards the first author gave a proof of this conjecture, which we will present in this paper. The precise meaning of (1.1) is this: the left-hand side of (1.1) is expressed as $f^{-n}F_n(f, f', \ldots, f^{(n)}, \alpha)$ for some polynomial $F_n(X_0, X_1, \ldots, X_n, Y) \in \mathbb{Z}[X_0, X_1, \ldots, X_n, Y]$ (see Lemma 2.2), and (1.1) asserts that $F_n = 0$ as a polynomial over $\mathbb{Z}$.

It should be noted that a slightly different identity:

$$\sum_{i=0}^{n} \binom{n}{i}(D^i f^{-\alpha})(D^{n-i} f^{\alpha}) = 0 \qquad (n = 1, 2, 3, \ldots), \tag{1.2}$$

can quite easily be established by using Leibniz's formula. Indeed, the left-hand side of (1.2) is nothing but $D^n(f^{-\alpha}f^{\alpha}) = 0$. As we shall see in this paper, however, the verification of (1.1) requires more difficult arguments.

By taking f to be various functions, the identity (1.1) produces many interesting formulas, only one of which is presented below. Set $f = (1-x^2)^{-1}$ and $\alpha = \nu - 1/2$. Then after some computations, (1.1) yields the following quadratic relations

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between Gegenbauer's polynomials (see e.g. [1] for their definition):

$$\sum_{i=0}^{n}(i+2\nu)_{n-1}C_i^{\nu}(x)C_{n-i}^{1-n-\nu}(x)=0 \qquad (n=1,2,3,\dots), \tag{1.3}$$

where $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ for any positive integer n. Finally we refer to [3] for another application of (1.1) to the Hamiltonian structure of the fourth Painlevé equation in several variables.

2. Proof

Lemma 2.1. *Let $P_n(X_0, X_1, \dots, X_n, Y) \in \mathbb{Z}[X_0, X_1, \dots, X_n, Y]$ be inductively defined by $P_1(X_0, X_1, Y) = X_1$ and*

$$\begin{aligned}P_{n+1}(X_0, X_1, \dots, X_{n+1}, Y) &= (Y-n)X_1P_n(X_0, X_1, \dots, X_n, Y)\\ &\quad + X_0\sum_{i=0}^{n}X_{i+1}\frac{\partial P_n}{\partial X_i}(X_0, X_1, \dots, X_n, Y)\\ &\qquad\qquad (n=1,2,3,\dots).\end{aligned}$$

Then P_n is of degree $n-1$ with respect to Y and weighted homogeneous of degree n with respect to $X_0, X_1, \dots, X_n$, where $\deg X_i = i$. Moreover,

$$D^nf^{\alpha} = \alpha f^{\alpha-n}P_n(f, f', \dots, f^{(n)}, \alpha) \qquad (n=1,2,3,\dots). \tag{2.1}$$

Proof. The first part of the lemma is easy to see. We show (2.1) by induction on n. It is clear that (2.1) holds for $n=1$. Assume that (2.1) holds for n. Then we have

$$\begin{aligned}D^{n+1}f^{\alpha} &= D(D^nf^{\alpha}) = \alpha D\{f^{\alpha-n}P_n(f, f', \dots, f^{(n)}, \alpha)\}\\ &= \alpha(\alpha-n)f^{\alpha-n-1}f'P_n(f, f', \dots, f^{(n)}, \alpha)\\ &+ \alpha f^{\alpha-n}DP_n(f, f', \dots, f^{(n)}, \alpha)\\ &= \alpha f^{\alpha-(n+1)}\{(\alpha-n)f'P_n(f, f', \dots, f^{(n)}, \alpha)\\ &+ f\sum_{i=0}^{n}f^{(i+1)}\frac{\partial P_n}{\partial X_i}(f, f', \dots, f^{(n)}, \alpha)\}\\ &= \alpha f^{\alpha-(n+1)}P_{n+1}(f, f', \dots, f^{(n+1)}, \alpha).\end{aligned}$$

Hence (2.1) remains true for $n+1$. Thereby the induction is complete. □

Lemma 2.2. *The left-hand side of* (1.1) *is expressed as $f^{-n}F_n(f, f', \dots, f^{(n)}, \alpha)$, where $F_n(X_0, X_1, \dots, X_n, Y) \in \mathbb{Z}[X_0, X_1, \dots, X_n, Y]$ is given by $F_1(X_0, X_1, Y) = 0$ and*

$$\begin{aligned}F_n(X_0, \dots, X_n, Y) &= P_n(X_0, \dots, X_n, Y) - P_n(X_0, \dots, X_n, -Y-n)\\ &- \sum_{i=1}^{n-1}(Y+i)\binom{n}{i}P_i(X_0, \dots, X_i, -Y-i)P_{n-i}(X_0, \dots, X_{n-i}, Y+i)\\ &\qquad\qquad (n=2,3,4,\dots).\end{aligned}$$

Proof. Apply Lemma 2.1 to the left-hand side of (1.1). □

Lemma 2.3.

