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THE CENTRAL LIMIT THEOREM FOR LACUNARY SERIES

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ABSTRACT. In this paper, the central limit theorem for lacunary trigonometric series is proved. Two gap conditions by Erdős and Takahashi are extended and unified. The criterion for the Fourier character of lacunary series is also given.

1. INTRODUCTION

It is well known that the lacunary trigonometric series $\sum a_i \cos(2\pi n_i\omega + \phi_i)$ behaves like random series when $\{n_i\}$ increases very fast. For example, if $\{n_i\}$ has Hadamard gaps, i.e. $n_{i+1}/n_i > q > 1$, the series converges or diverges almost everywhere according as $\sum a_i^2$ converges or diverges. (Kolmogorov [3] and Zygmund [10].) It is also known that the series is not a Fourier series of integrable function when $\sum a_i^2 = \infty$. (Zygmund [10].)

As to the central limit theorem for the series with Hadamard gaps, Salem-Zygmund [4] proved: If $A_n = \left(\frac{1}{2}\sum_{i=1}^{n} a_i^2\right)^{1/2} \to \infty$ and $a_n = o(A_n)$ are satisfied, then

$$\frac{1}{A_n}\sum_{i=1}^{n} a_i \cos(2\pi n_i\omega + \phi_i) \xrightarrow{\mathcal{D}} N_{0,1} \tag{1.1}$$

holds on probability space $\left(\Omega, d\omega/|\Omega|\right)$, when $\Omega \subset [\,0,1\,]$ has positive measure. Here $|\cdot|$ denotes Lebesgue measure, $N_{0,1}$ the standard normal distribution, and $\xrightarrow{\mathcal{D}}$ convergence in law.

Erdős [1] relaxed the gap condition to

$$n_{i+1}/n_i > 1 + c_i\big/\sqrt{i} \quad \text{where} \quad c_i \to \infty, \tag{1.2}$$

and proved (1.1) for $a_n \equiv 1$. Takahashi [6] proved that $a_n \equiv 1$ can be relaxed to

$$A_n \to \infty \quad \text{and} \quad a_n = O\big(A_n\big/\sqrt{n}\,\big). \tag{1.3}$$

Takahashi [7] also proved (1.1) assuming

$$n_{i+1}/n_i > 1 + c/i^{\alpha}, \quad A_n \to \infty \quad \text{and} \quad a_n = o(A_n/n^{\alpha}), \tag{1.4}$$

where $c > 0$ and $0 \le \alpha \le 1/2$. It should be noted that there is no implication between (1.2) and (1.3), and (1.4). Indeed, if we put $\alpha = 1/2$, the gap condition

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of (1.4) is weaker than (1.2), but if we put $a_n \equiv 1$, we must put $\alpha < 1/2$ in (1.4), which is stronger than (1.2).

From (1.1), by the way, we can deduce that the series is not a Fourier series. Therefore, the series is not a Fourier series under (1.4). Takahashi [8] proved that this claim remains valid even if we relax $a_n = o(A_n/n^\alpha)$ to $a_n = O(A_n/n^\alpha)$. Under this condition, (1.1) does not hold generally. A counterexample was constructed by Takahashi [9]. Previously, Erdős [1] had noted the existence of such an example for $\alpha = 1/2$.

Although these results have been considered to be best possible, we still have the following examples excluded from the above scheme: Under the conditions

$$n_{i+1}/n_i > 1 + c/\sqrt{i \log i}, \quad A_n \to \infty \quad \text{and} \quad a_n = o\big(A_n/\sqrt{n \log n}\,\big),$$

the central limit theorem (1.1) holds. Even if we relax the last condition to $a_n = O\big(A_n/\sqrt{n \log n}\,\big)$, the series is not a Fourier series, but there are counterexamples for (1.1).

In this note we introduce a more general gap condition and prove theorems including all the above results and examples.

Theorem 1. *Let us suppose the following conditions:*

$$\lambda(i) > \lambda \quad \textit{for some} \quad \lambda > 0; \tag{1.5}$$

$$\lambda(i+1) - \lambda(i) = o(1); \tag{1.6}$$

$$n_{i+1}/n_i > 1 + c/\lambda(i) \quad \textit{for some} \quad c > 0; \tag{1.7}$$

$$a_n = o\big(A_n/\lambda(n)\big) \quad \textit{and} \quad A_n \to \infty. \tag{1.8}$$

Then (1.1) holds on any $\Omega \subset [\,0,1\,]$ with positive measure.

