FINITELY GENERATED GROUP RING UNITS

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Abstract. We give a classification of nilpotent groups $G$ for which the unit group of the integral group ring is finitely generated.

For a group $G$ let $U_{ZG}$ denote the group of invertible elements in the integral group ring $ZG$. This is a subject of intensive study; see Sehgal [5].

The natural question arises as to when $U_{ZG}$ is finitely generated. It is easy to see that if this is the case, then $G$ must be finitely generated. Yet the question has remained unanswered even for the (simplest nonabelian) case of nilpotent groups. In the literature one can find a few partial results. The first is

1. Theorem (Sehgal [4], p. 194). Suppose that $G$ is an extension of a torsion group $T$ by a torsion free nilpotent group. If $ZG$ has no nilpotent elements, then $U_{ZG} = (U_{ZT}) \cdot G$.

It is well known that $U_{ZT}$ is finitely generated for any finite group $T$ and hence Theorem 1 implies that $U_{ZG}$ is finitely generated if $ZG$ contains no nilpotent elements. On the other hand we have a result obtained recently by Marciniak and Sehgal [2] who proved that $U_{ZG}$ is not finitely generated for $G = Z \times D_8$, where $D_8$ is the dihedral group of order 8. Another result of this sort is due to Mirowicz [3]. He showed that $U_{Z[D_\infty]}$ is not finitely generated for the infinite dihedral group $D_\infty$.

In the present paper we determine when $U_{ZG}$ is finitely generated for nilpotent groups $G$. In what follows we denote by $T(G)$ the set of torsion elements in $G$ and by $Z(G)$ the center of $G$. It is well known for finitely generated nilpotent groups $G$ that $T(G)$ is a finite characteristic subgroup which is the product of its Sylow subgroups. For the details concerning nilpotent groups we refer the reader to [1]. Our main result reads:

2. Theorem. Let $G$ be a nilpotent group. Then $U_{ZG}$ is finitely generated if and only if either $G$ is finite or $G$ is finitely generated and there are no nilpotent elements in $ZG$.

By using Sehgal’s characterization of group rings which have no nilpotent elements (see Theorem 10) one can formulate this result in purely group theoretic terms as follows.
3. Corollary. Let $G$ be a nilpotent group. Then $UZG$ is finitely generated if and only if either $G$ is finite or $G$ is finitely generated, all finite subgroups of $G$ are normal and its torsion part $T(G)$ satisfies one of the following conditions:

(i) $T(G)$ is abelian, or

(ii) $T(G) = A \times E \times K_8$ where $E$ is an elementary abelian 2-group and $A$ is a finite abelian group of odd order $m$ such that the multiplicative order of 2 mod $m$ is odd.

We start with four auxiliary results.

4. Lemma. Let $G$ be a nilpotent group with a finite normal $p$-subgroup $H$. Then the ideal $\Delta(G, H) = \ker(F_pG \to F_p[G/H])$ is nilpotent.

Proof. We proceed by induction on $|H|$. If $H = \langle t \rangle t^p = 1$, then

$$\Delta(G, H) = (F_pG)(t - 1) = (t - 1)(F_pG)$$

and hence for any $\alpha = \beta_i(t - 1) \in \Delta(G, H) (1 \leq i \leq p)$ we have $\alpha_1 \cdots \alpha_p \in F_pG(t - 1)^p = F_pG(t^p - 1) = 0$. Therefore the lemma is true when $|H| = p$.

Suppose now $|H| = p^n$, $n > 1$. As $G$ is nilpotent and $H \lhd G$, so $H / Z(G) \neq 1$ and hence there exists a central subgroup $C \triangleleft H$ with $|C| = p$. For $\phi : F_pG \to F_p[G/C]$ we clearly have $\phi(\Delta(G, H)) \subseteq \Delta(G/C, H/C)$. By the inductive hypothesis, $\Delta(G/C, H/C)$ and $\Delta(G/C)$ are nilpotent, hence

$$(\Delta(G, H))^{st} \subseteq (\Delta(G, C))^{st} = 0$$

for suitable $s, t$. \hfill $\Box$

5. Lemma. Let $G$ be a finitely generated nilpotent group of Hirsch rank 1. Then $G$ is a semidirect product of its finite subgroup $T(G)$ with an infinite cyclic group.

Proof. $G/T(G)$ is a finitely generated torsion free nilpotent group of Hirsch rank 1, so it is infinite cyclic. \hfill $\Box$

6. Lemma. If $G$ is a finitely generated nilpotent group with an infinite and normal subgroup $H$, then there exists a subgroup $H_1 < H$, $H_1 \lhd G$, such that $H/H_1 \cong \mathbb{Z}$.

