

THE UNIFORM EXISTENCE OF SOLUTIONS
FOR TWO-DIMENSIONAL AND THREE-DIMENSIONAL
SEMILINEAR WAVE EQUATIONS WITH OSCILLATORY DATA

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ABSTRACT. For two-dimensional and three-dimensional semilinear wave equations, I prove the uniform existence of solutions with oscillatory initial data. Hence I solve an “open question” in one paper of J. L. Joly, G. Metivier and J. Rauch.

Nonlinear geometric optics provide an asymptotic description in the limit $\varepsilon \rightarrow 0$ of solutions to the oscillatory initial value problem

$$(1) \quad \square u + f(u, \nabla u) = 0, \square := \partial_t^2 - \Delta$$

$$(2) \quad \partial_t^j u(0, x) = \varepsilon^{1-j} a_j(x) e^{\frac{i\varphi(x)}{\varepsilon}}, \quad j = 0, 1,$$

where

$$(3) \quad a_j \in C_0^\infty(B(0, R_0); \mathbb{C}), \varphi \in C^\infty(\mathbb{R}^n; \mathbb{R}),$$

and

$$(4) \quad d\varphi(x) \neq 0, \text{ when } x \in \text{supp} a_0 \cup \text{supp} a_1.$$

In particular, one has existence on an ε independent time interval and this is true for all dimensions. J. L. Joly, G. Metivier and J. Rauch showed that for $n \geq 4$, one consequence of focusing effects is that this uniform domain of existence may not exist when $d\varphi$ vanishes in the support of the initial data. When the dimension $n = 1$, classical estimates of Haar type imply uniform local solvability. The explicit formulas for the solutions show that in dimensions no larger than 3 the family of solutions of the linear initial value problem with the same initial data is uniformly bounded on spacetime for $0 < \varepsilon < 1$. We state this as a proposition.

Proposition. *If $n \leq 3$ and u^ε is the family of solutions of $\square u^\varepsilon = 0$ with initial data satisfying (2), (3) (and not necessarily (4)), then for any $T > 0$ the family u^ε is bounded in $L^\infty([0, T] \times \mathbb{R}^n)$.*

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Proof. Thanks to finite speed of propagation together with the method of descent, it suffices to prove the case $n = 3$. By the formula of solution, we have

$$\begin{aligned} u^\varepsilon(t, x) &= \frac{1}{4\pi t} \iint_{\Sigma_{i=1}^3(\xi_i - x_i)^2 = t^2} a_1(\xi) e^{i\frac{\varphi(\xi)}{\varepsilon}} dS \\ &\quad + \frac{1}{4\pi} \frac{\partial}{\partial t} \left[\frac{1}{t} \iint_{\Sigma_{i=1}^3(\xi_i - x_i)^2 = t^2} \varepsilon a_0(\xi) e^{i\frac{\varphi(\xi)}{\varepsilon}} dS \right] \\ &= I + II. \end{aligned}$$

We set $\xi_1 = x_1 + t\cos\theta\sin\varphi$, $\xi_2 = x_2 + t\sin\theta\sin\varphi$, $\xi_3 = x_3 + t\cos\varphi$, then

$$I = \frac{t}{4\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} a_1(x_1 + t\cos\theta\sin\varphi, x_2 + t\sin\theta\sin\varphi, x_3 + t\cos\varphi) e^{i\frac{\varphi(x_1 + t\cos\theta\sin\varphi, x_2 + t\sin\theta\sin\varphi, x_3 + t\cos\varphi)}{\varepsilon}} \sin\theta d\theta d\varphi.$$

So $|I| \leq \frac{\pi}{4} \|a_1(x)\|_0 t$

$$\begin{aligned} II &= \frac{\varepsilon}{4\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} a_0(x_1 + t\cos\theta\sin\varphi, x_2 + t\sin\theta\sin\varphi, x_3 + t\cos\varphi) \\ &\quad \times e^{i\frac{\varphi(x_1 + t\cos\theta\sin\varphi, x_2 + t\sin\theta\sin\varphi, x_3 + t\cos\varphi)}{\varepsilon}} \sin\theta d\theta d\varphi \\ &\quad + \frac{t}{4\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left\{ \varepsilon \left[\partial_1 a_0(x_1 + t\cos\theta\sin\varphi, x_2 + t\sin\theta\sin\varphi, x_3 + t\cos\varphi) \cos\theta\sin\varphi \right. \right. \\ &\quad \left. \left. + \partial_2 a_0(x_1 + t\cos\theta\sin\varphi, x_2 + t\sin\theta\sin\varphi, x_3 + t\cos\varphi) \sin\theta\sin\varphi \right. \right. \\ &\quad \left. \left. + \partial_3 a_0(x_1 + t\cos\theta\sin\varphi, x_2 + t\sin\theta\sin\varphi, x_3 + t\cos\varphi) \cos\varphi \right] \right. \\ &\quad \left. + i a_0(x_1 + t\cos\theta\sin\varphi, x_2 + t\sin\theta\sin\varphi, x_3 + t\cos\varphi) \right. \\ &\quad \times \left[\partial_1 \varphi(x_1 + t\cos\theta\sin\varphi, x_2 + t\sin\theta\sin\varphi, x_3 + t\cos\varphi) \cos\theta\sin\varphi \right. \\ &\quad \left. + \partial_2 \varphi(x_1 + t\cos\theta\sin\varphi, x_2 + t\sin\theta\sin\varphi, x_3 + t\cos\varphi) \sin\theta\sin\varphi \right. \\ &\quad \left. \left. + \partial_3 \varphi(x_1 + t\cos\theta\sin\varphi, x_2 + t\sin\theta\sin\varphi, x_3 + t\cos\varphi) \cos\varphi \right] \right\} \\ &\quad \times e^{i\frac{\varphi(x_1 + t\cos\theta\sin\varphi, x_2 + t\sin\theta\sin\varphi, x_3 + t\cos\varphi)}{\varepsilon}} \sin\theta d\theta d\varphi. \end{aligned}$$

