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THE WEAK CLOSURE OF THE SET OF LEFT TRANSLATION OPERATORS

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Abstract. It is known that for an amenable locally compact group G, 0 is not in the weak closure of $\{\lambda(g) : g \in G\}$ of $VN(G)$. In this paper, it is proved that the converse of this is true. In other words, if G is a non-amenable locally compact group, then 0 is in the weak closure of $\{\lambda(g) : g \in G\}$. This answers several questions of Ülger. Applications to the algebra $C_\delta^*(G)$ and the dual of the reduced group C^*-algebra are obtained.

1. Introduction

Let G be a locally compact group, and $A(G)$ the Fourier algebra of G. An element u in $A(G)$ can be represented as

$$u = f * \check{g},$$

where $f, g \in L^2(G)$, and $\check{g}(s) = g(s^{-1})$ for $s \in G$. The norm of u is given by

$$\|u\| = \inf\{\|f\|_2\|g\|_2; f, g \in L^2(G) \text{ and } u = f * \check{g}\}.$$

Actually, the above infimum can be attained.

For $f \in L^2(G)$ and $x \in G$, the function $\lambda(x)f$ is defined by

$$\lambda(x)f(y) = f(x^{-1}y),\ y \in G.$$

So $\lambda(x)$ is a linear operator on $L^2(G)$ with norm 1. Let $VN(G)$ and $C_\delta^*(G)$ be the von Neumann subalgebra and C^*-subalgebra of $\mathcal{B}(L^2(G))$ generated by $\{\lambda(x) : x \in G\}$, respectively. It turns out that the predual of $VN(G)$ is $A(G)$; see [3].

Let $M(G)$ denote the bounded Borel complex measures on G with convolution as multiplication. Since $\mu * g \in L^2(G)$ for every $\mu \in M(G)$ and $g \in L^2(G)$, one can check that $M(G) \subseteq VN(G)$. In particular, $L^1(G) \subseteq VN(G)$. Let $C_\lambda^*(G)$ be the norm closure of $L^1(G) \subseteq VN(G)$. It is called the reduced C^*-algebra of G. The Banach space dual of $C_\lambda^*(G)$ is identified with a space of continuous functions on G, denoted by $B_\lambda(G)$. The Fourier-Stieltjes algebra $B(G)$ consists of continuous functions u on G with the form

$$u(x) = \langle \pi(x)\xi, \eta\rangle$$

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for some unitary representation π of G on a Hilbert space H and $\xi, \eta \in H$. It is known that $A(G) \subseteq B_\lambda(G) \subseteq B(G)$, and both $A(G)$ and $B_\lambda(G)$ are ideals of $B(G)$. See [3].

Note that the spaces $A(G), B_\lambda(G), C^*_\lambda(G)$ and $VN(G)$ have p-analogues for $1 < p < \infty$, namely, $A_p(G), W_p(G), PF_p(G)$ and $PM_p(G)$; see [2], [5], [9].

In [4], Granirer proved that for an amenable locally compact group G, only finite subsets of $\{\lambda(x) : x \in G\}$ can be relatively weak compact in $VN(G)$. In fact, it is a consequence of the following observation: if G is amenable, then $0 \notin w - cl\{\lambda(x) : x \in G\}$. The proof of the latter was also included in [4]. Actually, these results and some interesting applications of them (see [4], [6]) were proved in the general setting of their p-analogues.

On the other hand, for an infinite free set E of the free group F_2 on two generators, $l^2(E^k)$ can be embedded into $VN(F_2), k = 1, 2, \ldots$; see [8], [10]. In particular, for each $k, \{\lambda(g); g \in E^k\}$ is a relatively weak compact set in $VN(F_2)$ and it is easy to see that $0 \in w - cl\{\lambda(g) : g \in E^k\}$. To our knowledge, there is no p-version of this in the literature.

The purpose of this paper is to establish a characterization of amenability based on the above discussion. More precisely, in Theorem 2.7 we show that if G is a non-amenable locally compact group, then $0 \in w - cl\{\lambda(x) : x \in G\}$.

In a recent paper [13], Ülger made an intensive study of the weak topology of the spectrum of commutative Banach algebras. He conjectured that G (which is the spectrum of $A(G)$; see [3]) is weakly closed in $VN(G)$ if and only if G is amenable. He also asked what would be the situation for $PM_p(G)$ with $1 < p < \infty$. Notice that G is weakly closed in $PM_p(G)$ if and only if $0 \notin w - cl\{\lambda(x) : x \in G\}$; see [4], [13]. So Theorem 2.7 and its p-version solve all the questions mentioned above.