Proof. Consider the upper central sequence of $G$: $G_0 < G_1 < G_2 < \cdots$ and the subgroups $D_i = G_i \cap H$. Let $i$ be the (least) integer satisfying $|H/D_i| = \infty$ and $|H/D_{i+1} < \infty$. Then $D_{i+1}/D_i$ is infinite, finitely generated and lies in the center of $G/D_i$. Besides, $|(H/D_i): (D_{i+1}/D_i)| < \infty$. Thus we can find a group $D_{i+\frac{1}{2}}$ such that $D_i \leq D_{i+\frac{1}{2}} < D_{i+1}$ and $D_{i+1}/D_{i+\frac{1}{2}} \cong \mathbb{Z}$. From Lemma 5 it follows that $H/D_{i+\frac{1}{2}}$ is a semidirect product of $\mathbb{Z}$ with its subgroup $T$ of torsion elements which, by the way, is normal in $G/D_{i+\frac{1}{2}}$. Hence we may define

$$H_1 = \pi^{-1}(T)$$

where $\pi : G \to G/D_{i+\frac{1}{2}}$ is the natural homomorphism. \hfill $\Box$

7. Lemma. Let $R$ be a ring of characteristic $p$ and $J$ its nilpotent ideal. Then $1 + J$ is a nilpotent $p$-group.

Proof. If $y \in J$, then $(1 + y)^{-1} = 1 - y + y^2 - \cdots \in 1 + J$, hence $1 + J$ is a group. Next observe that $(1 + J^k, 1 + J^l) \subseteq 1 + J^{k+l}$. Indeed, if $a \in J^k$ and $b \in J^l$, then

$$[1 - a, 1 - b] = (1 + a + a^2 + \cdots)(1 + b + b^2 + \cdots)(1 - a)(1 - b) = 1 + (\text{a sum of products in which both } a \text{ and } b \text{ occur}) \in 1 + J^{k+l}.$$

So our inclusion holds and
consequently $1 + J$ is a nilpotent group. To prove that $1 + J$ is a $p$-group notice that if $a \in J$, then $(1 + a)^p = 1 + a^p = 1$ for $n$ sufficiently large. 

**Definition.** A group ring unit $u \in ZG$ is unipotent if $(u - 1)^n = 0$ for some $n \geq 1$.

**Definition.** A subgroup $U < UZG$ is $p$-nice if $Z(G) < U$ and $G$ has a finite normal $p$-subgroup $V$ such that the image of $U$ in $Z[G/V]$ consists of trivial units only.

**8. Theorem.** Suppose $U < UZH$ is $p$-nice and the center $Z(H)$ is of finite index in $H$. If $U$ is finitely generated, then any infinite set of unipotent units in $U$ has two elements $u, v$ such that $p(u - v)$.

**Proof.** We have two commuting diagrams of natural maps:

$$
\begin{align*}
ZH & \longrightarrow Z[H/V] \\
\downarrow \rho & \downarrow \rho \\
F_p H & \longrightarrow F_p[H/V]
\end{align*}
\quad
\begin{align*}
UZH & \longrightarrow UZ[H/V] \\
\downarrow \rho & \downarrow \rho \\
UF_p H & \longrightarrow UF_p[H/V]
\end{align*}
\]

Let $\bar{U} = \rho(U) < UF_p H$ and $I = \ker(\pi: F_p H \to F_p[H/V]) = \Delta(H, V)$. As $\pi(\bar{U}) = \rho\pi(U) \leq \pm H/V$, so we have an exact sequence

$$1 \to (1 + I) \cap \bar{U} \to \bar{U} \to \pm H/V.$$

Let $Z = Z(H)$. We obtain an exact sequence

$$1 \to ((1 + I) \cap \bar{U})/((1 + I) \cap \bar{U} \cap Z) \to \bar{U}/Z \to \pm H/(Z \cdot V)$$

with $H/(Z \cdot V)$ finite. As $\bar{U}/Z$ is finitely generated, so is $((1 + I) \cap \bar{U})/((1 + I) \cap \bar{U} \cap Z)$. But $(1 + I) \cap \bar{U} \cap Z < (1 + I) \cap Z < V$, hence $(1 + I) \cap \bar{U} \cap Z$ is finite. Therefore $(1 + I) \cap \bar{U}$ is finitely generated. By Lemmas 4 and 7 it is a nilpotent $p$-group, hence it is finite.

