

A NOTE ON THE RAMSEY PROPERTY

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ABSTRACT. An elementary setting of the classical Ramsey property is given, which leads to simple proofs of the relevant theorems of Galvin-Prikry and Silver.

1. INTRODUCTION

The setting of the classical Ramsey property in this note is as follows: Some facts for arbitrary subsets of the space $[\mathbf{N}]$ of infinite subsets of \mathbf{N} are proved, which lead to easy proofs of the relevant theorems of Galvin-Prikry [4] and Silver [6]. Topological conditions are posed exactly at the points where they are necessary. Moreover, our proof of Silver's theorem is more direct and "constructive" than the previous ones.

The proof of Silver that every analytic subset of $[\mathbf{N}]$ is completely Ramsey uses deep metamathematical ideas involving forcing. A classical proof of Silver's theorem was given by Ellentuck [3] based on Marczewski's theorem that in a topological space the class of sets with the Baire property is closed under the Souslin operation. Also, Ellentuck obtained a topological characterization of the completely Ramsey sets, proving that a subset of $[\mathbf{N}]$ is completely Ramsey if and only if it has the Baire property in the topology introduced by him. This characterization played a substantial role in the subsequent development of qualitative Ramsey theory, see Carlson [1], Carlson-Simpson [2]. Finally a proof of Silver's theorem using Ellentuck's characterization of completely Ramsey sets and topological games is included in the book of A. Kechris [5, p. 155].

2. NOTATION AND DEFINITIONS

In this note a, b, \dots denote finite subsets of \mathbf{N} and A, B, L, \dots denote infinite subsets of \mathbf{N} . We set $a < A$ if $\max(a) < \min(A)$. We also set

$$[a, A] = \{a \cup L : L \subset A \text{ \& } a < L\}.$$

By $[\mathbf{N}]$ we denote the set of infinite subsets of \mathbf{N} endowed with the topology of pointwise convergence. If $X \subset [\mathbf{N}]$ we set $X^c = [\mathbf{N}] \setminus X$, \overline{X} denotes the closure of X in $[\mathbf{N}]$, and \widehat{X} denotes the set defined by

$$L \in \widehat{X} \Leftrightarrow \forall [a, A] (L \in [a, A] \Rightarrow [a, A] \cap X \neq \emptyset).$$

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The *Baire space* \mathcal{N} is the space $\mathbf{N}^{\mathbf{N}}$ endowed with the product topology, considering \mathbf{N} with the discrete topology, s denotes finite sequences of \mathbf{N} , and σ denotes infinite sequences of \mathbf{N} . For every s we denote by \mathcal{N}_s the set of members of \mathcal{N} that extend s .

Definition. A subset X of $[\mathbf{N}]$ is called *completely Ramsey* if for every a, A there exists $B \subset A$ such that either $[a, B] \subset X$ or $[a, B] \subset X^c$.

3. THE SETTING OF RESULTS

The following lemma is essentially contained in Galvin-Prikry [4] with different statement and terminology. We include a proof because of the different statement since it plays a substantial role in the sequel.

3.1. Lemma. *Let $X \subset [\mathbf{N}]$ be an arbitrary set. Then, for every a, A there exists $B \subset A$ such that either*

- (i) $[a, B] \subset X$ or
- (ii) $[a \cup b, L] \not\subset X$ for every $b \subset B$ and $L \subset B$.

Proof. We suppose that for every $B \subset A$, (i) fails and prove that (ii) holds for some $B \subset A$. The proof will be by induction. We assume without loss of generality that $a < A$.

Base. There exist $x_1 \in A$ and $A_2 \subset A$ such that for every $L \subset A_2$,

$$(*) \quad [a \cup \{x_1\}, L] \not\subset X.$$

Assume the contrary. We choose $y_1 \in A$. Then the pair (y_1, A) does not satisfy (*), so there exists $B_2 \subset A$ such that $[a \cup \{y_1\}, B_2] \subset X$. We choose $y_2 \in B_2$ with $y_2 > y_1$. Continuing in this way natural numbers and sets

$$y_1 < y_2 < \dots, \\ A = B_1 \supset B_2 \supset \dots$$

are constructed with $y_i \in B_i$ and $[a \cup \{y_i\}, B_{i+1}] \subset X$. We set $B = \{y_j : j = 1, 2, \dots\}$. Then $[a, B] \subset X$ which is absurd.

