LEVEL ONE REPRESENTATIONS OF $U_q(G_2^{(1)})$

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Abstract. We construct a level one representation of the quantum affine algebra $U_q(G_2^{(1)})$ by vertex operators from bosonic fields.

1. Introduction

Quantum affine algebras, the quantum groups associated to the affine Kac-Moody Lie algebras, provide an important underlying symmetry for the quantum Yang-Baxter equation [6] and quantum statistical models [11]. Explicit realizations of their representations are much needed in applications of quantum affine algebras. For instance, the Frenkel-Reshetikhin vertex operators [8] associated with the representations can be used to give solutions of the quantum Knizhnik-Zamolodchikov equation.

Lusztig first studied the abstract representations of quantum Kac-Moody algebras [20]. The program of constructing various representations was started in [7] for level one irreducible modules of ADE types, and subsequently twisted types were given in [12] and $B_n^{(1)}$ in [4]. Recently we have constructed symplectic quantum affine algebras in [17] for level one and in [16] for level $-1/2$. The case of $F_4^{(1)}$ can also be done similarly [15] using the idea of quantum Z-algebras [13, 15]. Besides the bosonic constructions, fermionic constructions were furnished in [10]. The $q$-Wakimoto construction was also known [21, 1, 22] afterwards. Other representations of classical quantum affine algebras have also been constructed [2]. The exceptional case of $G_2^{(1)}$ was the only case that has not been explicitly constructed.

The purpose of the paper is to give an explicit level one construction of the quantum affine algebra $U_q(G_2^{(1)})$ by vertex operators. The idea of the construction follows that of quantum Z-algebras [13, 15], which is a $q$-deformation of the classical ($q = 1$) Z-algebras [19, 18]. We construct some auxiliary vertex operators for the short root. This is parallel to the known constructions of the affine Lie algebra $G_2^{(1)}$ [5, 9, 23], though the specialization of $q = 1$ in our construction is new even in the classical case.

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The paper is organized as follows. In section two we review the quantum affine algebra \( U_q(G_2^{(1)}) \). Section three gives the Fock space representation of the quantum affine algebra \( U_q(G_2^{(1)}) \) stated in Theorem 3.1. Section four uses quantum vertex operator techniques to prove Theorem 3.1. In the proof of Serre relations we have to show a relation about certain symmetric functions, which is characteristic in the quantum affine algebras as noted in [12]. The Serre relations in \( G_2^{(1)} \) turn out to be the most complicated one among both untwisted and twisted cases, and actually capture all the existed phenomena in other types.

2. Quantum Affine Algebra \( U_q(G_2^{(1)}) \)

Let \( \alpha_i (i = 1, 2) \) be the simple roots of the simple Lie algebra \( G_2 \), and \( \lambda_i \) be the fundamental weight. Let \( P = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2 \) and \( Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 \) be the weight and root lattices. We then let \( \Lambda_i, i \in I = \{0, 1, 2\} \), be the fundamental weights for the affine Lie algebra \( G_2^{(1)} \), where \( \Lambda_1 = \lambda_1 + \Lambda_0 \), and \( \Lambda_i \) are the fundamental weights for the finite dimensional simple Lie algebra \( G_2 \). The nondegenerate symmetric bilinear form \( (\ , \) \) on \( \mathfrak{h}^* \) is given by

\[
(\alpha_i | \alpha_j) = d_{ij}, \quad (\delta | \alpha_i) = (\delta | \delta) = 0 \quad \text{for all } i, j,
\]

where \( (d_0, d_1, d_3) = (1, 1, 1/3) \) and \( A = (a_{ij}) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \).

Let \( q_i = q^{d_i} = q^{1/2} \alpha_i \), \( i \in I \). The quantum affine algebra \( U_q(G_2^{(1)}) \) is the associative algebra with 1 over \( \mathbb{C}(q^{i/6}) \) generated by the elements \( x_{ik}^\pm \), \( a_{ij} \), \( K_i^{\pm 1} \), \( \gamma^{\pm 1/2} \), \( q^\pm (i, j, \cdots, n, k \in \mathbb{Z}, l \in \mathbb{Z} \setminus \{0\}) \) with the following defining relations [6, 3, 14]:

\[
\gamma^{\pm 1/2}, u = 0 \quad \text{for all } u \in \mathfrak{u},
\]

\[
[a_{ik}, a_{jl}] = \delta_{k+l,0} \frac{[(\alpha_i | \alpha_j)k]}{k} \gamma_k - \gamma^{-k},
\]

\[
[a_{ik}, K_j^{\pm 1}] = [q^d, K_j^{\pm 1}] = 0,
\]

