PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 2, February 1999, Pages 615–618
S 0002-9939(99)04554-2

EXTENSIONS OF PERFECT GO-SPACES AND σ-DISCRETE DENSE SETS

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(Communicated by Alan Dow)

Abstract. In this paper, we prove that if a perfect GO-space X has a σ-discrete dense set, then X has a perfect linearly ordered extension. This answers a problem raised by H. R. Bennett, D. J. Lutzer and S. Purisch. And the result is also a partial answer to an old problem posed by H. R. Bennett and D. J. Lutzer.

1. Introduction

A *GO-space (generalized ordered space)* is a triple $\langle X, \tau, \leqslant\rangle$, where $\langle X, \leqslant\rangle$ is a linearly ordered set, τ a topology on X which is T_1 and has the base consisting of open sets which are order-convex. If we denote the usual interval topology on X by λ, then $\langle X, \lambda \leqslant\rangle$ is called a *LOTS (linearly ordered topological space).* We say that $\langle X, \lambda \leqslant\rangle$ is an *underlying LOTS* of the GO-space $\langle X, \tau, \leqslant\rangle$. If a GO-space $\langle X, \tau, \leqslant\rangle$ can be topologically embedded in a LOTS $\langle Y, \lambda, \prec\rangle$, then the LOTS $\langle Y, \lambda, \prec\rangle$ is called an *orderable extension* of the GO-space $\langle X, \tau, \leqslant\rangle$ and if $\leqslant = \prec |X$, then the LOTS $\langle Y, \lambda, \prec\rangle$ is called a *linearly ordered extension* of the GO-space $\langle X, \tau, \leqslant\rangle$. It is an interesting question whether a topological property on a GO-space can be reflected on some of its orderable extensions. It is known that for separability, metrizability and paracompactness the answers to this question are affirmative (cf. [1]). But the following question posed by H. R. Bennett and D. J. Lutzer remains open.

Problem 1 ([1]). Is it true that any perfect GO-space has a perfect orderable extension?

In [6] and [7] the author with T. Miwa and Y.-Z. Gao has proved that there exists a perfect GO-space which cannot be densely embedded in any perfect LOTS. On the other hand any perfect GO-space with the underlying LOTS satisfying local perfectness can be embedded in a perfect LOTS. Recently H. R. Bennett, D. J. Lutzer and S. Purisch studied dense subspaces of GO-spaces; they posed the following question.

Problem 2 ([2]). Is it true that a GO-space with a σ-discrete dense subset can be embedded in a LOTS with a σ-discrete dense subset?

Received by the editors January 7, 1997 and, in revised form, May 26, 1997.
1991 *Mathematics Subject Classification.* Primary 54F05, 54D35; Secondary 54F65, 54A10.
Key words and phrases. σ-discrete set, perfect, LOTS, GO-space, extension.

The aim of the present paper is to give a solution to Problem 2 in the affirmative and to point out relations between Problem 1 and an older set theoretic problem (see Section 3).

For a GO-space $\langle X, \tau, \leqslant \rangle$, let

λ be the interval topology on $\langle X, \leqslant \rangle$,
$I = \{x \in X \mid \{x\} \in \tau - \lambda\}$,
$R = \{x \in X - I \mid [x, \rightarrow) \in \tau - \lambda\}$,
$L = \{x \in X - I \mid (\leftarrow, x] \in \tau - \lambda\}$,
$E = X - (R \cup L \cup I)$.

It is well-known that a GO-space topology on $\langle X, \leqslant \rangle$ can be determined by the sets I, R, L, E. So we denote the GO-space $\langle X, \tau, \leqslant \rangle$ by $GO_X(R, E, I, L)$ and write $X = GO(R, E, I, L)$, simply saying X is a GO-space. By 'discrete' we always mean 'closed discrete'.

2. Main results

To prove our results, we state a known result proved by the author.