With Theorem 2.7, we can get an improvement to the following result of Bédos [1, Theorem 1]: G is amenable if and only if there exists a nonzero multiplicative linear functional on $C^*_\delta(G)$, the C^*-algebra generated by $\{\lambda(x) : x \in G\}$. We are able to show in Theorem 2.9 that: G is amenable if and only if there exists a linear functional ϕ on $C^*_\delta(G)$, such that $\inf\{|\langle\phi, \lambda(x)\rangle| : x \in G\} > 0$. Note that the "if" part is the interesting part.

It is well known that a locally compact group G is amenable if and only if $B(G) = B_\lambda(G)$. This is equivalent to saying that $1_G \in B_\lambda(G)$, where 1_G is the constant 1 function on G. Finally, as an application of Theorem 2.7 we prove that if G is a non-amenable group, then there is a net $\{x_\alpha\}$ in G such that

$$u(x_\alpha) \to 0$$

for every $u \in B_\lambda(G)$. This is formally stronger than that of $1_G \notin B_\lambda(G)$.

Every result in this paper has a p-version.

2. The main results

We will begin with some discussions on the relatively weak compact sets of $\{\lambda(x) : x \in G\}$.

The first result is contained in the proof of Lemma 1.2 of Granirer [4]. We give a different proof here.

Lemma 2.1. *Let G be a locally compact group. If there exists an infinite set $S \subseteq G$ such that $\{\lambda(x) : x \in S\}$ is relatively weak compact in $VN(G)$, then $0 \in w-cl\{\lambda(x) : x \in G\}$.*

Proof. Choose a sequence of distinct elements $\{x_n\}$ of S such that $\lambda(x_n) \to T \in VN(G)$ in weak topology.

If $\{x_n\}$ has a subsequence (which will again be denoted by $\{x_n\}$) such that $x_n \to x_0 \in G$, then $\lambda(x_n) \to \lambda(x_0)$ in w^*-topology (i.e., $u(x_n) \to u(x_0)$ for each $u \in A(G)$). So, $T = \lambda(x_0)$. Since $\lambda(x_n) \to \lambda(x_0)$ weakly, there is a convex combination of some $\lambda(x_{n_1}), \ldots, \lambda(x_{n_k})$, say $\sum_{i=1}^{k} c_i\lambda(x_{n_i})$, such that

$$\|\lambda(x_0) - \sum_{i=1}^{k} c_i\lambda(x_{n_i})\| < 1.$$

Since there exists $u \in A(G)$ such that $u(x_0) = 1, u(x_{n_i}) = 0, i = 1, \ldots, k$, and $\|u\| = 1$ (so $\langle\lambda(x_0) - \sum_{i=1}^{k} c_i\lambda(x_{n_i}), u\rangle = 1$), we have reached a contradiction.

So $x_n \to \infty$. Now for any $u \in A(G)$,

$$\langle T, u\rangle = \lim_{n\to\infty}\langle\lambda(x_n), u\rangle = \lim_{n\to\infty} u(x_n) = 0.$$

So we conclude that $T = 0$. □

Remark. We avoided the use of the facts that (i) $SpecA(G) = G$ and (ii) $VN(G)$ has a topological invariant mean.

As usual, we denote by coE the convex hull of a subset E of a linear space.

Lemma 2.2. *If G is amenable, then for any $T \in co\{\lambda(x) : x \in G\}, \|T\| = 1$.*

Proof. Theorem 3.2.2 of Greenleaf [7] states that if $\mu \in M(G), \mu \geq 0$, then $\|\lambda(\mu)\| = \|\mu\|$. Here if $T = \sum_{i=1}^{n} c_i\lambda(x_i)$, then $T = \lambda(\mu)$ where $\mu = \sum_{i=1}^{n} c_i\delta_{x_i}$. □

The next Corollary is essentially Lemma 1.2 of Granirer [4].

Corollary 2.3. *If G is amenable, then*
(i) $0 \notin w - cl\{\lambda(x) : x \in G\}$.
(ii) Only finite subsets of $\{\lambda(x) : x \in G\}$ can be relatively weak compact.

