SOME REMARKS ON THE REAL RANK OF NON-UNITAL C*-ALGEBRAS

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ABSTRACT. For a non-unital C*-algebra \( A \), let \( A^\sim \) be the C*-algebra obtained from \( A \) by adjoining an identity. In this paper we show that
\[
RR(\mathcal{C}_0(X) \otimes A) = RR(\mathcal{C}_0(X) \otimes A^\sim),
\]
where \( X \) is a locally compact Hausdorff space with \( RR(\mathcal{C}_0(X)) \leq 1 \).

1. Introduction

For a unital C*-algebra \( A \), L.G. Brown and G.K. Pedersen [2] define the real rank of \( A \), denoted by \( RR(A) \), as the least integer \( n \) such that for each \((n+1)\)-tuple \( (x_0, \ldots, x_n) \) of self-adjoint elements in \( A \) and every \( \varepsilon > 0 \), there exists an \((n+1)\)-tuple \( (y_0, \ldots, y_n) \) in \( A_{sa} \) such that \( \sum_{j=0}^{n} y_j^2 \) is invertible and
\[
\|x_k - y_k\| < \varepsilon \quad (k = 0, \ldots, n).
\]
M.A. Rieffel [5] defines the stable rank of \( A \), denoted by \( sr(A) \), as the least integer \( n \) such that for each \( n \)-tuple \( (x_1, \ldots, x_n) \) of elements in \( A \) and every \( \varepsilon > 0 \), there exists an \( n \)-tuple \( (y_1, \ldots, y_n) \) in \( A \) such that \( \sum_{j=1}^{n} y_j^* y_j \) is invertible and
\[
\|x_k - y_k\| < \varepsilon \quad (k = 1, \ldots, n).
\]
If \( A \) is non-unital, then we define the real rank of \( A \) and the stable rank of \( A \) by \( RR(A^\sim) \) and \( sr(A^\sim) \) respectively, where the C*-algebra \( A^\sim \) is obtained from \( A \) by adjoining an identity.

In [2], L.G. Brown and G.K. Pedersen show that
\[
RR(C(X)) = \dim X
\]
for a unital commutative C*-algebra \( C(X) \) of all the continuous functions on a compact Hausdorff space \( X \), where \( \dim X \) means the covering dimension of \( X \). For every compact space \( X \) and every paracompact \( Y \), not both empty, we have
\[
\dim(X \times Y) = \dim X + \dim Y
\]
(see [3]). In [4], M. Nagisa, H. Osaka and N.C. Phillips show that
\[
RR(C_0(X) \otimes A) \leq \dim X + RR(A)
\]
for any C*-algebra and any locally compact, σ-compact, Hausdorff space, where \( C_0(X) \) means all the continuous functions vanishing at infinity on \( X \). We remark that a pathological phenomena occurs without the assumption of countability. Strictly speaking, for a non-unital C*-algebra \( A \), they use the fact \( \text{RR}(C_0(X) \otimes A) \leq \text{RR}(C_0(X) \otimes A^\sim) \) and show that

\[
\text{RR}(C_0(X) \otimes A^\sim) \leq \dim X + \text{RR}(A).
\]

In this note, we consider whether \( \text{RR}(C_0(X) \otimes A) \) differs from \( \text{RR}(C_0(X) \otimes A^\sim) \) or not. Our result is

\[
\text{RR}(C_0(X) \otimes A) = \text{RR}(C_0(X) \otimes A^\sim)
\]

for any non-unital C*-algebra \( A \) and any locally compact Hausdorff space \( X \) with \( \text{RR}(C_0(X)) \leq 1 \).

2. Main result

We shall denote the one-point compactification of a locally compact Hausdorff space \( X \) by \( X \cup \{\infty\} \); then we can identify \( C_0(X)^\sim \) and \( C(X \cup \{\infty\}) \). We shall also denote, by \( C_0(X) \otimes A \), all the \( A \)-valued continuous functions vanishing at infinity on \( X \). Then we can canonically identify \( C_0(X, A) \) and the C*-tensor product \( C_0(X) \otimes A \) of \( C_0(X) \) and a C*-algebra \( A \). We use \( q_A \) to denote the natural quotient map \( A^\sim \ni a + \lambda 1 \mapsto \lambda \in \mathbb{C} \cong A^\sim/A \).