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}[\alpha+i]_k=0 \qquad (k=0,1,\ldots,n-1),$$

where $[\alpha]_0=1$ *and* $[\alpha]_k=\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)$ *for any positive integer* k.

Proof. Consider the function $u(x)=x^\alpha(1-x)^n$. The binomial theorem yields

$$u(x)=\sum_{i=0}^{n}(-1)^i\binom{n}{i}x^{\alpha+i}.$$

Differentiating this k-times, we have

$$D^k u=\sum_{i=0}^{n}(-1)^i\binom{n}{i}[\alpha+i]_k x^{\alpha+i-k}. \tag{2.2}$$

Since u has a zero of order n at $x=1$, we have

$$D^k u|_{x=1}=0 \qquad (k=0,1,\ldots,n-1). \tag{2.3}$$

Substituting $x=1$ in (2.2) and using (2.3), we establish Lemma 2.3. □

Lemma 2.4.

$$g_n(\alpha):=\sum_{i=0}^{n}\frac{(-1)^i}{\alpha+i}\binom{n}{i}f^{-\alpha-i}(D^n f^{\alpha+i})=0 \qquad (n=1,2,3,\ldots).$$

Proof. In view of Lemma 2.1, we set $p_n(\alpha)=P_n(f,f',\ldots,f^{(n)},\alpha)$. Since $p_n(\alpha)$ is of degree $n-1$ with respect to α, it can be expressed as

$$p_n(\alpha)=\sum_{k=0}^{n-1}p_{n,k}[\alpha]_k, \tag{2.4}$$

where $p_{n,k}$ are polynomials of $f,f',\ldots,f^{(n)}$. Then it follows that

$$\begin{aligned} g_n(\alpha)&=\sum_{i=0}^{n}\frac{(-1)^i}{\alpha+i}\binom{n}{i}f^{-\alpha-i}\cdot(\alpha+i)f^{\alpha+i-n}p_n(\alpha+i)\\ &=f^{-n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{k=0}^{n-1}p_{n,k}[\alpha+i]_k\\ &=f^{-n}\sum_{k=0}^{n-1}p_{n,k}\left\{\sum_{i=0}^{n}(-1)^i\binom{n}{i}[\alpha+i]_k\right\}=0, \end{aligned}$$

where Lemma 2.1, (2.4) and Lemma 2.3 are used to obtain the first, second and fourth equalities, respectively. The proof is complete. □

Lemma 2.5. *Let* I *be an open interval in* $\mathbb{R}$. *For any* $\mathbb{C}$*-valued, smooth, nowhere vanishing function* $f\in C^\infty(I)$ *and* $\alpha\in\mathbb{C}\setminus\{0,-1,-2,\ldots\}$, *the identity* (1.1) *holds in* $C^\infty(I)$.

Proof. Let $\langle\cdot,\cdot\rangle : C^\infty(I)\times C_0^\infty(I)\to\mathbb{C}$ be the pairing defined by

$$\langle f,\phi\rangle=\int_I f(x)\phi(x)dx,$$

where $C_0^\infty(I)$ is the set of all smooth functions in I with compact support. We consider $\langle (D^i f^{-\alpha-i})(D^{n-i} f^{\alpha+i}), \phi\rangle$. Integration by parts yields

$$\begin{aligned}\langle (D^i f^{-\alpha-i})(D^{n-i} f^{\alpha+i}), \phi\rangle &= \langle D^i f^{-\alpha-i}, (D^{n-i} f^{\alpha+i})\phi\rangle \\ &= -\langle D^{i-1} f^{-\alpha-i}, D\{(D^{n-i} f^{\alpha+i})\phi\}\rangle \\ &= \cdots = (-1)^i \langle f^{-\alpha-i}, D^i\{(D^{n-i} f^{\alpha+i})\phi\}\rangle.\end{aligned}$$