In Theorem 1, the next condition is implicitly assumed:

$$\sum_{i=1}^{\infty} \frac{1}{\lambda^2(i)} = \infty. \tag{1.9}$$

Or, more precisely, the existence of $\{a_i\}$ satisfying (1.8) is equivalent to (1.9). Indeed, if (1.9) is false, the contradiction $A_n^2 = O\big(\frac{1}{2}\sum_{i=1}^n A_i^2/\lambda^2(i)\big) = o(A_n^2)$ follows. In case (1.9) is valid, we can construct $\{a_i\}$ satisfying (1.8), by putting $A_n = \exp\big(\frac{1}{2}\sum_{i=1}^n e_i\big)$ and $a_n^2 = (A_n^2 - A_{n-1}^2)/2$, where $\{e_i\}$ satisfies $0 < e_i = o\big(1/\lambda^2(i)\big)$ and $\sum_{i=1}^\infty e_i = \infty$.

Although Theorem 1 contains assumptions that generalize (1.4), we can derive the next corollary which assumes the generalization of (1.2) and (1.3). The condition (1.9) is again implicitly assumed for $\{\Lambda(i)\}$.

Corollary. *Let us suppose the following conditions:*

$$\Lambda(i) > 0 \quad \textit{and} \quad \Lambda(i) \to \infty;$$
$$\Lambda(i+1) - \Lambda(i) = O(1);$$
$$n_{i+1}/n_i > 1 + c_i/\Lambda(i) \quad \textit{where} \quad c_i \to \infty;$$
$$a_n = O\big(A_n/\Lambda(n)\big) \quad \textit{and} \quad A_n \to \infty.$$

Then (1.1) holds on any $\Omega \subset [\,0,1\,]$ with positive measure.

Actually, we will prove that the assumption of Theorem 1 is equivalent to that of the Corollary. Thus, our result extends and unifies previous results.

Since $a_n = o(A_n)$ is necessary for (1.1) (Salem-Zygmund [4]), if we assume (1.8), the condition (1.5) is indispensable for Theorem 1.

The next theorem asserts that (1.8) cannot be weakened.

Theorem 2. *Suppose that* $\{\lambda(i)\}$ *satisfies* $\lambda(i) > 0$*, (1.6) and (1.9). Then there exist sequences* $\{n_i\}$ *and* $\{a_i\}$ *satisfying (1.7),*

$$a_n = O\big(A_n/\lambda(n)\big) \quad \text{and} \quad A_n \to \infty, \tag{1.10}$$

such that the central limit theorem (1.1) does not hold on $\Omega = [\,0, 1\,]$.

Finally, we state a result on the Fourier character of the series. Since (1.11) is always true under Hadamard's gap condition, it includes that of Zygmund [10].

Theorem 3. *Let us assume (1.5), (1.6), (1.7), (1.9),*

$$a_n = O\big(A_n \ell(A_n)/\lambda(n)\big) \quad \text{and} \quad A_n \to \infty, \tag{1.11}$$

where $\ell(x) = \sqrt{\log x \log\log x \ldots \log^{(\gamma)} x}$ *and* $\gamma \in \mathbf{N}$*. Then the series diverges almost surely and is neither a Fourier series nor a Fourier-Stieltjes series.*

Before closing the introduction, we note that the same results for lacunary Walsh series can be proved in the same way.

2. The central limit theorem

Let us put $n_0 = 1$ and $\lambda(0) = 2\lambda$, and introduce the following notation:

$$\begin{aligned} &p(0) = 0, \qquad p(k) = \max\{\, i \mid n_i \le 2^k \,\}\ (k \ge 1), \qquad l(k) = p(k+1) - p(k), \\ &P(k) = \mathbf{N} \cap \big(p(k), p(k+1)\big], \qquad \mu(k) = \max_{i \in P(k)} \lambda(i), \qquad \nu(k) = \min_{i \in P(k)} \lambda(i). \end{aligned}$$

Since $\{n_i\}$ diverges to infinity, $\{p(k)\}$ does also. If $p(k) + 1 < p(k+1)$, we have

$$2 > \frac{n_{p(k+1)}}{n_{p(k)+1}} > \prod_{i=p(k)+1}^{p(k+1)-1} \left(1 + \frac{c}{\lambda(i)}\right) > 1 + \sum_{i=p(k)+1}^{p(k+1)-1} \frac{c}{\lambda(i)} > 1 + c\frac{l(k) - 1}{\mu(k)}.$$

From this and $\mu(k) > \lambda$, it follows that

$$l(k) = O\big(\mu(k)\big). \tag{2.1}$$

By (1.6), we have $\lambda(i) = o(i)$ and $\mu(k) = o\big(p(k+1)\big)$. Applying this to (2.1) we get

$$p(k+1) \sim p(k). \tag{2.2}$$

Applying (1.6) and (2.1), we have

$$0 \le \frac{\mu(k) - \nu(k)}{\mu(k)} \le \sum_{i=p(k)}^{p(k+1)-1} \frac{\big|\lambda(i+1) - \lambda(i)\big|}{\mu(k)} = o\left(\frac{l(k)}{\mu(k)}\right) = o(1).$$

This implies $\mu(k) \sim \nu(k)$, and hence $\mu(k) \sim \lambda\big(p(k{+}1)\big)$ and $\mu(k{+}1) \sim \lambda\big(p(k{+}1){+}1\big)$ follow. Since $\lambda(i+1) \sim \lambda(i)$ is clear from (1.5) and (1.6), we have

$$\mu(k+1) \sim \mu(k). \tag{2.3}$$

The next lemma plays an important role in the proof.

