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METRIZABLE AND $\mathbb{R}$-METRIZABLE BETWEENNESS SPACES

JURAJ ŠIMKO

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ABSTRACT. It is proved that the theory of the class of all betweenness spaces metrizable by real-valued metrics does not coincide with the theory of the class of all betweenness spaces metrizable by metrics taking values in any ordered field. This solves a problem raised by Mendris and Zlatoš.

Let d be a metric on a nonempty set A taking values in an ordered field F. The ternary betweenness relation T_d on A is defined by

$$T_d(x, y, z) \ \Leftrightarrow\ d(x, y) + d(y, z) = d(x, z).$$

A first-order structure (A, T) with a single ternary relation T is called a metrizable betweenness space if $T = T_d$ for some metric d on A; it is called an $\mathbb{R}$-metrizable betweenness space if $T = T_d$ for some real-valued d.

As proved by R. Mendris and P. Zlatoš in [1], the class $\mathcal{M}$ of all metrizable betweenness spaces, i.e., the class of all first-order structures of the form (A, T_d), where d is a metric on A, is a universal elementary one. On the other hand, being not closed under elementary extensions, the class $\mathcal{M}_0$ of all $\mathbb{R}$-metrizable betweenness spaces is not elementary. However, the question of whether $\mathcal{M}$ is the least elementary class containing $\mathcal{M}_0$ or, equivalently, the question of whether $\operatorname{Th}\mathcal{M}_0 = \operatorname{Th}\mathcal{M}$ remained open. In this short note we will answer that question negatively.

Let $|xy| \leqslant |uv|$ stand as the abbreviation for the formula

$$(T(x, y, v)\ \&\ T(u, x, y)) \vee (T(x, y, u)\ \&\ T(v, x, y)).$$

Obviously, in a metrizable betweenness space (A, T_d), $|xy| \leqslant |uv|$ implies $d(x, y) \leqslant d(u, v)$ for any $x, y, u, v \in A$.

Further, let $\phi(x, y, z, u, v)$ be the formula expressing that x, y, z, u, v are five distinct elements, and $T(x, y, z)$, $T(u, x, y)$, $T(u, z, v)$ are the only non-trivial betweenness relations among them.

Finally, let us denote by θ the sentence:

$$\big(\exists\, x, y, z, u, v\big)\phi(x, y, z, u, v)\ \&\ \big(\forall\, x, y, z, u, v\big)\big[\phi(x, y, z, u, v) \\ \Rightarrow\ \big((\exists\, z')(z' \neq v\ \&\ T(z, z', v)\ \&\ |xy| \leqslant |zz'|)\big)\big].$$

Theorem 1. $\neg\theta \in \operatorname{Th}\mathcal{M}_0 \setminus \operatorname{Th}\mathcal{M}$; *consequently, the least elementary class containing $\mathcal{M}_0$ is a proper subclass of $\mathcal{M}$.*

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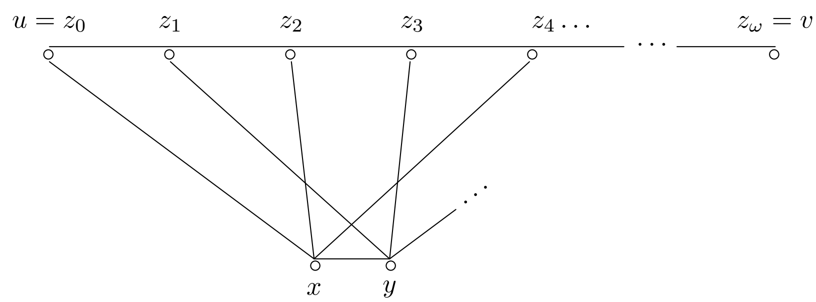

Diagram 1

Proof. (1) $\neg\theta \in \operatorname{Th}\mathcal{M}_0$: Let (A,T) be a metrizable betweenness space satisfying θ, let d be any metric on A inducing T, and let x, y, z, u, v be the elements of A guaranteed by θ. By induction one can construct a sequence $(z_n)_{n<\omega}$ in A, such that $z_0 = u$, $z_1 = z$ and $z_{n+1} = z'_n$ for $n \geqslant 1$, i.e., $z_{n+1} \neq v$, $T(z_n, z_{n+1}, v)$ and $|xy| \leqslant |z_n z_{n+1}|$. Then both $T(u, z_m, z_n)$ and $T(z_m, z_n, v)$ can easily be verified for any $m \leqslant n < \omega$. Therefore,

$$d(u,v) = d(u,z_1) + d(z_1,z_2) + \cdots + d(z_{n-1},z_n) + d(z_n,v) \geqslant n\cdot d(x,y)$$

for each $n < \omega$. As $d(x,y) > 0$, d cannot take values in an Archimedean ordered field.