Hence $|II| \leq \frac{\pi}{4} (\|a_0(x)\|_0 + t \|\nabla_x a_0(x)\|_0 + t \|a_0(x)\|_0 \sup_{|x| \leq R_0} |\nabla_x \varphi(x)|)$.

Therefore we complete the proof. \square

The next result shows that this implies uniform local solvability for nonlinear problems when the nonlinearity depends only on u and not on ∇u .

In [1], J. L. Joly, G. Metivier and J. Rauch posed the following questions:

“If $n = 2$ and $\square u = f(u)$ with f polynomially bounded, then the domain of smooth existence does not shrink with ε . Is the same true for arbitrary smooth f ?”

“If $n = 3$, investigate the analogous questions for scalar wave equations with or without derivatives in the nonlinearities.”

The next theorem answers the first question in the affirmative and proves uniform solvability for the second when the nonlinear term is independent of ∇u .

Theorem. Suppose that $n \leq 3, T_0 > 0$ and that

$$\{u_L^\varepsilon : 0 < \varepsilon < 1\} \in C^\infty([0, T_0] \times \mathbb{R}^n)$$

is a family of solutions of D'Alembert's equation

$$\square u^\varepsilon = 0$$

which is bounded in $L^\infty([0, T_0] \times \mathbb{R}^n)$. Then there is a $T_1 \in (0, T_0]$ so that for $0 < \varepsilon < 1$ the solution u_ε of

$$\square u_\varepsilon + f(u_\varepsilon) = 0, \partial_t^j u_\varepsilon(0, x) = \partial_t^j u_L^\varepsilon(0, x), j = 0, 1,$$

exists for $0 \leq t \leq T_1$. This solution belongs to $C^\infty([0, T_1] \times \mathbb{R}^n)$ and is bounded in $L^\infty([0, T_1] \times \mathbb{R}^n)$.

Proof. For each ε there is a maximal solution $u_\varepsilon \in C^\infty([0, T(\varepsilon)] \times \mathbb{R}^n)$. Let

$$M := \sup_\varepsilon \|u_L^\varepsilon\|_{L^\infty([0, T_0] \times \mathbb{R}^n)} < \infty$$

and

$$C_1 := \sup\{|f(w)| : |w| \leq 2M\} < \infty.$$

The fundamental solution E of the wave equation defined by

$$\square E = 0, E(0, x) = 0, E_t(0, x) = \delta$$

is a nonnegative measure with

$$\langle E, [0, t] \times \mathbb{R}^n \rangle = C_2 t < \infty.$$

Write

$$u_\varepsilon = u_L^\varepsilon + v^\varepsilon$$

where v^ε satisfies

$$\square v^\varepsilon = f(u_L^\varepsilon + v^\varepsilon), v^\varepsilon(0, x) = v_t^\varepsilon(0, x) = 0.$$

It follows that as long as $|v^\varepsilon| \leq M$ on $[0, t] \times \mathbb{R}^n$ one has

$$\|v^\varepsilon\|_{L^\infty([0, t] \times \mathbb{R}^n)} \leq t C_1 C_2.$$

Choose $T_1 \in (0, T_0]$ so that $T_1 C_1 C_2 \leq \frac{M}{2}$. Standard local existence and regularity theorems imply that v^ε exists and is smooth throughout $0 \leq t \leq T_1$ where it is bounded in absolute value by $\frac{3M}{2}$. \square

Remark. If w_ε and w satisfy the following equation respectively:

$$\square w_\varepsilon = 0, w_\varepsilon(0, x) = \varepsilon a_0(x) e^{i \frac{\varphi(x)}{\varepsilon}}, \partial_t w_\varepsilon(0, x) = a_1(x) e^{i \frac{\varphi(x)}{\varepsilon}},$$

$$\square w + f(w) = 0, w(0, x) = \partial_t w(0, x) = 0,$$

then for T_1 in the theorem, we can easily prove

$$\sup_{0 \leq t \leq T_1} |\nabla_{x,t}(u_\varepsilon - w_\varepsilon - w)|_{L^2([0, T_1] \times \mathbb{R}^n)} \rightarrow 0$$

as $\varepsilon \rightarrow 0$.