The following lemma is shown in [2], [6].

**Lemma 1.** Let \( 0 \to J \to A \to A/J \to 0 \) be an exact sequence of C*-algebras. Then \( \text{RR}(A) = 0 \) if and only if \( \text{RR}(J) = \text{RR}(A/J) = 0 \) and any projection in \( A/J \) can be lifted to a projection in \( A \).

The following proposition is essentially proved in [4]. Here, we extend the original one to the case that a C*-algebra \( A \) is non-unital and a space \( X \), instead of the interval, is general.

**Proposition 2.** Let \( X \) be a locally compact Hausdorff space with \( \text{RR}(C_0(X)) \geq 1 \). Then \( \text{RR}(C_0(X) \otimes A) \geq 1 \) for any C*-algebra \( A \).

**Proof.** For any \( x \in X \), the map \( C_0(X) \otimes A \cong C_0(X, A) \ni f \mapsto f(x) \in A \) is the surjective *-homomorphism. This implies that \( \text{RR}(C_0(X) \otimes A) \geq \text{RR}(A) \). So we may assume that \( \text{RR}(A) = 0 \), and we consider the following exact sequence of C*-algebras:

\[
0 \to C_0(X) \otimes A \to C(X \cup \{\infty\}) \otimes A \xrightarrow{\varphi} A \to 0,
\]

where \( \varphi \) is defined by \( \varphi(f) = f(\infty) \) for all \( f \in C(X \cup \{\infty\}) \otimes A \). Clearly any projection in \( A \) can be lifted to a projection in \( C(X \cup \{\infty\}) \otimes A \). This means that \( \text{RR}(C(X \cup \{\infty\}) \otimes A) = 0 \), if \( \text{RR}(C_0(X) \otimes A) = 0 \). So it is enough to show that \( \text{RR}(C(X \cup \{\infty\}) \otimes A) \geq 1 \) for a compact Hausdorff space \( X \).

From the assumption \( \text{RR}(C(X)) \geq 1 \), there exists a self-adjoint element \( f \) in \( C(X) \) and a positive number \( \delta \) such that, if \( g \) is in \( C(X)_{s.a} \), with \( \|f - g\| < \delta \), then \( g(y) = 0 \) for some \( y \in X \). Since \( X \) is compact, we can find a positive constant \( K \) such that \( f(x) > -K \) for any \( x \in X \). Let \( a \) be a positive element in \( A \) with \( \|a\| = 1 \). We define a self-adjoint element \( F \in (C(X) \otimes A)^\sim \) by

\[
F(x) = (f(x) + K)a - K1 \quad (x \in X),
\]
where we identify \((C(X) \otimes A)^\sim\) and the set \(\{ g \in C(X, A^\sim) \mid q_A(g(\cdot)) \text{ is a constant function on } X \}\). Let us denote, for any element \(b \in A\), the set \(\{ \lambda \in \mathbb{C} \mid b - \lambda 1 \text{ is not invertible} \}\) by the symbol \(Sp(b)\). Then \(f(x) = \max Sp(F(x))\). In fact, by the positivity of \(\lambda\) for some \(y\) function on \(\otimes\)

Proposition 2, we have \(RR(C_\Delta in M\mathcal{C}n\mathcal{C})\).

\[ RR(C_\Delta in M\mathcal{C}n\mathcal{C}) \]

In this identification, we can see \((\mathcal{C}_\Delta in M\mathcal{C}n\mathcal{C})\).

We shall prove only the case that \(\| A \| \leq 0\).

\[ \| A \| \leq 0 \]

Proof. We shall prove only the case that \(X\) is non-compact. We can similarly prove the case that \(X\) is compact.