\[
q^d x_{ik}^+ q^{-d} = q^k x_{ik}^+, \quad q^d a_{il} q^{-d} = q^l a_{il},
\]

\[
K_i x_{jk}^+ K_i^{-1} = q^{\pm (\alpha_i | \alpha_j) x_{jk}^+},
\]

\[
[a_{ik}, x_{jl}^\pm] = \pm \frac{[(\alpha_i | \alpha_j)k]}{k} \gamma_{\pm k/2} x_{ik+l}^\pm,
\]

\[
(z - q^{\pm (\alpha_i | \alpha_j)}) X_i^\pm(z) X_j^\pm(w) + (w - q^{\pm (\alpha_i | \alpha_j)}) X_j^\pm(w) X_i^\pm(z) = 0,
\]

\[
[X_i^+(z), X_j^-(w)] = \frac{\delta_{ij}}{q_i - q_i^{-1}} \left( \psi_i(w \gamma^{1/2}) \delta(\frac{w \gamma}{z}) - \varphi_i(w \gamma^{-1/2}) \delta(\frac{w \gamma^{-1}}{z}) \right).
\]

where \( X_i^\pm(z) = \sum_{n \in \mathbb{Z}} x_{i,n} z^{-n-1} \), \( \psi_{im} \) and \( \varphi_{im} (m \in \mathbb{Z}_{\geq 0}) \) are defined by

\[
\sum_{m=0}^{\infty} \psi_{im} z^{-m} = K_i \exp \left( (q_i - q_i^{-1}) \sum_{k=1}^{\infty} a_{ik} z^{-k} \right),
\]

\[
\sum_{m=0}^{\infty} \varphi_{i,-m} z^m = K_i^{-1} \exp \left( -(q_i - q_i^{-1}) \sum_{k=1}^{\infty} a_{i,-k} z^k \right),
\]
Let \( (a_i) \) be an auxiliary simple root isomorphic to \( \alpha_2 \). We define the Fock module \( \mathcal{F} \) as the tensor product of the symmetric algebra generated by \( a_i(-n), b(-n), c(-n) \) \( (n \in \mathbb{N}) \) and the twisted group algebra \( \mathbb{C}\{P + \mathbb{Z}\beta_2\} \) generated by \( e^\alpha, e^\beta \) subject to the relation
\[
e^{\alpha_1}e^{\alpha_2} = -e^{\alpha_2}e^{\alpha_1}, \quad e^\alpha e^\beta = e^\beta e^\alpha, \quad e^\alpha e^\alpha = e^\alpha e^\alpha,
\]
where \( \alpha \in P \), and \( \beta \) is an element of the auxiliary lattice \( \mathbb{Z}\alpha_2 \) (another copy of the sublattice generated by the short root \( \alpha_2 \)). In the following we reserve \( \beta \) to denote an element from this auxiliary lattice.

The element 1 is the vacuum state. We define the action by
\[
a_i(n).1 = 0 \quad (n > 0), \quad b_i(n).1 = 0 \quad (n > 0),
\]
The elements \( a_i(0) \) and \( b(0) \) act as differential operators by
\[
a_i(0)e^\alpha = (\alpha_i(0))e^\alpha, \quad b(0)e^\beta = -\frac{2}{3}e^\beta.
\]
As usual we define the normal product as the ordered product by moving annihilation operators \( a_i(n), b(n), a_i(0), b(0) \) to the left.

Let’s introduce the following operators:
\[
Y_i^+(z) = \exp(\sum_{n=1}^{\infty} a_i(-n) q^{\frac{\alpha_i}{2}} z^n) \exp(\sum_{n=1}^{\infty} a_1(n) q^{\frac{\alpha_1}{2}} z^{-n}) e^{\alpha_1} z^{(a_1)(0)},
\]
\[
Y_2^+(z) = \exp(\sum_{n=1}^{\infty} a_i(-n) + b(-n) q^{\frac{\alpha_i}{2}} z^n) \exp(\sum_{n=1}^{\infty} a_2(n) + b(n) q^{\frac{\alpha_2}{2}} z^{-n}) e^{\alpha_2} z^{2(0)},
\]
\[
U_\pm(z) = \exp(\sum_{n=1}^{\infty} [n/3] b(\pm n) z^{\mp n} q^{\pm b(0)/2}),
\]
\[
W_\pm(z) = \exp(\sum_{n=1}^{\infty} [n/3] c(\pm n) z^{\mp n}).
\]
Theorem 3.1. The space $F$ is a $U_q(G_2^{(1)})$-module of level one under the action defined by $\gamma \mapsto q, K_i \mapsto q^{\alpha_i(0)}, a_{im} \mapsto a_i(m), q^d \mapsto q^d$, and

\[
X_1^\pm(z) \mapsto Y_1^\pm(z), \quad X_2^\pm(z) \mapsto \frac{\pm Y_2^\pm(z)}{q_2 - q_2} \left( U_\pm(q^{\mp 5/6} z) W_\pm(q^{\mp 1/2} z)^{\pm 1} - U_\pm(q^{\mp 5/6} z) W_\pm(q^{\pm 1/2} z)^{\mp 1} \right).
\]