Proof. If $0 \in w - cl\{\lambda(x); x \in G\}$, then 0 is in the norm closed convex hull of $\{\lambda(x) : x \in G\}$. This contradicts Lemma 2.2. That proves part (i). Part (ii) follows from Lemma 2.1. □

One of our main results is to show that the converse of (i) of Corollary 2.3 is true. We begin with a construction of a linear functional on $VN(G)$ that is positive and away from 0 on the set $\{\lambda(x) : x \in G\}$.

Proposition 2.4. *If $0 \notin w - cl\{\lambda(x) : x \in G\}$, then there exist an $\epsilon_0 > 0$ and $\phi \in VN(G)^*$ with $\|\phi\| \leq 1$ such that $\langle\phi, \lambda(x)\rangle \geq \epsilon_0$ for every $x \in G$.*

Proof. Suppose that $0 \notin w - cl\{\lambda(x) : x \in G\}$. Then there exist $\phi_1, \ldots, \phi_k \in VN(G)^*$, all are of norm 1, and an $\epsilon > 0$ such that

$$\{\lambda(x) : x \in G\} \cap \left(\bigcap_{i=1}^{k}\{F \in VN(G) : |\langle\phi_i, F\rangle| < \epsilon\}\right) = \emptyset.$$

If for each i we denote

$$P_i = \{x \in G : |\langle \phi_i, \lambda(x)\rangle| \geq \epsilon\},$$

then $G = \bigcup_{i=1}^k P_i$. This gives that

$$\sum_{i=1}^{k} |\langle \phi_i, \lambda(x)\rangle| \geq \epsilon$$

for every $x \in G$.

By Goldstine's theorem, we can choose a net $\{u_{\alpha,i}\}$ in $A(G)$ such that $u_{\alpha,i} \xrightarrow{w^*} \phi_i$ and $\|u_{\alpha,i}\| \leq 1, i = 1, \ldots, k$. Assume that

$$u_{\alpha,i} = f_{\alpha,i} * \check{g_{\alpha,i}}$$

for some $f_{\alpha,i}, g_{\alpha,i} \in L^2(G)$ and $\|f_{\alpha,i}\|_2 \|g_{\alpha,i}\|_2 \leq 1$.

For each $1 \leq i \leq k$, let $v_{\alpha,i} = |f_{\alpha,i}| * \check{|g_{\alpha,i}|}$. Then $v_{\alpha,i} \in A(G)$ and $\|v_{\alpha,i}\| \leq 1$. By taking subnets if necessary, we assume that for each i, $v_{\alpha,i} \xrightarrow{w^*} \tilde{\phi}_i$ for some $\tilde{\phi}_i \in VN(G)^*$. Evidently, $\|\tilde{\phi}_i\| \leq 1$. For every $x \in G$, we have

$$\begin{aligned} |\langle \phi_i, \lambda(x)\rangle| &= \lim_{\alpha} |\langle \lambda(x), u_{\alpha,i}\rangle| \\ &= \lim_{\alpha} |u_{\alpha,i}(x)| \\ &\leq \lim_{\alpha} v_{\alpha,i}(x) \\ &= \lim_{\alpha} \langle \lambda(x), v_{\alpha,i}\rangle \\ &= \langle \tilde{\phi}_i, \lambda(x)\rangle, \end{aligned}$$

and hence

$$\sum_{i=1}^{k} \langle \tilde{\phi}_i, \lambda(x)\rangle \ \geq \epsilon.$$

The proposition is proved if we set $\phi = \frac{1}{k}\sum_{i=1}^k \tilde{\phi}_i$ and $\epsilon_0 = \frac{1}{k}\epsilon$. □

Lemma 2.5. *If* $0 \notin w - cl\{\lambda(x) : x \in G\}$, *then for any* $T = \sum_{i=1}^n c_i \lambda(x_i)$ *in* $co\{\lambda(x) : x \in G\}, \|T\| = 1$.

Proof. Let ϕ and ϵ_0 be as in Proposition 2.4. For any $T = \sum_{i=1}^n c_i \lambda(x_i)$ in $co\{\lambda(x) : x \in G\}$,

$$\begin{aligned} \|T\| &\geq \langle \phi, T\rangle \\ &= \sum_{i=1}^{n} c_i \langle \phi, \lambda(x_i)\rangle \geq \epsilon_0. \end{aligned}$$

Notice that $T^k \in co\{\lambda(x) : x \in G\}$ if $T \in co\{\lambda(x) : x \in G\}$ and k is a positive integer. So,

$$\epsilon_0 \leq \|T^k\| \leq \|T\|^k.$$

And hence $\|T\| = 1$. □

The following is Proposition 9.8 of [12].