For a locally compact, non-compact Hausdorff space \(X\), we can identify \((C_0(X) \otimes A^\sim)^\sim\) and

\[ \{ F \in C(X \cup \{ \infty \}, A^\sim) \mid F(\infty) = q_A(F(\infty))1 \} \]

In this identification, we can see \((C_0(X) \otimes A)^\sim\) as

\[ \{ F \in C(X \cup \{ \infty \}, A^\sim) \mid F(\infty) = q_A(F(x))1, \text{ for all } x \in X \} \]

We consider the following split exact sequence of C*-algebras:

\[ 0 \longrightarrow C_0(X) \otimes A \longrightarrow (C_0(X) \otimes A^\sim)^\sim \overset{\pi_j}{\longrightarrow} C(X \cup \{ \infty \}) \longrightarrow 0, \]

where \(\pi_j(F)(x) = q_A(F(x))\) and \(j(f)(x) = f(x)1\) for \(F \in (C_0(X) \otimes A^\sim)^\sim, f \in C_0(X \cup \{ \infty \})\) and \(x \in X \cup \{ \infty \}\). Clearly we have

\[ RR(C_0(X) \otimes A) \leq RR(C_0(X) \otimes A^\sim). \]

At first, we examine the case that the real rank of \(C_0(X) \otimes A\) is zero. By Proposition 2, we have \(RR(C_0(X)) = 0\) and it is clear that any projection in \(C(X \cup \{ \infty \})\) can be lifted to a projection in \((C_0(X) \otimes A^\sim)^\sim\). So we get that \(RR((C_0(X) \otimes A^\sim)) = 0\) by Lemma 1.

In the case that \(RR(C_0(X) \otimes A) = n \geq 1\), we shall show that \(RR(C_0(X) \otimes A^\sim) \leq n\). Let \(F_0, F_1, \ldots, F_n\) be self-adjoint elements in \((C_0(X) \otimes A^\sim)^\sim\), and set

\[ g_0 = \pi(F_0), g_1 = \pi(F_1), f_2 = \pi(F_2), \ldots, f_n = \pi(F_n). \]

Since \(RR(C(X \cup \{ \infty \})) \leq 1\), for any \(\varepsilon > 0\) we can choose \(F_0, F_1 \in C(X \cup \{ \infty \})\) such that \(\| g_0 - f_0 \| < \varepsilon, \| g_1 - f_1 \| < \varepsilon\) and \(f_2^2 + f_1^2 > 0\). We consider an element \(\Delta\) in \(M_n(\mathbb{C}(X \cup \{ \infty \}))\) as follows:

\[
\Delta = \begin{pmatrix}
\frac{f_0}{k_0} & \frac{f_1}{k_0} & \frac{f_2}{k_0} & \frac{f_3}{k_0} & \cdots & \frac{f_{n-1}}{k_0} & \frac{f_n}{k_0} \\
\frac{f_0}{k_1} & \frac{f_1}{k_1} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{f_0}{k_2} & -\frac{f_1}{k_2} & \frac{f_2^2}{k_2} & \frac{f_3^2}{k_2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-\frac{f_0}{k_m} & -\frac{f_1}{k_m} & -\frac{f_2}{k_m} & -\frac{f_3}{k_m} & \cdots & \frac{f_{n-1}}{k_m} & \frac{f_n}{k_m} \\
\end{pmatrix},
\]
where \( k_0 = \sqrt{f_0^2 + f_1^2 + \cdots + f_n^2} \), \( k_1 = \sqrt{f_0^2 + f_1^2} \) and
\[
k_l = \sqrt{(f_0^2 + \cdots + f_{l-1}^2)(f_0^2 + \cdots + f_l^2)} \quad (2 \leq l \leq n).
\]
Then we have
\[
 t \Delta \Delta = I_{n+1} \quad \text{and} \quad \frac{1}{k_0} \Delta \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]
We define self-adjoint elements \( G_0, G_1, \ldots, G_n \) in \( (C_0(X) \otimes A^\sim)^{\sim} \) by the following relation:
\[
(j \otimes id_{n+1})(\frac{1}{k_0} \Delta) \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_n \end{pmatrix} = \begin{pmatrix} G_0 \\ G_1 \\ \vdots \\ G_n \end{pmatrix}.
\]
Then we can easily check
\[
q_A(G_0(x)) = 1, \quad q_A(G_1(x)) = 0, \ldots, q_A(G_n(x)) = 0
\]
for any \( x \in X \cup \{\infty\} \), that is, \( G_0, G_1, \ldots, G_n \) belong to \( (C_0(X) \otimes A)^{\sim} \). So we can choose self-adjoint elements \( \tilde{G}_0, \tilde{G}_1, \ldots, \tilde{G}_n \) in \( (C_0(X) \otimes A)^{\sim} \) satisfying \( \|G_l - \tilde{G}_l\| < (1/||k_0||)\varepsilon \) for \( 0 \leq l \leq n \), and \( \sum_{l=0}^{n}(\tilde{G}_l)^2 \) is invertible. If we put
\[
\begin{pmatrix} \tilde{F}_0 \\ \tilde{F}_1 \\ \vdots \\ \tilde{F}_n \end{pmatrix} = (j \otimes id_{n+1})(k_0 t \Delta) \begin{pmatrix} \tilde{G}_0 \\ \tilde{G}_1 \\ \vdots \\ \tilde{G}_n \end{pmatrix},
\]
then we get \( \tilde{F}_l \in (C_0(X) \otimes A^\sim)^{\sim} \) with \( \|F_l - \tilde{F}_l\| < \varepsilon \) for \( 0 \leq l \leq n \), and \( \sum_{l=0}^{n}(\tilde{F}_l)^2 = \sum_{l=0}^{n}(\tilde{G}_l)^2 \) is invertible. \( \square \)

