

UNCONDITIONAL BASIC SEQUENCE IN $L^p(\mu)$ AND ITS l^p -STABILITY

LIHUA YANG

(Communicated by J. Marshall Ash)

ABSTRACT. This paper is concerned with unconditional basic sequences in $L^p(\mu)$. We prove that, under some conditions, a sequence in $L^p(\mu)$ is a bounded unconditional basic sequence if and only if it is l^p -stable. At last the results are applied to the shift-invariant basic sequences generated by a finite subset of $L^p(R^s)$, which is very important in wavelet analysis.

1. INTRODUCTION

Unconditional basic sequences are very important in the basis theory of Banach spaces. It is a motivation for the development of the recent wavelet analysis. In general, it is difficult to verify whether a given basic sequence is unconditional. Therefore, it is meaningful to find more practical but equivalent conditions. In $L^2(\mu)$ it is well known that a sequence is a bounded unconditional basic sequence if and only if it is l^2 -stable. We want to know whether the result can be extended to $L^p(\mu)$ ($1 \leq p \leq \infty$). In this paper, we mainly study some properties of unconditional basic sequences and then prove that, under some conditions, a bounded basic sequence in $L^p(\mu)$ is unconditional if and only if it is l^p -stable. Finally, the results are applied to the shift-invariant basic sequence generated by a finite subset of $L^p(R^s)$, which is very important in wavelet analysis.

Let (Ω, Σ, μ) be a σ -finite (positive) measure space, $L^p(\mu) := L^p(\Omega, \Sigma, \mu)$ ($1 \leq p \leq \infty$) the space of all p -integrable functions on (Ω, Σ, μ) with norm defined by

$$\|f\|_p := \begin{cases} (\int_{\Omega} |f(x)|^p d\mu(x))^{1/p}, & f \in L^p(\mu) \ (1 \leq p < \infty), \\ \text{ess sup}_{x \in \Omega} |f(x)|, & f \in L^\infty(\mu) \ (p = \infty). \end{cases}$$

As usual, for $1 \leq p \leq \infty$, $l^p := l^p(\mathbf{N})$ (\mathbf{N} denotes the set of all natural numbers) denotes the space of all sequences a such that $\|a\|_p < \infty$, where

$$\|a\|_p := \begin{cases} (\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, & 1 \leq p < \infty, \\ \sup_{j \in \mathbf{N}} |a_j|, & p = \infty. \end{cases}$$

For simplicity, let $l := l(\mathbf{N})$ denote the space of all sequences and

$$l_0 := l_0(\mathbf{N}) := \{a \in l \mid a_j \neq 0 \text{ for only finite } j\}.$$

Received by the editors October 14, 1996 and, in revised form, May 21, 1997.

1991 *Mathematics Subject Classification*. Primary 46B20.

Key words and phrases. Unconditional basic sequence, l^p -stability, wavelet analysis.

Let X be a Banach space. A sequence $\{x_j\}_{j=1}^\infty \subset X$ is called a (Schauder) basis if $\forall x \in X$, there exists unique sequence $a \in l$ such that $x = \sum_{j=1}^\infty a_j x_j$.

A Schauder basis $\{x_j\}_{j=1}^\infty$ of a Banach space X is called an unconditional basis if $\forall \{\theta_j\}_{j=1}^\infty \in l$, the convergence of $\sum_{j=1}^\infty a_j x_j$ in X implies $\sum_{j=1}^\infty \theta_j a_j x_j$ converges in X provided $\theta_j = \pm 1$; a sequence $\{x_j\}_{j=1}^\infty$ is called a (or: an unconditional) basic sequence in a Banach space X if it is a (or: an unconditional) basis of a closed subspace of X .

A sequence $\{x_j\}_{j=1}^\infty \subset X$ is called bounded if

$$0 < \inf_{j \in \mathbf{N}} \|x_j\| \leq \sup_{j \in \mathbf{N}} \|x_j\| < \infty.$$

$\{x_j\}_{j=1}^\infty \subset X$ is called l^p -stable ($1 \leq p \leq \infty$) if there exist positive constants c_p and C_p such that $\forall a \in l_0$,

$$c_p \|a\|_p \leq \left\| \sum_{j=1}^\infty a_j x_j \right\| \leq C_p \|a\|_p$$

i.e., simply

$$\left\| \sum_{j=1}^\infty a_j x_j \right\| \sim \|a\|_p.$$

Hereafter, $A \sim B$ means there exist positive constants c, C such that $cA \leq B \leq CA$. The preceding definitions can be found in [SYL], [JM] and [JIA].

The paper is organized as follows. In section 2, we give an intuitive characterization and an equivalent norm for unconditional basic sequences. In section 3 the equivalence between l^p -stability and unconditionality is studied. Finally, the shift-invariant family generated by a finite set of $L^p(R^s)$ is studied in section 4.

2. UNCONDITIONAL BASIC SEQUENCE IN $L^p(\mu)$

The classical theory of unconditional bases in general Banach spaces contains rich contents. Some sufficient and necessary conditions for unconditional bases and their corresponding properties have been established (see [SYL]). In this paper, we prove the following theorem at first, which gives an intuitive characterization on unconditional basic sequences.