Lemma 2.6. *If* $\|T\| = 1$ *for all* $T \in co\{\lambda(x) : x \in G\}$, *then* G *is amenable.*

Now, we are in a position to state our first main result.

Theorem 2.7. *If G is a non-amenable group, then $0 \in w - cl\{\lambda(x) : x \in G\}$ in $VN(G)$.*

Proof. It follows directly from Lemma 2.5 and Lemma 2.6. □

Remark. The methods described above are adoptable to the case $1 < p < \infty$. Therefore, the above theorem has a p-analogue.

It is known that G is the spectrum of the algebra $A_p(G)$ $(1 < p < \infty)$; see [3], [9]. Ülger conjectured that group G is amenable if and only if the spectrum G of $A(G)$ is weakly closed in $VN(G)$, this is the question (d)-(ii) of [13]. The other two related questions in [13] are:

(e)-(i) If G is weakly closed in $VN(G)$, is G weakly closed in $A_p(G)^*(= PM_p(G))$ for all p with $1 < p < \infty$?

In the case where G is weakly closed in $A_p(G)^*$,

(e)-(ii) Is the closedness of G in $(A_p(G)^*, weak)$ an intrinsic property of the algebra $A_p(G)$?

As observed in [4], [13], G is weakly closed in $PM_p(G)$ is equivalent to saying that $0 \notin w - cl\{\lambda(x) : x \in G\}$ in $PM_p(G)$. Thus, Theorem 2.7 and its p-analogue answer all three questions positively. In other words, we have

Corollary 2.8. *G is amenable if and only if G is weakly closed in $PM_p(G), 1 < p < \infty$.*

A characterization of amenable locally compact groups due to Bédos is stated as: G is amenable if and only if there exists a nonzero multiplicative linear functional on $C^*_\delta(G)$; see [1, Theorem 1]. Our following theorem improves Bédos' result.

Theorem 2.9. *G is amenable if and only if there exists a linear functional ϕ on $C^*_\delta(G)$ and a $\delta > 0$, such that $|\langle \phi, \lambda(x) \rangle| \geq \delta$ for all $x \in G$.*

Proof. If G is amenable, then there exists a bounded approximate identity $\{u_\alpha\}$ in $A(G)$. Let ϕ be the restriction of a w^*-limit of $\{u_\alpha\}$ in $VN(G)^*$. Then this ϕ satisfies the requirement.

Conversely, suppose there exists a linear functional ϕ on $C^*_\delta(G)$ with the condition in the statement of the Theorem. Let $\tilde{\phi}$ be an extension of ϕ to $VN(G)$. If G is not amenable, then by Theorem 2.7, there must be a net $\{x_\alpha\}$ of G such that

$$\lambda(x_\alpha) \to 0$$

weakly in $VN(G)$. In particular,

$$\langle \phi, \lambda(x_\alpha) \rangle = \langle \tilde{\phi}, \lambda(x_\alpha) \rangle \to 0.$$

This is a contradiction. □

Let $C^*_\lambda(G)$ be the reduced group C^*-algebra of G and $B_\lambda(G)$ its dual. It is well known that G is amenable if and only if the constant 1 function is in $B_\lambda(G)$. Using Theorem 2.7 and a result of Granirer, we are able to obtain a formally stronger result in characterizing the amenability of locally compact groups.

Theorem 2.10. *Let G be a locally compact group. Then the following are equivalent:*

(i) G is non-amenable.

(ii) There exists a net $\{x_\alpha\}$ of G such that

$$u(x_\alpha) \to 0$$

for each $u \in B_\lambda(G)$.

Proof. By Theorem 2.7, there exists a net $\{x_\alpha\}$ of G such that

$$\lambda(x_\alpha) \to 0$$

weakly in $VN(G)$, if G is non-amenable.

Lemma 1.1 of Granirer [4] states that if $u \in B_\lambda(G)$, then

$$\mu \longmapsto \langle u, \mu \rangle = \int_G u(x) d\mu(x)$$

is a continuous linear functional on the normed linear space $(M(G), \| \cdot \|_{VN})$. So,

$$u(x_\alpha) \to 0$$

as desired.

Conversely, (ii) implies that the constant 1 function does not belong to $B_\lambda(G)$, and hence G cannot be amenable. □

Remark. Both Theorem 2.9 and 2.10 have p-versions.

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