In the above theorem, for a space \( X \) with \( RR(C_0(X)) \geq 2 \), it is not true that \( RR(C_0(X) \otimes A) = RR(C_0(X) \otimes A^\sim) \) in general. For example, let us denote by \( I \) the \([0,1] \)-interval and by \( K \) the \( C^* \)-algebra of all the compact operators. Then we have
\[
RR(C(I \times I) \otimes K) = 1
\]
(see [1], [2]). But, by the fact \( C(I \times I) \otimes K^\sim / C(I \times I) \otimes K \cong C(I \times I) \), we have
\[
RR(C(I \times I) \otimes K^\sim) \geq 2.
\]

We can apply the above argument for real rank to that for stable rank. Then we have the following statement:

**Proposition 4.** Let \( X \) be a locally compact Hausdorff space, and \( A \) a non-unital \( C^* \)-algebra.

(1) If \( sr(C_0(X)) = 1 \), then
\[
\operatorname{sr}(C_0(X) \otimes A) = \operatorname{sr}(C_0(X) \otimes A^\sim).
\]
Then we have

$$\text{sr}(C_0(X)) = 2$$ and $$\text{sr}(C_0(X) \otimes A) \geq 2$$, then

$$\text{sr}(C_0(X) \otimes A) = \text{sr}(C_0(X) \otimes A^\sim).$$

Proof. We use the identification as stated in the proof of Theorem 3. From the split exact sequence of $$C^*$$-algebras

$$0 \longrightarrow C_0(X) \otimes A \longrightarrow (C_0(X) \otimes A^\sim) \overset{\pi}{\longrightarrow} C(X \cup \{\infty\}) \longrightarrow 0,$$

we have $$\text{sr}(C_0(X) \otimes A) \leq \text{sr}(C_0(X) \otimes A^\sim).$$

First, we examine the case that $$\text{sr}(C_0(X)) \leq 2$$ and $$\text{sr}(C_0(X) \otimes A) \geq 2$$. Suppose that $$\text{sr}(C_0(X) \otimes A) = n$$. We shall show that $$\text{sr}(C_0(X) \otimes A^\sim) \leq n$$. Let $$F_1, F_2, \ldots, F_n$$ be elements in $$(C_0(X) \otimes A^\sim)^n$$, and set

$$g_1 = \pi(F_1), \ g_2 = \pi(F_2), \ f_3 = \pi(F_3), \ldots, f_n = \pi(F_n).$$

Since $$\text{sr}(C(X \cup \{\infty\})) \leq 2$$, for any $$\varepsilon > 0$$ we can choose $$f_1, f_2 \in C(X \cup \{\infty\})$$ such that $$\|g_1 - f_1\| < \varepsilon$$, $$\|g_2 - f_2\| < \varepsilon$$ and $$|f_1|^2 + |f_2|^2 > 0$$. We consider an element $$\Delta$$ in $$M_n(C(X \cup \{\infty\}))$$ as follows:

$$\Delta = \begin{pmatrix}
\frac{\bar{f}_1}{k_1} & \frac{\bar{f}_2}{k_1} & \frac{\bar{f}_3}{k_1} & \cdots & \frac{\bar{f}_{n-1}}{k_1} & \frac{\bar{f}_n}{k_1} \\
-\frac{\bar{f}_1}{k_2} & \frac{\bar{f}_2}{k_2} - \frac{|f_1|^2 + |f_2|^2}{k_3} & 0 & \cdots & 0 & 0 \\
-\frac{\bar{f}_1}{k_3} & \frac{\bar{f}_2}{k_3} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-\frac{\bar{f}_1}{k_n} & \frac{\bar{f}_2}{k_n} & \cdots & \cdots & \frac{\bar{f}_n}{k_n} & -\frac{f_1 f_2}{k_n} \sum_{i=1}^{n-1} \frac{|f_i|^2}{k_i}
\end{pmatrix},$$

where $$\bar{f}$$ means the complex conjugate of $$f$$,

$$k_1 = \sqrt{|f_1|^2 + \cdots + |f_n|^2}, \quad k_2 = \sqrt{|f_1|^2 + |f_2|^2}$$

and

$$k_l = \sqrt{(|f_1|^2 + \cdots + |f_{l-1}|^2)(|f_1|^2 + \cdots + |f_l|^2)} \quad (3 \leq l \leq n).$$

Then we have

$$\sum \Delta = I_n \quad \text{and} \quad \frac{1}{k_1} \Delta \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

For elements $$G_1, G_2, \ldots, G_n$$ in $$(C_0(X) \otimes A)^\sim$$ which are defined by

$$(j \otimes id_n)(\frac{1}{k_1} \Delta) \begin{pmatrix} F_1 - j(g_1) + j(f_1) \\ F_2 \\ \vdots \\ F_n \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{pmatrix},$$

we can choose elements $$\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n$$ in $$(C_0(X) \otimes A)^\sim$$ satisfying $$\|G_l - \tilde{G}_l\| < (1/\|k_l\|)\varepsilon$$ for $$1 \leq l \leq n$$, and $$\sum_{l=1}^{n} (\tilde{G}_l)^2$$ is invertible. Then we can get the desired
approximants \( \tilde{F}_1, \ldots, \tilde{F}_n \) by the relation
\[
\begin{pmatrix}
\tilde{F}_1 \\
\tilde{F}_2 \\
\vdots \\
\tilde{F}_n \\
\end{pmatrix} = (j \otimes \text{id}_n)(k_1 \Delta) \begin{pmatrix}
\tilde{G}_1 \\
\tilde{G}_2 \\
\vdots \\
\tilde{G}_n \\
\end{pmatrix},
\]

Next, we consider the case that \( \text{sr}(C_0(X) \otimes A) = 1 \) and \( \text{sr}(C_0(X)) = 1 \). The above argument can be applied to this case. There, \( \Delta \) is a \( 1 \times 1 \) matrix \( \frac{f}{|f|} \) for an invertible element \( f \) in \( C(X \cup \{\infty\}) \).

We also remark that
\[
\text{sr}(C(I^4) \otimes \mathbb{K}) = 2 \quad \text{and} \quad \text{sr}(C(I^4) \otimes \mathbb{K}^-) \geq \text{sr}(C(I^4)) = 3.
\]
So, in the case \( \text{sr}(C_0(X)) \geq 3, \)
\[
\text{sr}(C_0(X) \otimes A) = \text{sr}(C_0(X) \otimes A^-)
\]
does not hold in general.

In [4], the following fact is proved: if \( \text{sr}(C(I^n) \otimes A) = 1 \) for a unital C*-algebra \( A \), then we have \( n = 0 \) or \( 1 \) (i.e. \( \text{sr}(C(I^n)) = 1 \)). For a non-unital C*-algebra \( A \) and a locally compact Hausdorff space \( X \), we do not know whether \( \text{sr}(C_0(X) \otimes A) = 1 \) implies \( \text{sr}(C_0(X)) = 1 \) or not.

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References


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