Applying Leibniz's formula to $D^i\{(D^{n-i} f^{\alpha+i})\phi\}$, we obtain

$$\langle (D^i f^{-\alpha-i})(D^{n-i} f^{\alpha+i}), \phi\rangle = (-1)^i \sum_{j=0}^{i} \binom{i}{j} \langle f^{-\alpha-i}(D^{n-j} f^{\alpha+i}), D^j\phi\rangle. \tag{2.5}$$

Let h denote the left-hand side of (1.1). Then (2.5) yields

$$\begin{aligned}\langle h, \phi\rangle &= \sum_{i=0}^{n} \frac{1}{\alpha+i}\binom{n}{i} \langle (D^i f^{-\alpha-i})(D^{n-i} f^{\alpha+i}), \phi\rangle \\ &= \sum_{i=0}^{n} \frac{(-1)^i}{\alpha+i}\binom{n}{i} \sum_{j=0}^{i}\binom{i}{j} \langle f^{-\alpha-i}(D^{n-j} f^{\alpha+i}), D^j\phi\rangle \\ &= \sum_{j=0}^{n} \left\langle \sum_{i=j}^{n} \frac{(-1)^i}{\alpha+i}\binom{n}{i}\binom{i}{j} f^{-\alpha-i}(D^{n-j} f^{\alpha+i}), D^j\phi \right\rangle \\ &= \sum_{j=0}^{n} \binom{n}{j} \left\langle \sum_{i=j}^{n} \frac{(-1)^i}{\alpha+i}\binom{n-j}{i-j} f^{-\alpha-i}(D^{n-j} f^{\alpha+i}), D^j\phi \right\rangle \\ &= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left\langle \sum_{i=0}^{n-j} \frac{(-1)^i}{(\alpha+j)+i}\binom{n-j}{i} f^{-(\alpha+j)-i}(D^{n-j} f^{(\alpha+j)+i}), D^j\phi \right\rangle \\ &= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \langle g_{n-j}(\alpha+j), D^j\phi\rangle,\end{aligned}$$

where $g_n(\alpha)$ is defined in Lemma 2.4. Since $g_{n-j}(\alpha+j) = 0$ for $j = 0, 1, \dots, n-1$ (see Lemma 2.4) and $g_0(\alpha+n) = 1/(\alpha+n)$, we have

$$\langle h, \phi\rangle = \frac{(-1)^n}{\alpha+n}\int_I (D^n\phi)dx = \frac{(-1)^n}{\alpha+n}\left[D^{n-1}\phi\right]_I = 0.$$

Hence $\langle h, \phi\rangle = 0$ for any $\phi \in C_0^\infty(I)$. The fundamental lemma in variational theory implies $h = 0$. The proof is complete. □

Remark 2.6. Lemma 2.5 is restated that for any nowhere vanishing function $f \in C^\infty(I)$ and $\alpha \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, the identity $F_n(f, f', \dots, f^{(n)}, \alpha) = 0$ holds in $C^\infty(I)$, where $F_n(X_0, X_1, \dots, X_n, Y) \in \mathbb{Z}[X_0, X_1, \dots, X_n, Y]$ is defined in Lemma 2.2.

Proof of (1.1). As is mentioned in Section 1, in order to establish (1.1), it suffices to show that $F_n(X_0, X_1, \dots, X_n, Y) = 0$ in $\mathbb{Z}[X_0, X_1, \dots, X_n, Y]$. Choose a transcendental number $\alpha \in \mathbb{C}$ and set $f = \Gamma$, the Gamma function. Let I be an open interval in $\mathbb{R}$ on which Γ is nowhere vanishing. Then Lemma 2.5 together with Remark 2.6 implies

$$F_n(\Gamma, \Gamma', \dots, \Gamma^{(n)}, \alpha) = 0. \tag{2.6}$$

On the other hand, O. Hölder [2] proved that the Gamma function does not satisfy any algebraic differential equation. To be more precise, $\Gamma, \Gamma', \dots, \Gamma^{(n)}, \dots$ are algebraically independent over the field $\mathbb{C}(x)$. In particular, if α is a transcendental number, then $\Gamma, \Gamma', \dots, \Gamma^{(n)}$ and α are algebraically independent over $\mathbb{Z}$. Therefore (2.6) implies that $F_n(X_0, X_1, \dots, X_n, Y) = 0$ in $\mathbb{Z}[X_0, X_1, \dots, X_n, Y]$. The proof is complete. □

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