Lemma 1. *For any given integers j, k, h, and q satisfying*

$$j < k \quad \text{and} \quad p(j) + 1 < h \le p(j+1) < p(k) + 1 < q \le p(k+1),$$

the number of solutions (n_r, n_i) of the equation

$$n_q - n_r = n_h - n_i \quad \text{where} \quad p(j) < i < h \quad \text{and} \quad p(k) < r < q,$$

is at most $2^{j-k+1}\mu(k)/c$.

Proof. If (n_r, n_i) is a solution, we have

$$n_r = n_q - (n_h - n_i) > n_q - 2^j > n_q(1 - 2^{j-k}) \ge n_q(1 + 2^{j-k+1})^{-1}.$$

Let us denote the least (or greatest) index of n_r's by m_1 (or m_2). Dividing $n_q \ge n_{m_2+1}$ by $n_q(1 + 2^{j-k+1})^{-1} \le n_{m_1}$, we have

$$1 + 2^{j-k+1} \ge \frac{n_{m_2+1}}{n_{m_1}} \ge \prod_{m=m_1}^{m_2}\Big(1 + \frac{c}{\lambda(m)}\Big) \ge 1 + \frac{c(m_2 - m_1 + 1)}{\mu(k)}.$$

□

The next lemma can be proved in the same way.

Lemma 2. *For any given integers j, k, h, and q satisfying*

$$j + 1 < k \quad \text{and} \quad p(j+1) < h \le p(j+2) < p(k+1) < q \le p(k+2),$$

the number of solutions (n_r, n_i) of the equation

$$n_q - n_r = n_h - n_i \quad \text{where} \quad p(j) < i \le p(j+1) \quad \text{and} \quad p(k) < r \le p(k+1),$$

is at most $2^{j-k+2}\mu(k)/c$.

In the proof of Theorem 1, we assume $\phi_i = 0$ to simplify notation. The general case can be proved in the same way. We apply the following result ([5]).

Theorem A. *Let $\{d_i\}$ be a sequence of real numbers and put*

$$f_n(\omega) = \sum_{i=1}^{n} d_i \cos 2\pi i\omega, \quad \Delta_k = f_{2^{k+1}} - f_{2^k}, \quad D_n = \left(\frac{1}{2}\sum_{i=1}^{n} d_i^2\right)^{1/2},$$

and $B_k = D_{2^{k+1}}$. Suppose that the following conditions are satisfied:

$$\int_0^1 \left| \frac{1}{B_n^2}\sum_{k=1}^{n}\big(\Delta_k^2(\omega) + 2\Delta_k(\omega)\Delta_{k+1}(\omega)\big) - 1 \right| d\omega \to 0, \tag{2.4}$$

$$B_n \to \infty \quad \text{and} \quad \sup_{\omega\in[0,1]}\big|\Delta_n(\omega)\big| = o(B_n). \tag{2.5}$$

Then for any $\Omega \subset [0,1]$ with positive measure, the law of f_n/D_n on $\big(\Omega, d\omega/|\Omega|\big)$ converges weakly to the standard normal distribution.

We apply this by putting $\Delta_k(\omega) = \sum_{i\in P(k)} a_i \cos 2\pi n_i\omega$ and $B_k = A_{p(k+1)}$. Let us put $C_k = \|\Delta_k\|$ where $\|\cdot\|$ denotes $L^2[0,1]$-norm. Obviously we have $C_k^2 = B_k^2 - B_{k-1}^2$. By (1.8) and (2.1), we have

$$\sup_{\omega\in[0,1]}\big|\Delta_k(\omega)\big| \le \sum_{i\in P(k)} |a_i| \le l(k)\max_{i\in P(k)} |a_i| = o\left(\mu(k)\frac{B_k}{\nu(k)}\right) = o(B_k). \tag{2.6}$$

This implies $C_k = o(B_k)$ and $B_{k+1}^2 / B_k^2 = \left(1 - C_{k+1}^2 / B_{k+1}^2\right)^{-1} = 1 + o(1)$. The next estimate follows from the Schwarz inequality:

$$\sum_{q \in P(k)} |a_q| \leq l^{1/2}(k) C_k = O\big(\mu^{1/2}(k) C_k\big). \tag{2.7}$$