4. PROOF OF THE THEOREM

We now prove the theorem by checking that the action satisfies the Drinfeld relations. It is clear that (2.2-2.6) are true by the construction. The relation (2.7) follows from the definition of $Y_i^\pm(z)$ and the commutativity among $\alpha_i(n), b(n)$ and $c(n)$. So we only need to show (2.8) and (2.9).

We first compute the operator product expansions for $Y_i^\pm(z)$:

\[
Y_i^\pm(z) Y_i^\pm(w) = : Y_i^\pm(z) Y_i^\pm(w) : \cdot \exp(-\sum_{n=1}^{\infty} \frac{[\alpha_i, \alpha_j] n}{n!} q^{5/6} (\frac{w}{z})^n z^{(\alpha_i, \alpha_j)},
\]

(4.1)\]

\[
Y_i^\pm(z) Y_j^\pm(w) = : Y_i^\pm(z) Y_j^\pm(w) : \cdot \exp(\sum_{n=1}^{\infty} \frac{[\alpha_i, \alpha_j] n}{n!} (\frac{w}{z})^n z^{- (\alpha_i, \alpha_j)}).
\]

For $\epsilon = \pm = \pm 1$ we define

\[
X_2^\pm(z) = Y_2^\pm(z) U_\epsilon (q^{-5/6} z) W_\epsilon (q^{-1/2} z), \quad X_2^\mp(z) = Y_2^\mp(z) U_\epsilon (q^{5/6} z) W_\epsilon (q^{1/2} z)^{-1},
\]

so that

\[
X_2^\pm(z) = \frac{1}{q_2 - q_2} (X_2^\pm(z) - X_2^\mp(z)).
\]

Note that for $i = j = 1$ the relation (2.8) follows from the $sl(2)$ case. For $(\alpha_i, \alpha_j) = -1$ (i.e. $i \neq j$) equations (4.1) become

(4.2) \[ Y_i^\pm(z) Y_j^\pm(w) = : Y_i^\pm(z) Y_j^\pm(w) : (z - q^{1/2} w)^{-1}, \]

(4.3) \[ Y_i^\pm(z) Y_j^\pm(w) = : Y_i^\pm(z) Y_j^\pm(w) : (z - q^{1/2} w), \]

which implies that for $i \neq j$

(4.4) \[ (z - q^{1/2} w) X_i^\pm(z) X_j^\pm(w) = (q^{1/2} z - w) X_j^\pm(w) X_i^\pm(z), \]

(4.5) \[ [X_i^\pm(z), X_j^\pm(w)] = 0, \]

where the latter is one case of relation (2.9).

To prove the remaining case of (2.8) we compute

(4.6) \[ X_2^\pm(z) X_2^\pm(w) = : X_2^\pm(z) X_2^\pm(w) : \frac{z - w}{z - q^{1/2} w} q^{(1+\epsilon)/6}. \]

Then we immediately get the “+” case of relation (2.8) for $i = j = 2$. The “−” case is shown similarly.

In relation (2.9), again we only need to show the cases involved with the short root $\alpha_2$, since the proof of $[X_i^\pm(z), X_i^- w)]$ is quite similar to that of type $A$ in [12]. Observe that

(4.7) \[ X_2^\pm(z) X_2^- w) = : X_2^\pm(z) X_2^- w). \]
Thus we reduce the relation to the commutators $[X_{2r}^+(z), X_{2r}^-(w)]$. We compute that

$$[X_{2r}^+(z), X_{2r}^-(w)] = :X_{2r}^+(z)X_{2r}^-(w): = \frac{z - q^{5/3}w - w - q^{-5/3}z}{z - qw} q^{-1/3}$$

Similarly we can prove that

$$[X_{2r}^+(z), X_{2r}^-(w)] = (q^{1/3} - q^{-1/3}) \phi_2(qz^{-1/2}) \delta\left(\frac{qw}{z}\right)$$