(2) $\neg\theta \notin \operatorname{Th}\mathcal{M}$: It suffices to find an $(A,T) \in \mathcal{M}$ satisfying θ. Let A consist of the chain $u = z_0 < z_1 < \cdots < z_n < z_{n+1} < \cdots < z_\omega = v$, ordered by the type $\omega+1$, and two other elements x, y. Let T contain all the triples of the form (a,a,b) $(a,b \in A)$, (x,y,z_{2n+1}), (z_{2n},x,y) $(n<\omega)$, (z_m,z_n,z_α) $(m<n<\alpha\leqslant\omega)$, and the reversed ones — see Diagram 1.

One can check that (A,T) satisfies θ. On the other hand, (A,T) is metrizable in any non-Archimedean ordered field F. Indeed, identifying $n<\omega$ with $n\cdot 1 \in F$ and ω with an arbitrary element of F bigger than any n, the metric $d : A\times A \to F$ can be defined by, say,

$$d(x,y) = 1, \qquad d(z_m,z_n) = 2|m-n|, \qquad d(z_n,v) = \omega - 2n,$$
$$d(x,z_{2n}) = d(y,z_{2n+1}) = d(x,v) = d(y,v) = \omega, \qquad d(x,z_{2n+1}) = d(y,z_{2n}) = \omega+1,$$

for all $m,n<\omega$. Then $T = T_d$ is plain. □

Thus we see that $\mathcal{M}$ is not the least elementary class containing $\mathcal{M}_0$. Nevertheless, $\mathcal{M}$ is still a certain closure of $\mathcal{M}_0$.

Theorem 2. *$\mathcal{M}$ is both the least universal class and the least universal-existential class containing $\mathcal{M}_0$, i.e.,* $\operatorname{Th}_\forall\mathcal{M}_0 = \operatorname{Th}_{\forall\exists}\mathcal{M}_0 = \operatorname{Th}\mathcal{M}$.

Proof. Let $\mathcal{K}$ denote the least universal-existential class containing $\mathcal{M}_0$. Obviously, $\mathcal{K} \subseteq \mathcal{M}$. It suffices to show that every countable $(A,T) \in \mathcal{M}$ belongs to $\mathcal{K}$. Each (A,T) can be written as the union of a chain of its finite substructures. Since $\mathcal{M}$ and $\mathcal{M}_0$ contain the same finite structures and the universal-existential class $\mathcal{K}$ is closed under chain unions, $(A,T) \in \mathcal{K}$. Hence $\operatorname{Th}_{\forall\exists} = \mathcal{M}_0 = \operatorname{Th}\mathcal{M}$. Since $\operatorname{Th}\mathcal{M} \subset \operatorname{Th}\mathcal{M}_0$, we have $\operatorname{Th}_\forall\mathcal{M} \subseteq \operatorname{Th}_\forall\mathcal{M}_0$, therefore, since $\operatorname{Th}_\forall\mathcal{M} = \operatorname{Th}\mathcal{M}$ (by part (ii) of the Theorem in [1]), $\operatorname{Th}\mathcal{M} \subseteq \operatorname{Th}_\forall\mathcal{M}_0$. The statement of the theorem now follows from $\operatorname{Th}_\forall\mathcal{M}_0 \subseteq \operatorname{Th}_{\forall\exists}\mathcal{M}_0 = \operatorname{Th}\mathcal{M}$. □

As $\neg\theta$ is equivalent to an existential-universal sentence, the counterexample of Theorem 1 is the best possible.

References

[1] R. Mendris and P. Zlatoš, *Axiomatization and undecidability results for metrizable betweenness relations*, Proc. Amer. Math. Soc. **123** (1995), 873–882. MR **95d:**03008

Department of Mathematics, Faculty of Chemical Technology, Slovak Technical University, 812 37 Bratislava, Slovakia

E-mail address: simko@cvt.stuba.sk