To prove (2.4), we divide Δ_k^2 into three parts; putting

$$\begin{aligned} U_k &= \frac{1}{2} \sum_{q \in P(k)} a_q \sum_{r \in P(k)} a_r \cos 2\pi(n_q + n_r)\omega \quad \text{and} \\ V_k &= \sum_{p(k) < r < q \leq p(k+1)} a_q a_r \cos 2\pi(n_q - n_r)\omega, \end{aligned}$$

we have $\Delta_k^2 - C_k^2 = U_k + V_k$. From (2.6), it follows that

$$\|U_k\| \leq \frac{1}{2} \sum_{q \in P(k)} |a_q| C_k = o(B_k C_k) \quad \text{and} \quad \|V_k\| = o(B_k C_k).$$

Since $\{U_k\}$ is an orthogonal sequence, we have

$$\Big\| \sum_{k=1}^n U_k \Big\|^2 = \sum_{k=1}^n \big\|U_k\big\|^2 = o\bigg(B_k^2 \sum_{k=1}^n C_k^2\bigg) = o(B_k^4).$$

Noting this and $\big\|\sum (U_k + V_k)\big\|^2 \leq 2\big\|\sum U_k\big\|^2 + 2\big\|\sum V_k\big\|^2$, we have

$$\Big\| \sum_{k=1}^n \big(\Delta_k^2 - C_k^2\big) \Big\|^2 = o(B_k^4) + 4 \sum_{1 \leq j < k \leq n} \int_0^1 V_k(\omega) V_j(\omega)\, d\omega.$$

In a similar way, we can prove

$$\Big\| \sum_{k=1}^n \Delta_k \Delta_{k+1} \Big\|^2 = o(B_k^4) + 4 \sum_{1 \leq j < k \leq n} \int_0^1 W_k(\omega) W_j(\omega)\, d\omega,$$

where $W_k = \sum_{q \in P(k+1)} a_q \sum_{r \in P(k)} a_r \cos 2\pi(n_q - n_r)\omega$. By Lemma 1, (1.8) and (2.7), we have

$$\begin{aligned} \left| \int_0^1 V_k(\omega) V_j(\omega)\, d\omega \right| &\leq \sum_{q \in P(k)} |a_q| \sum_{h \in P(j)} |a_h| \max_{r \in P(k)} |a_r| \max_{i \in P(j)} |a_i| \frac{2^{j-k+1} \mu(k)}{c} \\ &= o\big(B_k B_j \mu^{1/2}(k) \mu^{-1/2}(j) 2^{j-k} C_k C_j\big). \end{aligned}$$

Because of (2.3), for large enough M, we have $\mu(k)/\mu(j) \leq M2^{k-j}$. Hence we have

$$\begin{aligned}
\sum_{1\leq j<k\leq n} \int_0^1 V_k(\omega)V_j(\omega)\,d\omega &= o(B_n^2)\sum_{k=2}^{n} C_k \sum_{j=1}^{k-1} \sqrt{2}^{j-k} C_j \\
&= o(B_n^2)\sum_{k=2}^{n} C_k \Bigl(\sum_{j=1}^{k-1} \sqrt{2}^{j-k} C_j^2\Bigr)^{1/2} \Bigl(\sum_{j=1}^{k-1} \sqrt{2}^{j-k}\Bigr)^{1/2} \\
&= o(B_n^2)\Bigl(\sum_{k=2}^{n} C_k^2\Bigr)^{1/2} \Bigl(\sum_{k=2}^{n}\sum_{j=1}^{k-1} \sqrt{2}^{j-k} C_j^2\Bigr)^{1/2} \\
&= o(B_n^3)\Bigl(\sum_{j=1}^{n-1} C_j^2 \sum_{k=j+1}^{n} \sqrt{2}^{j-k}\Bigr)^{1/2} \\
&= o(B_n^4).
\end{aligned}$$

Similarly we have $\sum_{1\leq j<k\leq n}\int_0^1 W_k(\omega)W_j(\omega)\,d\omega = o(B_n^4)$. These estimates yield

$$\Bigl\|\sum_{k=1}^{n}\bigl(\Delta_k^2 - C_k^2\bigr)\Bigr\| = o(B_n^2) \quad \text{and} \quad \Bigl\|\sum_{k=1}^{n}\Delta_k\Delta_{k+1}\Bigr\| = o(B_n^2),$$

which imply (2.4).

Next we prove the Corollary. Let us put $\rho(i) = 1/\bigl(n_{i+1}/n_i - 1\bigr)$. Let $\Delta x(i)$ denote $x(i+1) - x(i)$.