In fact, set $V_\varepsilon = u_\varepsilon - w_\varepsilon - w$, then

$$\square V_\varepsilon + f(u_\varepsilon) - f(w) = 0, V_\varepsilon(0, x) = \partial_t V_\varepsilon(0, x) = 0.$$

By the energy estimate for the wave equation we get

$$(5) \quad \sup_{0 \leq t \leq T_1} (|\partial_t V_\varepsilon|_{L^2(\mathbb{R}^n)}^2 + |\nabla_x V_\varepsilon|_{L^2(\mathbb{R}^n)}^2) \leq C \int_0^{T_1} |f(u_\varepsilon) - f(w)|_{L^2(\mathbb{R}^n)}^2 dt \leq C_{T_1}$$

where C_{T_1} is independent of ε . Hence V_ε is uniformly bounded in $C([0, T_1], H^1(\mathbb{R}^n))$ and $H^1([0, T_1] \times \mathbb{R}^n)$. Because V_ε has compact support, then for any subsequence V_{ε_j} of V_ε , there exists the subsequence V_{ε_j} of V_ε and $V \in L^\infty([0, T_1], H^1(\mathbb{R}^n)) \cap H^1([0, T_1] \times \mathbb{R}^n)$, such that $V_{\varepsilon_j} \rightharpoonup V, H^1([0, T_1] \times \mathbb{R}^n)$. By Rellich's theorem, then $V_{\varepsilon_j} \rightarrow V, L^2([0, T_1] \times \mathbb{R}^n)$. Using $w_\varepsilon \rightarrow 0, L^2([0, T_1] \times \mathbb{R}^n)$, then $V_{\varepsilon_j} \rightarrow V + w, L^2([0, T_1] \times \mathbb{R}^n)$. Since $u_\varepsilon, w_\varepsilon, w$ are uniformly bounded in $[0, T] \times \mathbb{R}^n$, then $V \in L^\infty([0, T_1] \times \mathbb{R}^n)$ and $f(u_\varepsilon) - f(w) \rightarrow f(V + w) - f(w), L^2([0, T_1] \times \mathbb{R}^n)$. Therefore $V \in L^\infty([0, T_1] \times \mathbb{R}^n) \cap H^1([0, T_1] \times \mathbb{R}^n)$ is the solution in the sense of distribution for the following equation:

$$\square V + f(V + w) - f(w) = 0, V(0, x) = \partial_t V(0, x) = 0$$

By the energy estimate of weak solution, then for any $t \in [0, T_1]$ we have

$$\begin{aligned} |\partial_t V(t, \cdot)|_{L^2(\mathbb{R}^n)}^2 + |\nabla_x V(t, \cdot)|_{L^2(\mathbb{R}^n)}^2 &\leq C \int_0^t |f(V + w) - f(w)|_{L^2(\mathbb{R}^n)}^2 dt \\ &\leq C \int_0^t |V|_{L^2(\mathbb{R}^n)}^2 dt \leq C(R_0 + T_1)^2 \int_0^t |\nabla_x V|_{L^2(\mathbb{R}^n)}^2 dt. \end{aligned}$$

So $V = 0$. Because V_{ε_j} is the arbitrary subsequence of V_ε , then

$$V_\varepsilon \rightarrow 0, L^2([0, T_1] \times \mathbb{R}^n) \quad \text{and} \quad f(u_\varepsilon) - f(w) \rightarrow 0, L^2([0, T_1] \times \mathbb{R}^n).$$

By (5) we know $\sup_{0 \leq t \leq T_1} |\nabla_{x,t}(u_\varepsilon - w_\varepsilon - w)|_{L^2(\mathbb{R}^n)} \rightarrow 0$. Because the geometric optics expansion of w_ε is known (for example see [2] or [3]), then we can obtain the valid geometric optics expansion of u_ε (the meaning of validity may refer to [4]).

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REFERENCES

1. J. L. Joly, G. Metivier, J. Rauch, *Coherent and focusing multidimensional nonlinear geometric optics*, Ann. Sci. Ecole Norm. Sup. **28** (1995), 51-113. MR **95k**:35035
2. L. Hormander, *The Analysis of Linear Partial Differential Operators*, Springer-Verlag, Berlin-New York **2** (1983). MR **85g**:35002b
3. J. L. Joly, G. Metivier, J. Rauch, *Nonlinear oscillations beyond caustics*, Comm. Pure Appl. Math., Vol. XIII (1996), 443-527. MR **96m**:35195
4. J. L. Joly, G. Metivier, J. Rauch, *Focusing at a point and absorption of nonlinear oscillations*, Trans. Amer. Math. Soc. **347** (1995), 3921-3971. MR **95m**:35024

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