Let $S \subseteq U$ be an infinite set of unipotent units. As 1 is the only unipotent unit in $\pm H/V \subset Z[H/V]$ so $\pi(u) = 1$ for all $u \in S$. Hence $\bar{\pi}(u) = \rho\pi(u) = 1$ and $\rho(u) \in (1 + I) \cap \bar{U}$. It follows that $\rho(u) = \rho(v)$ for some $u, v \in S$.

For an element $\alpha \in ZH$ let $C(\alpha)$ denote the centralizer of $\alpha$ in $H$, i.e., $C(\alpha) = \{h \in H: h\alpha = \alpha h\}$.

**9. Proposition.** Suppose $ZG$ contains a nilpotent element $\eta \neq 0$. If there exist a group $H$ with $|H : Z(H)| < \infty$ and a homomorphism $\phi: G \to H$ such that $\phi(\eta) \neq 0$, $|\phi(C(\eta))| = \infty$ and $\phi(UZG)$ is $p$-nice in $UZH$, then $UZG$ is not finitely generated.

**Proof.** We may assume that not all coefficients of $\eta$ are divisible by $p$. Set $U = \phi(UZG)$. Let $h_1, h_2, \ldots \in C(\eta)$ map to different elements in $H$. Consider $X = \{1 + h, i: i = 1, 2, \ldots \} \subseteq UZG$. Let overbar denote images under $\phi$. If $u = 1 + \bar{h_1}\bar{\eta}$, $v = 1 + \bar{h_j}\bar{\eta} \in U$, then $u - v = (\bar{h_1} - \bar{h_j})\bar{\eta}$. Now, for every $h_i$ we are able to choose infinitely many elements $h_j, h_2, \ldots$ such that $\mathrm{supp}(\bar{h_j}\bar{\eta}) \cap \mathrm{supp}(\bar{h_j}\bar{\eta}) = \emptyset$ and hence $p\bar{j}(u - v)$ for every $u, v \in S$. Our proposition now follows from Theorem 8.

We now use Theorem 8 to classify finitely generated nilpotent groups $G$ for which $UZG$ is not finitely generated. Recall first the following
10. **Theorem** (Sehgal [4], p. 176). Let $G$ be a finitely generated nilpotent group. The group ring $\mathbb{Z}G$ has no nilpotent elements if every finite subgroup of $G$ is normal and either

(i) $T(G)$ is abelian, or

(ii) $T(G) = A \times E \times K_8$ where $E$ is an elementary abelian 2-group and $A$ is a finite abelian group of odd order $m$ such that the multiplicative order of 2 mod $m$ is odd.

Now, using Theorems 8 and 10, we prove the following

11. **Theorem.** If $G$ is an infinite nilpotent group and $\mathbb{Z}G$ contains a nonzero nilpotent element, then $\mathbb{U} \mathbb{Z}G$ is not finitely generated.

**Proof.** If $\mathbb{U} \mathbb{Z}G$ is generated by $u_1, \ldots, u_m$, then $G$ is generated by $\bigcup_{i=1}^m \text{supp}(u_i)$. Therefore we can assume that $G$ is finitely generated. We know that every finitely generated nilpotent group is torsion free-by-finite so we can find a normal torsion free subgroup $H_1 \triangleleft G$ such that $|G : H_1| < \infty$. Lemma 6 assures the existence of a normal in $G$ subgroup $H_2 < H_1$ such that $H_1/H_2 \cong \mathbb{Z}$. Then the Hirsch rank of $G/H_2$ is 1.

From Theorem 10 it follows that we are to consider two cases. First let us assume that there exists a finite subgroup $E \triangleleft G$ which is not normal. As $T(G)$ is the product of its Sylow subgroups we may assume that $E = \langle y \rangle$ where $o(y) = p^n$ for some prime $p$. We can always find an element $t \in G$ such that $t^{-1}Et \neq E$ and $o(t) = \infty$ where $\bar{t}$ is the image of $t$ in $G/H_2$. This is so because $|T(G), H_2| = 1$ and for any elements $z, x \in G$ such that $o(z) = \infty$ and $x^{-1}Ex \neq E$ one of the elements $x, xz, z$ fits. Let $T(G) = S_p \times S_{p'}$ where $S_p$ is the Sylow $p$-subgroup of $T(G)$.