We now assume $0 < \Lambda(i) \to \infty$, $\Delta\Lambda(i) = O(1)$, $\rho(i) = o\bigl(\Lambda(i)\bigr)$ and $\rho(i) \leq \Lambda(i)$, and hereafter construct $\lambda(i)$ which satisfies $2\lambda \vee \rho(i) \leq \lambda(i)$, $\lambda(i) = o\bigl(\Lambda(i)\bigr)$ and $\Delta\lambda(i) = o(1)$. The conditions of Theorem 1 are clearly derived from these. Put $\lambda = \frac{1}{2}\inf_i \Lambda(i)$. Let us first construct sequences $1 = i_0 < i_1 < i_2 < \cdots$ and $\{\Lambda_0(i)\}$, $\{\Lambda_1(i)\}$, $\{\Lambda_2(i)\}$, ... such that

$$\rho(i) = o\bigl(\Lambda_n(i)\bigr) \quad \text{and} \quad \Lambda_n(i) \to \infty \quad (i \to \infty,\ n \geq 0), \tag{2.8}$$

$$2\lambda \vee \rho(i) \leq \Lambda_n(i) \leq \Lambda_{n-1}(i) \text{ and } \Delta\Lambda_n(i) = \frac{1}{2}\Delta\Lambda_{n-1}(i) \quad (n \geq 1,\ i \geq i_n), \tag{2.9}$$

$$\Lambda_{n-1}(i) \leq \frac{2}{3}\Lambda_{n-2}(i) \quad (n \geq 2,\ i \geq i_n). \tag{2.10}$$

These sequences are constructed inductively in n. First we put $i_0 = 1$ and $\Lambda_0(i) = \Lambda(i)$. It is clear that (2.8) is satisfied for $n = 0$. After i_{n-1} and $\{\Lambda_{n-1}(i)\}$ have been constructed, we define i_n and $\{\Lambda_n(i)\}$ as follows: We can take $j > i_{n-1}$ such that

$$\rho(i) \leq \Lambda_{n-1}(i)/2 \quad \text{and} \quad \Lambda_{n-2}(i_{n-1}) \leq \Lambda_{n-2}(i)/3 \quad (i \geq j). \tag{2.11}$$

(The second condition must be omitted in case $n = 1$.) Let us take $i_n \geq j$ such that $\Lambda_{n-1}(i_n) = \min_{i\geq j}\Lambda_{n-1}(i)$ holds, and define $\Lambda_n(i)$ by

$$\Lambda_n(i) = \begin{cases} \Lambda_{n-1}(i), & i < i_n, \\ \bigl(\Lambda_{n-1}(i) + \Lambda_{n-1}(i_n)\bigr)/2, & i \geq i_n. \end{cases}$$

By definition, (2.11) holds if we put $j = i_n$, and $\Lambda_{n-1}(i) \geq \Lambda_n(i_n)$ holds for all $i \geq i_n$. Therefore $\Lambda_n(i) \leq \Lambda_{n-1}(i)$ holds for $i \geq i_n$, and the rest of (2.9) is clear from definition and the first inequality of (2.11). (2.10) follows from the last inequality of (2.11). By definition, (2.8) is clear, and the sequences are well constructed.

If we put $\lambda(i) = \Lambda_n(i)$ for $i_n \le i < i_{n+1}$, it satisfies $2\lambda \vee \rho(i) \le \lambda(i)$, $\Delta\lambda(i) = \Delta\Lambda_n(i) = \left(\frac{1}{2}\right)^n \Delta\Lambda(i) = o(1)$ and $\lambda(i) = \Lambda_n(i) \le \Lambda_{n-1}(i) \le \left(\frac{2}{3}\right)^{n-1}\Lambda(i) = o\big(\Lambda(i)\big)$.

Finally, we derive Theorem 1 from the Corollary. By this we see that Theorem 1 and the Corollary are equivalent. We now assume (1.5), (1.6), (1.7) or $\rho(i) = O\big(\lambda(i)\big)$, and (1.8), and derive the conditions of the Corollary. Conditions (1.5) and (1.8) imply $a_i/A_i \to 0$. We can therefore take an increasing sequence $\widetilde{\lambda}(i)$ of positive numbers such that $\widetilde{\lambda}(i)a_i/A_i \to 0$ and $\Delta\widetilde{\lambda}(i) = o(1)$. If we put $\lambda_0(i) = \lambda(i) + \widetilde{\lambda}(i)$, we have $0 < \lambda_0(i) \to \infty$, $\Delta\lambda_0(i) = o(1)$, $\rho(i) = O\big(\lambda_0(i)\big)$ and $\lambda_0(i)a_i/A_i \to 0$. Next we construct $1 = i_0 < i_1 < i_2 < \cdots$ and $\{\lambda_1(i)\}$, $\{\lambda_2(i)\}$, $\ldots$ such that $\lambda_n(i) \ge \lambda_{n-1}(i)$,

$$0 < \lambda_n(i) \to \infty, \quad \Delta\lambda_n(i) = o(1), \quad \text{and} \quad \lambda_n(i)a_i/A_i \to 0 \quad (i \to \infty, n \ge 0),$$

$$|\Delta\lambda_n(i)| \le 1,\ \Delta\lambda_n(i) = 2\Delta\lambda_{n-1}(i),\ \left|\frac{\lambda_n(i)a_i}{A_i}\right| \le 1\ (i \ge i_n, n \ge 1),$$