Induction step. Assume that natural numbers and sets

$$x_1 < x_2 < \dots < x_k, \\ (**) \quad A = A_1 \supset A_2 \supset \dots \supset A_k \supset A_{k+1}$$

have been constructed such that for every $i \leq k$, $x_i \in A_i$ and for every $L \subset A_{i+1}$ and $b \subset \{x_1, \dots, x_i\}$, $[a \cup b, L] \not\subset X$. Then for every $b \subset \{x_1, \dots, x_k\}$ and for every $L \subset A_{k+1}$, $[a \cup b, L] \not\subset X$. Now we can find x_{k+1} and A_{k+2} extending (**), since otherwise repeating the previous argument we find $b \subset \{x_1, \dots, x_k\}$ and $B \subset A_{k+1}$ such that $[a \cup b, B] \subset X$ which is absurd. Thus

$$x_1 < x_2 < \dots, \\ A = A_1 \supset A_2 \supset \dots$$

are constructed such that every initial segment satisfies (**). We set $B = \{x_j : j = 1, 2, \dots\}$. Then B satisfies condition (ii). \square

Condition (ii) says exactly that $[a, B] \subset \widehat{X}^c$. Hence as $X \subset \widehat{X} \subset \overline{X}$ we have the following.

3.2. Corollary. *Every closed subset of $[\mathbf{N}]$ is completely Ramsey.*

The next lemma suggests the definition of completely Ramsey sets of Section 2.

3.3. Lemma. *Let $\{X_i\}$ be an arbitrary sequence of subsets of $[\mathbf{N}]$. Let also a, A be sets. We suppose that for every $L \subset A$,*

$$[a, L] \not\subset \bigcap_{i=1}^{\infty} X_i.$$

Then there exist $b \subset A$ with $b > a$, $B \subset A$ and i_0 such that

$$(*) \quad [a \cup b, L] \not\subset X_{i_0}$$

for every $L \subset B$.

Proof. Assume the contrary. We choose $x_1 \in A = A_1$ with $x_1 > a$. Then the triple $(x_1, 1, A)$ does not satisfy $(*)$, so there exists $A_2 \subset A$ with $[a \cup \{x_1\}, A_2] \subset X_1$. Suppose that natural numbers and sets

$$(**) \quad \begin{aligned} &x_1 < x_2 < \dots < x_k, \\ &A = A_1 \supset A_2 \supset \dots \supset A_k \supset A_{k+1} \end{aligned}$$

have been constructed such that for every $i \leq k$, $x_i \in A_i$ and for every nonempty $b \subset \{x_1, \dots, x_i\}$ we have $[a \cup b, A_{i+1}] \subset X_j$ for all $j \leq i$. Then for every nonempty $b \subset \{x_1, \dots, x_k\}$ we have $[a \cup b, A_{k+1}] \subset X_j$ for every $j \leq k$. Using a repeated argument we find x_{k+1} and A_{k+2} extending $(**)$. The set $B = \{x_i : i = 1, 2, \dots\}$ constructed in this way satisfies $[a, B] \subset \bigcap_{i=1}^{\infty} X_i$ which is a contradiction, and the proof is complete. \square

3.4. Corollary. *If $\{R_i\}$ is a sequence of completely Ramsey sets, then the union $\bigcup_{i=1}^{\infty} R_i$ is a completely Ramsey set.*

Proof. Let a, A be sets. Let us assume that there does not exist $B \subset A$ with $[a, B] \subset \bigcup_{i=1}^{\infty} R_i$. Then by Lemma 3.1 there exists $B_1 \subset A$ with $[a \cup b, L] \not\subset \bigcup_{i=1}^{\infty} R_i$ for every $b \subset B_1$ and every $L \subset B_1$. If we assume that there does not exist $B \subset B_1$ with $[a, B] \subset \bigcap_{i=1}^{\infty} R_i^c$, then by Lemma 3.3 there exist $b \in B_1$ with $b > a$, $B' \subset B_1$ and i_0 such that $[a \cup b, L] \not\subset R_{i_0}^c$ for every $L \subset B'$. Since R_{i_0} is completely Ramsey, there exists $B_2 \subset B'$ with $[a \cup b, B_2] \subset R_{i_0} \subset \bigcup_{i=1}^{\infty} R_i$, a contradiction. \square

From Corollaries 3.2 and 3.4 the next theorem follows.