Finally we use the quantum vertex operator calculus [12] to prove the Serre relations. The case $(a_{12} = -1)$ is similar to that of $U_q(A_1^{(1)})$ [12]. We only check the other one $(a_{ij} = -3)$ in the “+” case:

$$(4.8) \sum_{\sigma \in S_4} \sigma \left( X_i^+(w)X_2^+(z_1)X_2^+(z_2)X_2^+(z_3)X_2(z_4) \right)$$

Recalling that $X_i^+(z) = \frac{1}{q^2 - q^{-2}} (X^+_j(z) - X^+_j(z))$ and using (4.1-4.6) and Wick's theorem, we can reduce the left-hand side of (4.8) to a linear combination of operator product terms $X_1^+(w)X_{2r_1}(z_1)X_{2r_2}(z_2)X_{2r_3}(z_3)X_{2r_4}(z_4)$; where $\epsilon_i = \pm$. Thus the Serre relation is equivalently reduced to four Serre-like relations grouped by the number of appearances of $X_{2r}^+(z_i)$ in the product. Due to (4.7) the most complicated contraction functions comes from the case when all $\epsilon_i = +$. All four subcases can be treated similarly. In the following we will only prove the case when $\epsilon_i = +$. Ignoring the factor $(q_2 - q_2^{-1})^{-4}$, the left-hand side of this Serre-like relation is $q^2 : X_i^+(w)X_2^+(z_1)X_2^+(z_2)X_2^+(z_3)X_2^+(z_4) :$ times the following expression:

$$\sum_{\sigma \in S_4} \sigma \prod_i \frac{1}{(w - q^{-1}z_i)(z_i - q^{-1}w)} \prod_{i < j} \frac{z_i - z_j}{z_i - q^{2/3}z_j}$$

$$\cdot \left[ (z_1 - q^{-1}w) \cdots (z_4 - q^{-1}w) + [4]_2 (w - q^{-1}z_1) (z_2 - q^{-1}w) \cdots (z_4 - q^{-1}w) + 4 \left( \frac{1}{2} \right)_2 (w - q^{-1}z_1) (w - q^{-1}z_2) (z_3 - q^{-1}w) (z_4 - q^{-1}w) \right]$$

where the symmetric group $S_4$ acts on the ring of rational functions in $z_i$ by permutations on the indices. The $q$-binomial identity implies that the coefficients of 1
and \(w^4\) are zero, and the expression in the brackets is then simplified to
\[
q_2^4 (q_2^{-6} - 1) \left[ q_2^{-1} w^3 (q_2^{-12} z_{1} - q_2^{-6} (1 + q_2^{-2} + q_2^{-4}) z_{2} + q_2^{-2} (1 + q_2^{-2} + q_2^{-4}) z_{3} - z_{4}) + w^2 (1 + q_2^{-2}) (q_2^{-12} z_{1} z_{2} - q_2^{-6} (1 + q_2^{-2}) z_{1} z_{3} + q_2^{-4} (1 + q_2^{-2} + q_2^{-4}) z_{1} z_{4}) + q_2^{-4} (1 + q_2^{-2} + q_2^{-4}) z_{2} z_{3} + q_2^{-4} (1 + q_2^{-2}) z_{2} z_{4} - z_{3} z_{4}) + q_2^{-1} w (q_2^{-12} z_{1} z_{2} z_{3} - q_2^{-6} (1 + q_2^{-2} + q_2^{-4}) z_{1} z_{2} z_{3} + q_2^{-2} (1 + q_2^{-2} + q_2^{-4}) z_{1} z_{3} z_{4}) + q_2^{-2} (1 + q_2^{-2} + q_2^{-4}) z_{1} z_{3} z_{4} - z_{2} z_{3} z_{4}) \right].
\]

Let \(f(z_1, z_2, z_3, z_4)\) denote the above expression. Since \(\prod_{i<j}(z_i - q_2^{-2}z_j)(z_i - q_2^{-2}z_j)\) is symmetric, we see that the Serre relation is equivalent to the following identity:
\[
(4.9) \quad \sum_{\sigma \in S_4} \text{sgn}(\sigma) z_{\sigma} \left( f(z_1, z_2, z_3, z_4) \prod_{i<j}(z_i - q_2^{-2}z_j) \right) = 0.
\]

We claim that (4.9) is true. Notice that it is enough to check only the coefficients of \(w\) and \(w^2\), and even these two are quite similar. The tedious checking of the coefficient of \(w\) shows that it is zero indeed. Thus the Serre relation is proved.

The constructed level one representation is reducible due to the presence of auxiliary bosons \(b(m)\) and \(c(m)\). All integrable irreducible level one modules are contained in the Fock representation and can be recovered by the technique of the screening operators.

**References**


LEVEL ONE REPRESENTATIONS OF $U_q(G^{(1)}_2)$


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