$$\lambda_{n-1}(i) \ge \frac{3}{2}\lambda_{n-2}(i) \quad (i \ge i_n, n \ge 2).$$

It can be achieved first by taking i_{n+1} to satisfy $\lambda_n(i_{n+1}) = \min_{i \ge i_{n+1}} \lambda_n(i)$,

$$\left|\frac{\lambda_n(i)a_i}{A_i}\right| \le \frac{1}{2}, \quad |\Delta\lambda_n(i)| \le \frac{1}{2} \quad \text{and} \quad \lambda_{n-1}(i) \ge 2\lambda_{n-1}(i_n) \quad (i \ge i_{n+1}),$$

and then putting $\lambda_{n+1}(i) = 2\lambda_n(i) - \lambda_n(i_{n+1})$ if $i \ge i_{n+1}$ and $\lambda_{n+1}(i) = \lambda_n(i)$ otherwise. If we put $\Lambda(i) = \lambda_n(i)$ for $n_i \le i < n_{i+1}$, we can verify the conditions on $\Lambda(i)$ in a similar way as before.

3. Construction of counterexamples

We may assume $\lambda(i) \to \infty$, since the condition $a_n = o(A_n)$ is necessary for (1.1). There is no loss of generality if we assume $c = 1$ and $\lambda(i) \ge 1$. Let us denote by $\|\cdot\|_\infty$ the sup-norm on $[0,1]$. Redefine $\{p(k)\}$ by

$$p(0) = 0 \quad \text{and} \quad p(k) = \max\left\{ j \;\middle|\; \sum_{i=1}^{j} \frac{1}{\lambda(i)} \le k. \right\} \quad (k \ge 1),$$

and define $l(k)$, $P(k)$, $\mu(k)$ and $\nu(k)$ as before by using new $\{p(k)\}$.

If $p(k) + 1 < p(k+1)$, we have

$$\frac{l(k)}{\nu(k)} + \frac{1}{\nu(k+1)} + \frac{1}{\nu(k-1)} \ge \sum_{i=p(k)}^{p(k+1)+1} \frac{1}{\lambda(i)} \ge 1 \ge \sum_{i=p(k)+1}^{p(k+1)} \frac{1}{\lambda(i)} \ge \frac{l(k)}{\mu(k)},$$

which implies $\liminf l(k)/\nu(k) \ge 1$ and $l(k) \le \mu(k)$. By using $l(k) \le \mu(k)$, in the same way as before, we can prove (2.2), $\mu(j) \sim \nu(j)$, and (2.3) in turn. Consequently we have $\mu(j) \sim \nu(j) \sim l(j) \to \infty$, and therefore we can take $j_0 \ge 1$ such that $\mu(j)/2 \le l(j) \le 2\nu(j)$ for $j \ge j_0$. We note that

$$\sum_{j=j_0}^{\infty} \frac{1}{l(j)} = \sum_{j=j_0}^{\infty} \frac{l(j)}{l^2(j)} \ge \sum_{j=j_0}^{\infty} \sum_{i \in P(j)} \frac{1}{4\lambda^2(i)} = \infty.$$

Let us put

$$a_i = \begin{cases} 1 & \text{if } i \le p(j_0), \\ A_{p(j)}/l(j) & \text{if } i \in P(j),\ j \ge j_0, \end{cases} \qquad \text{and} \quad b_j = a_{p(j+1)},$$

and define Δ_k as before. We easily have $A_{p(j_0)} > 0$, $a_i = O\big(A_i/\lambda(i)\big)$ and

$$A^2_{p(k+1)} = A^2_{p(j_0)} \prod_{j=j_0}^{k} \left(1 + \frac{1}{2l(j)}\right) \geq \frac{A^2_{p(j_0)}}{2} \sum_{j=j_0}^{k} \frac{1}{l(j)} \to \infty. \tag{3.1}$$

Let us take an increasing sequence $\{q(j)\}$ of integers such that

$$q(j_0) = p(j_0) + 1 \quad \text{and} \quad 2^{q(j+1)-q(j)} \geq \max\{\, 2l(j+1), \pi A_{p(j)} l^2(j) j^2 \,\},$$

and introduce $\{n_i\}$ by

$$n_{p(j)+l} = \begin{cases} 2^{p(j)+l} & \text{if } j < j_0,\ 1 \leq l \leq l(j), \\ 2^{q(j)} l & \text{if } j \geq j_0,\ 1 \leq l \leq l(j). \end{cases}$$

If $j \geq j_0$ and $1 \leq l \leq l(j)$, (1.7) is verified by noting $l(j) \leq 2\nu(j) \leq 2\lambda\big(p(j)+l\big)$;

$$\frac{n_{p(j)+l+1}}{n_{p(j)+l}} \geq 1 + \frac{1}{l(j)} \geq 1 + \frac{1}{\lambda(p(j)+l)} \quad \text{and} \quad \frac{n_{p(j)+1}}{n_{p(j)}} = \frac{2^{q(j)-q(j-1)}}{l(j)} \geq 2.$$