3.5. Theorem (Galvin-Prikry). *Every Borel subset of $[\mathbf{N}]$ is completely Ramsey.*

Also from Lemma 3.3 we have the following lemma:

3.6. Lemma. *If $\{R_i\}$ is a sequence of completely Ramsey sets and $[a, A] \subset \bigcup_{i=1}^{\infty} R_i$, then there exist $b \subset A$ with $b > a$, $B \subset A$ and i_0 such that $[a \cup b, B] \subset R_{i_0}$.*

Finally the following lemma is an immediate consequence of Lemma 3.1.

3.7. Lemma. *For every $X \subset [\mathbf{N}]$ the set \widehat{X} is completely Ramsey.*

We are now in a position to prove that analytic sets in $[\mathbf{N}]$ are completely Ramsey.

3.8. Theorem (Silver). *Every analytic subset of $[\mathbf{N}]$ is completely Ramsey.*

Proof. Let $X \subset [\mathbf{N}]$ be analytic and let a, A be sets. Also let $f: \mathcal{N} \rightarrow [\mathbf{N}]$ be a continuous function with $X = f(\mathcal{N})$. For every s we set $X_s = f[\mathcal{N}_s]$. Certainly, $X_\emptyset = X$. Then for every s we have

$$(1) \quad X_s = \bigcup_{i=1}^{\infty} X_{\langle s, i \rangle} \quad \text{and} \quad X = \bigcup_{\sigma} \bigcap_j X_{\sigma|j},$$

and since f is continuous

$$X = \bigcup_{\sigma} \bigcap_j \overline{X}_{\sigma|j}.$$

Since $X_s \subset \widehat{X}_s \subset \overline{X}_s$, we also have

$$(2) \quad X = \bigcup_{\sigma} \bigcap_j \widehat{X}_{\sigma|j}.$$

By Lemma 3.7 each \widehat{X}_s is completely Ramsey, so the set

$$\bigcup_s \left(\widehat{X}_s \setminus \bigcup_{i=1}^{\infty} \widehat{X}_{\langle s, i \rangle} \right)$$

is completely Ramsey. If we assume that there exists $B \subset A$ with

$$[a, B] \subset \bigcup_s \left(\widehat{X}_s \setminus \bigcup_{i=1}^{\infty} \widehat{X}_{\langle s, i \rangle} \right),$$

then, by Lemma 3.6, there exist $b \in B$ with $a < b$, $B' \subset B$ and s such that

$$[a \cup b, B'] \subset \widehat{X}_s \setminus \bigcup_{i=1}^{\infty} \widehat{X}_{\langle s, i \rangle},$$

which is impossible because of (1) and the definition of \widehat{X}_s . Therefore there exists $B_1 \subset A$ such that

$$(3) \quad [a, B_1] \cap \left[\bigcup_s \left(\widehat{X}_s \setminus \bigcup_{i=1}^{\infty} \widehat{X}_{\langle s, i \rangle} \right) \right] = \emptyset.$$

Let us assume that there does not exist $B \subset B_1$ with $[a, B] \subset (\widehat{X}_\emptyset)^c = (\widehat{X})^c \subset X^c$. Then there exists $B \subset B_1$ with $[a, B] \subset \widehat{X}_\emptyset$. We claim that $[a, B] \subset X$. Indeed, let $L \in [a, B]$. Then $L \in \widehat{X}_\emptyset$ and by (3) there exists i_1 with $L \in \widehat{X}_{\langle i_1 \rangle}$. Repeating this argument we find i_2 with $L \in \widehat{X}_{\langle i_1, i_2 \rangle}$. Continuing in this way we find a sequence $\sigma = \langle i_1, i_2, \dots \rangle$ with $L \in \bigcap_j \widehat{X}_{\sigma|j} \subset X$, and the proof of the theorem is complete. \square

3.9. Remark. Ellentuck's topology on $[\mathbf{N}]$ is the topology with base the family of sets of the form $[a, B]$. The above lemmas and corollaries can be used to obtain easily the characterization of Ellentuck that a subset of $[\mathbf{N}]$ is completely Ramsey if and only if it has the Baire property in Ellentuck's topology. Also, for every X , the set \widehat{X} is a minimal completely Ramsey cover of X with respect to the σ -ideal (Lemma 3.6) of the Ramsey null sets, i.e. of the completely Ramsey sets Y with the property $\forall [a, A] ([a, A] \cap Y^c \neq \emptyset)$.

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