Now we put: $H = G/\langle H_2 \cdot S_{p'} \rangle$, $\phi : G \to H$ is the natural homomorphism, $\eta = (1 - y)t(1 + y + \cdots + y^{p^n} - 1)$, $V = \phi(S_p) \simeq S_p$, $U = \phi(\mathbb{U} \mathbb{Z}G)$ and check that all assumptions of Proposition 9 are satisfied.

We have the following diagram of natural homomorphisms:

\[
\begin{array}{ccc}
\mathbb{Z}G & \longrightarrow & \mathbb{Z}[G/T(G)] \\
\downarrow & & \downarrow \\
\mathbb{Z}H = \mathbb{Z}[G/(H_2 \cdot S_{p'})] & \longrightarrow & \mathbb{Z}[G/(H_2 \cdot T(G))] = \mathbb{Z}[H/V]
\end{array}
\]

and $\mathbb{Z}[G/T(G)]$ has only trivial units because $G/T(G)$ is torsion free nilpotent. Also $H < U$, hence $U$ is $p$-nice. It is clear that $\phi(\eta) \neq 0$. As $T(G)$ is finite we can find an integer $k$ such that $t^k$ commutes with $T(G)$ and from the construction of $\eta$ it follows that we have an infinite subset $\{t^{kl} : l = 1, 2, \ldots\} \subseteq \phi(C(\eta))$. Finally, $|H : \mathbb{Z}(H)| \lt \infty$, because $H$ has Hirsch rank 1. Hence all assumptions of Proposition 9 are satisfied and so $\mathbb{U} \mathbb{Z}G$ is not finitely generated.

Let us now suppose that every finite subgroup of $G$ is normal. As $\mathbb{Z}G$ contains a nonzero nilpotent element we can, by Theorem 10, assume that $T(G) = A \times E \times K_8$, where $E$ is an elementary abelian 2-group and $A$ is a finite abelian group of odd order $m$ such that the multiplicative order of 2 mod $m$ is even. Recall that we have found torsion free normal subgroups $H_1$ and $H_2$ such that $G/H_1$ is finite and $H_1/H_2 \simeq \mathbb{Z}$.

Let $|A| = m$. If $m = p_1^{n_1}p_2^{n_2} \cdots p_k^{n_k}$, then there is at least one $1 \leq i \leq k$ such that the multiplicative order of 2 mod $p_i^{n_i}$ is even. Indeed, otherwise $2^{l_i} \equiv 1 \pmod{p_i^{n_i}}$ for some odd integers $l_i$. But then $2^{l_1l_2 \cdots l_k} \equiv 1 \pmod{m}$ and the multiplicative
order of 2 mod $m$ would be odd. Hence we can choose $i$ so that the multiplicative order of 2 mod $p^a_i$ is even. Put $p = p_i$ and let $S_p$ be the Sylow $p$-subgroup of $T(G)$ (i.e., of $A$). Let $S_{2',p'}$ be the product of all remaining Sylow subgroups of $A$. Theorem 2 ensures the existence of a nonzero nilpotent element $\eta$ in $\mathbb{Z}[S_p \times K_8]$.

Now we put: $H = G/(H_2 \cdot S_{2',p'})$, $\phi: G \to H$ is the natural homomorphism, $V = \phi(S_p) \simeq S_p$, $U = \phi(U\mathbb{Z}G)$.

As in the previous case we prove that the assumptions of Proposition 9 are satisfied. This time we have the following diagram of natural homomorphisms:

\[
\begin{array}{ccc}
\mathbb{Z}G & \longrightarrow & \mathbb{Z}[G/(S_{2',p'} \cdot S_p)] \\
\downarrow & & \downarrow \\
\mathbb{Z}H = \mathbb{Z}[G/(H_2 \cdot S_{2',p'})] & \longrightarrow & \mathbb{Z}[G/(H_2 \cdot S_{2',p'} \cdot S_p)] = \mathbb{Z}[H/V]
\end{array}
\]

The torsion subgroup of $G/(S_{2',p'} \cdot S_p)$ is $E \times K_8$. It is generally known that there are only trivial units in $\mathbb{Z}[E \times K_8]$. Because we assumed that there are no nontrivial, finite, not normal subgroups in $G$ we may apply Theorem 10 to conclude that there are only trivial units in $\mathbb{Z}[G/(S_{2',p'} \cdot S_p)]$. Also $H < U$ and therefore $U$ is $p$-nice. Like before we can show that $\phi(\eta) \neq 0$ and $|\phi(C(\eta))| = \infty$. Applying Proposition 9 we obtain that $U\mathbb{Z}G$ is not finitely generated.

The main result, Theorem 2, easily follows from the above theorem.

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References

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