By using the Dirichlet kernel, we see that there exists an absolute constant $c_0 > 0$ such that for all integers m and l, $P\big(\big|\sum_{j=1}^{l} \cos 2\pi m j\omega\big| > l/e\big) \geq c_0/l$. Applying this, we have $P\big(|\Delta_j| \geq b_j l(j)/e\big) \geq c_0/l(j)$. Note that $b_j l(j)/e = A_{p(j)}/e$ for $j \geq j_0$. If we put $J_m = \{\, j = j_0, \dots, m \mid e A_{p(j)} \geq A_{p(m+1)} \,\}$, we have

$$\sum_{j=1}^{m} P\left(|\Delta_j| \geq \frac{A_{p(m+1)}}{e^2}\right) \geq \sum_{j \in J_m} P\left(|\Delta_j| \geq \frac{A_{p(j)}}{e}\right) \geq \sum_{j \in J_m} \frac{c_0}{l(j)}.$$

Since we have $A_{p(m+1)}/A_{p(j)} \leq \exp\big(\frac{1}{2}\sum_{k=j+1}^{m} 1/l(k)\big)$, by putting $J'_m = \big\{\, j = j_0, \dots, m \;\big|\; \sum_{k=j+1}^{m} 1/l(k) \leq 2 \,\big\}$, we have $J'_m \subset J_m$ and hence

$$\sum_{j=1}^{m} P\left(|\Delta_j| \geq \frac{A_{p(m+1)}}{e^2}\right) \geq \sum_{j \in J'_m} \frac{c_0}{l(j)} \to 2c_0. \tag{3.2}$$

Let $\omega = \sum_{r=1}^{\infty} 2^{-r} d_r(\omega)$ be the dyadic expansion of ω, and put

$$\widetilde{X}_j(\omega) = \sum_{i \in P(j)} a_i \cos 2\pi n_i \left(\sum_{r=q(j)+1}^{q(j+1)} 2^{-r} d_r(\omega) \right) \quad \text{and} \quad X_j = \widetilde{X}_j - E\widetilde{X}_j.$$

Clearly, $\{X_j\}$ is an independent sequence. Because of $E\Delta_j = 0$ and the estimate

$$\|\Delta_j - \widetilde{X}_j\|_\infty \leq b_j \sum_{i \in P(j)} 2\pi n_i 2^{-q(j+1)} \leq \pi b_j l^2(j) 2^{q(j)-q(j+1)} \leq 1/j^2,$$

we have $\big\|\Delta_j - X_j\big\|_\infty \leq 2/j^2$. If we put $\sigma_j^2 = EX_j^2$ and $s_m^2 = \sum_{j=1}^{m} \sigma_j^2$, we get

$$|s_m - A_{p(m+1)}| \leq \sum_{j=1}^{m} \|\Delta_j - X_j\|_\infty \leq 4 \quad \text{and} \quad |\sigma_j - E^{1/2}\Delta_j^2| \leq \frac{2}{j^2}.$$

Combining these with (3.1) and $E\Delta_j^2 = A^2_{p(j)}/2l(j)$, we have

$$s_m \to \infty \quad \text{and} \quad \sigma_m = O\big(1/m^2\big) + E^{1/2}\Delta_m^2 = o(A_{p(m)}) = o(s_m). \tag{3.3}$$

If the central limit theorem (1.1) holds, then it holds for $\{X_i\}$. The Lindeberg theorem claims that (cf. Chapter XV of Feller [2]) under (3.3), the central limit theorem $s_m^{-1}\sum_{i=1}^m X_k \xrightarrow{\mathcal{D}} N_{0,1}$ implies the Lindeberg condition

$$s_m^{-2}\sum_{i=1}^m E\big(X_i^2 : |X_i| > \varepsilon s_m\big) \to 0 \quad \text{for all} \quad \varepsilon > 0.$$

From this we have

$$\lim_{m\to\infty}\sum_{i=1}^m P\big(|X_i| > \varepsilon s_m\big) = 0 \quad \text{for all} \quad \varepsilon > 0.$$

On the other hand, by (3.2) we have

$$0 < c_0 \le \sum_{i=1}^m P\big(|\Delta_i| > A_{p(m+1)}/e^2\big) \le \sum_{i=1}^m P\big(|X_i| > s_m/e^3\big)$$

for large m. These contradict each other.