Theorem 1 ([5])**.** *A perfect GO-space $X = GO(R, E, I, L)$ has a perfect linearly ordered extension if and only if there exists a σ-discrete subset F of X such that $X' = GO_X(\emptyset, X - F, F, \emptyset)$ is perfect.*

Lemma 2. *Let $X = GO(R, E, I, L)$ be a GO-space and Y the underlying LOTS of X. If D is a discrete subset of X, then there exists a discrete subset $D' \supset D$ of X such that D' is closed in Y.*

Proof. Let $D' = \mathrm{cl}_Y D$. It is sufficient to prove that D' is discrete in X.

For $x \in X$, if $x \in I$, $\{x\}$ is an open neighborhood of x in X which intersects D in at most one point.

If $x \in R$, there exists $y > x$ such that $[x, y) \cap D = \{x\}$ or $[x, y) \cap D = \emptyset$ since D is discrete in X. If $(x, y) \cap (D' - D) \neq \emptyset$, we would have $(x, y) \cap D \neq \emptyset$. So $[x, y) \cap D' = \{x\}$ or $[x, y) \cap D' = \emptyset$. Similarly if $x \in L$, we may choose a $y < x$ such that $(y, x] \cap D' = \{x\}$ or $(y, x] \cap D' = \emptyset$.

If $x \in E$, there exist y_0, y_1 with $y_0 < x < y_1$ such that $(y_0, y_1) \cap D = \{x\}$ or $(y_0, y_1) \cap D = \emptyset$. If $(y_0, y_1) \cap (D' - D) \neq \emptyset$, then $|(y_0, y_1) \cap D| > 1$. Therefore $(y_0, y_1) \cap D' = \{x\}$ or $(y_0, y_1) \cap D' = \emptyset$. Hence D' is discrete in X and closed in Y. ☐

Lemma 3 ([8])**.** *If a GO-space X has a σ-discrete dense subset, then X is perfect.*

Theorem 4. *Let $X = GO(R, E, I, L)$ be a perfect GO-space. If X has a σ-discrete dense subset F, then X has a perfect linearly ordered extension.*

Proof. Let Y be the underlying LOTS of X. Since F is σ-discrete in X, $F = \bigcup\{F_n \mid n \in \omega_0\}$, where F_n is discrete in X for each $n \in \omega_0$. By Lemma 2, for each $n \in \omega_0$, we may choose a discrete subset F'_n of X such that $F'_n \supset F_n$ and F'_n is closed in Y.

Put $F' = \bigcup\{F'_n \mid n \in \omega_0\}$. Then F' is an F_σ-set in Y and a σ-discrete subset of X. Consider the GO-space $X' = GO_X(\emptyset, X - F', F', \emptyset)$. We prove that X' is perfect. It is obvious that F'_n is closed in X' for each $n \in \omega_0$. So F' is σ-discrete in X'. Assume that $x \in X - F'$ and (y_0, y_1) is a neighborhood of x in X'. Since (y_0, y_1) is also open in X and $F \subset F'$ is dense in X, $(y_0, y_1) \cap F' \neq \emptyset$. Thus F' is a

σ-discrete dense subset of X'. It follows from Lemma 3 that X' is perfect. Hence by Theorem 1, X has a perfect linearly ordered extension. $\square$

Theorem 5. *If a GO-space $X = GO(R, E, I, L)$ has a σ-discrete dense subset, then X has a perfect linearly extension with a σ-discrete dense subset.*