4. Fourier character of lacunary series

We prove Theorem 3 for $\gamma = 2$. The general case can be proved by iterating the following argument γ times. Let us put $S_n(\omega) = \sum_{i=1}^n a_i\cos(2\pi n_i\omega + \phi_i)$. We may assume $\delta = \sup_i |a_i| < \infty$, otherwise the conclusion is clear. Let us put

$$r(0) = 0 \quad \text{and} \quad r(m) = \max\{\, i \mid A_i \le \delta m\} \quad (m \ge 1).$$

By $\delta^2 m^2 \ge A_{r(m)}^2 = A_{r(m)+1}^2 - a_{r(m)}^2/2 > \delta^2(m^2 - 1/2)$, we have $A_{r(m)} \sim \delta m$, $A_{r(m)}^2 - A_{r(m-1)}^2 \sim 2\delta^2 m$ and $A_{r(m)}^{-1} - A_{r(m-1)}^{-1} = O(m^{-2})$.

First we assume that $\{S_n\}$ converges on some set E with $|E| > 0$ and derive a contradiction. Let us put $b_i = a_i/A_i$, $B_m = \frac{1}{2}\sum_{i=1}^m b_i^2$ and $T_m(\omega) = \sum_{i=1}^m b_i\cos 2\pi n_i\omega$. Since each term of T_m is a product of the term of S_m and non-negative decreasing sequence $1/A_i$, by using the Abel's Theorem (cf. (2.4) of Chapter I in Zygmund [11]), $\{T_n\}$ converges on E. Since we have

$$B_{r(m)}^2 = \sum_{k=1}^m \sum_{i=r(k-1)+1}^{r(k)} \frac{a_i^2}{2A_i^2} \sim \sum_{k=1}^m \frac{A_{r(k)}^2 - A_{r(k-1)}^2}{A_{r(k)}^2} \sim \sum_{k=1}^m \frac{2}{k} \sim \log m^2 \sim \log A_{r(m)}^2,$$

we get $B_k^2 \sim \log A_k^2$, and hence we can prove

$$b_i = o\big(B_i\sqrt{\log B_i}/\lambda(i)\big) \quad \text{and} \quad B_i \to \infty. \tag{4.1}$$

Let us repeat the above argument, i.e., take a sequence $\{r'(m)\}$ as $B_{r'(m)} \sim m$ and define $c_i = b_i/B_i$, $C_k = \frac{1}{2}\sum_{i=1}^k c_i^2$ and $Z_m(\omega) = \sum_{i=1}^m c_i\cos 2\pi n_i\omega$. In the same way we can prove that $C_k^2 \sim \log B_k^2$ and hence

$$c_i = o\big(C_i/\lambda(i)\big) \quad \text{and} \quad C_i \to \infty, \tag{4.2}$$

and that Z_∞ converges on E and thereby $Z_m/C_m \to 0$ on E. By (4.2) we can apply Theorem 1 and conclude that Z_m/C_m converges to $N_{0,1}$ on E, which is a contradiction.

Next we assume that the series is a Fourier series or a Fourier-Stieltjes series. Let us take $\rho \in (1/2, 1)$. By the Riesz-Kolmogorov inequality (cf. (6.8) or (6.27) of Chapter VII in Zygmund [11]), we have $E|S_m|^\rho = O(1)$.

Let us redefine b_i as $b_i = a_i/A_{r(m+1)}$ if $r(m) < i \le r(m+1)$, and define B_k and T_m as before by using these b_i. In a similar way, we can prove $B_k^2 \sim \log A_k^2$ and (4.1). For any m, let us take n such that $r(n) < m \le r(n+1)$. By applying the Abel's partial summation (the Abel transformation) to $T_m = (S_m - S_{r(n)})/A_{r(n+1)} + \sum_{k=1}^n (S_{r(k)} - S_{r(k-1)})/A_{r(k)}$, we have

$$\begin{aligned} E\big|T_m\big|^\rho &= E\bigg|\frac{1}{A_{r(n+1)}}S_m + \sum_{k=1}^n \bigg(\frac{1}{A_{r(k)}} - \frac{1}{A_{r(k+1)}}\bigg)S_{r(k)}\bigg|^\rho \\ &\le \frac{1}{A^\rho_{r(n+1)}}E|S_m|^\rho + \sum_{k=1}^n \bigg(\frac{1}{A_{r(k)}} - \frac{1}{A_{r(k+1)}}\bigg)^\rho E|S_{r(k)}|^\rho. \end{aligned}$$

Since $\big(\frac{1}{A_{r(k)}} - \frac{1}{A_{r(k+1)}}\big)^\rho = O(k^{-2\rho})$ is summable in k, we have $E|T_m|^\rho = O(1)$.

Let us redefine c_i as $c_i = b_i/B_{r'(m+1)}$ if $r'(m) < i \le r'(m+1)$, and define C_k and Z_m as before. Then we have (4.2) and $E|Z_m|^\rho = O(1)$. Thus Z_m/C_m converges to $N_{0,1}$ in law, and to 0 in L^ρ-sense and hence in probability. This is a contradiction.

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