Proof. By the proof of Theorem 4, it is known that if the GO-space X has a σ-discrete dense subset, then X satisfies the conditions of Theorem 1. With R, L and I as in Section 1, by the proof of Theorem 1 (see [5]), there exists a σ-discrete set F of X such that $I \subset F \subset R \cup L \cup I$, and the perfect linearly ordered extension of X constructed in [5] has the form

$$\begin{aligned} P(X) &= (X \times \{0\}) \cup ((R - F) \times \{-1\}) \cup ((L - F) \times \{1\}) \\ &\quad \cup (I_0 \times (-1, 1)) \cup ((I_- \cup (F \cap R)) \times (-1, 0)) \cup ((I_+ \cup (F \cap L)) \times (0, 1)) \end{aligned}$$

where

$$\begin{aligned} I_- &= \{x \in I \mid \text{there is a } \ y \in X \text{ such that } x < y \text{ and } (x, y) = \emptyset\}, \\ I_+ &= \{x \in I \mid \text{there is a } \ y \in X \text{ such that } y < x \text{ and } (y, x) = \emptyset\}, \\ I_0 &= I - (I_- \cup I_+). \end{aligned}$$

Since

$$\begin{aligned} \mathcal{O} &= \{\{x\} \times (-1, 1) \mid x \in I_0\} \\ &\quad \cup \{\{x\} \times (-1, 0) \mid x \in I_- \cup (F \cap R)\} \cup \{\{x\} \times (0, 1) \mid x \in I_+ \cup (F \cap L)\} \end{aligned}$$

is a σ-discrete collection in $P(X)$ and every element of the collection has a countable dense subset, there exists a σ-discrete subset of $P(X)$ which is dense in each element of $\mathcal{O}$. For a point $\langle x, y\rangle \in P(X)$, if $x \notin F$, then the intersection of any neighborhood of $\langle x, y\rangle$ with $X \times \{0\}$ contains an interval of $X \times \{0\}$. It is easy to check that the σ-discrete subset of $X \times \{0\}$ is also σ-discrete in $P(X)$. Thus $P(X)$ has a σ-discrete dense subset. $\square$

3. Remark

By Theorem 4, we know that to find a counterexample to Problem 1, one must find a perfect GO-space which has no σ-discrete dense subset. But this is related to an old problem which is still open.

Problem 3 ([1])**.** Is there an example of a perfect GO-space in ZFC which does not have a σ-discrete dense subset ?

So if the answer to Problem 3 is 'no', then there exists no counterexample in ZFC to Problem 1. On the other hand, it is well-known that if we assume that there exists a Souslin line S, the existence of which is independent of ZFC, then S is a perfect LOTS which does not have a σ-discrete dense subset. However even under the assumption that Souslin line exists, any perfect GO-space with the Souslin line as the underlying LOTS does not serve as a counterexample to Problem 1 because by the result in [7], we know that any perfect GO-space with a perfect underlying LOTS has a perfect linearly extension.

References

1. H. R. Bennett and D. J. Lutzer, *Problems in perfect ordered space*, in: J. van Mill and G. M. Reed, eds, Open Problems in Topology (North-Holland, Amsterdam, 1990), 223–236. CMP 91:03
2. H. R. Bennett, D. J. Lutzer and S. D. Purisch, *On dense subspaces of generalized ordered spaces*, Topology Appl. (to appear).
3. R. Engelking, General Topology, (Heldermann, Berlin, rev.ed., 1989). MR **91c:**54001
4. D. J. Lutzer, On generalized ordered spaces, Dissertationes Math. 89(1971). MR **48:**3018
5. W.-X. Shi, *Perfect GO-spaces which have a perfect linearly ordered extension*, Topology Appl. **81** (1997), 23–33.
6. W.-X. Shi, T. Miwa and Y.-Z. Gao, *A perfect GO-space which cannot densely embed in any perfect orderable space*, Topology Appl. 66(1995), 241–249. MR **96i:**54026
7. ———, *Any perfect GO-space with the underlying LOTS satisfying local perfectness can embed in a perfect LOTS*, Topology Appl. 74(1996), 17–24. CMP 97:06
8. J. van Wouwe, GO-spaces and generalizations of metrizability, MC Tract no. 104, (Mathematical Center, Amsterdam, 1979). MR **80m:**54046

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