Theorem 1. *A basic sequence $\{x_n\}_{n=1}^\infty$ in a Banach space is unconditional if and only if, for any two disjoint finite subsets A, B of \mathbf{N} , one of the following conditions holds:*

- (i) $\|x + y\| \sim \|x - y\| \quad \forall x \in \text{span}\{x_j \mid j \in A\}, y \in \text{span}\{x_j \mid j \in B\}$.
- (ii) *There exists a constant $C > 0$ such that $\|x - y\| \geq C$ for any $x \in \text{span}\{x_j \mid j \in A\}, y \in \text{span}\{x_j \mid j \in B\}$ with $\|x\| = \|y\| = 1$.*

Proof. Let $\{x_n\}$ be an unconditional basis of Banach space X and A, B be two disjoint finite subsets of \mathbf{N} . Then there exists a constant $K > 0$ such that (see [SYL])

$$\|y\| \leq K \|x + y\| \quad \forall x \in \text{span}\{x_j \mid j \in A\}, y \in \text{span}\{x_j \mid j \in B\}.$$

Hence

$$\|x - y\| = \|(x + y) - 2y\| \leq (1 + 2K)\|x + y\|.$$

The inverse inequality can be proved similarly. (i) follows.

In the sequel, we prove (ii); we assume (i) holds now.

$\forall x \in \text{span}\{x_j \mid j \in A\}, y \in \text{span}\{x_j \mid j \in B\}$ with $\|x\| = \|y\| = 1$, it is easily seen that $\|x + y\| \geq \frac{1}{2}$ or $\|x - y\| \geq \frac{1}{2}$. If $\|x - y\| \geq \frac{1}{2}$, (ii) holds obviously; if $\|x + y\| \geq \frac{1}{2}$, then $\|x - y\| \sim \|x + y\| \geq \frac{1}{2}$, i.e. (ii) also holds.

Finally, let (ii) hold. We prove that $\{x_n\}$ is unconditional.

Let $x \in \text{span}\{x_j \mid j \in A\}, y \in \text{span}\{x_j \mid j \in B\}$ with $x \neq 0, y \neq 0$. Then there exists $C > 0$ such that

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geq C.$$

Choosing $\sigma > 0$ satisfying $0 < 2(1 - \sigma) \leq C$ and assuming $\|y\| \leq \|x\|$ without loss of generality, we have that

(1⁰). If $\|y\| \geq \sigma\|x\|$, then

$$\begin{aligned} \|x - y\| &= \left\| \frac{x}{\|x\|}\|x\| - \frac{y}{\|y\|}\|x\| + \frac{y}{\|y\|}\|x\| - y \right\| \\ &\geq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \cdot \|x\| - \left| \frac{\|x\|}{\|y\|} - 1 \right| \cdot \|y\| \\ &\geq C\|x\| - (\|x\| - \|y\|) \\ &\geq \frac{1}{2}C\|x\|. \end{aligned}$$

Hence

$$\frac{\|x + y\|}{\|x - y\|} \leq \frac{\|x\| + \|y\|}{\frac{1}{2}C\|x\|} \leq \frac{4}{C}.$$

(2⁰). If $\|y\| \leq \sigma\|x\|$, one easily gets

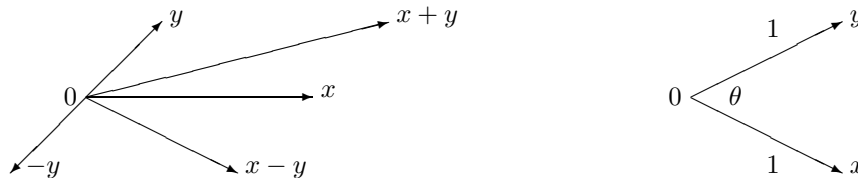
$$\frac{\|x + y\|}{\|x - y\|} \leq \frac{2\|x\|}{\|x\| - \|y\|} \leq \frac{2}{1 - \sigma}.$$

In any case, there always exists $K > 0$ such that $\|x + y\| \leq K\|x - y\|$. Hence

$$\|x\| \leq \frac{1}{2}\|x + (-y)\| + \frac{1}{2}\|x - (-y)\| \leq \frac{1}{2}(1 + K)\|x + y\|$$

which implies that $\{x_n\}$ is an unconditional basic sequence (see [SYL]). Theorem 1 follows. □

Remark. We give an intuitive explanation of the theorem as follows. A basic sequence $\{x_n\}_{n=1}^\infty$ in a Banach space is unconditional if and only if, for any two disjoint finite subsets A, B of \mathbf{N} , the included angle between the subspaces $\text{span}\{x_j \mid j \in A\}$ and $\text{span}\{x_j \mid j \in B\}$ is larger than a positive number independent of A and B .



Theorem 2. Let $\{x_j\}_{j=1}^\infty$ be a basic sequence in $L^p(\mu)$ ($1 \leq p \leq \infty$). Then it is unconditional if and only if

$$\left\| \sum_{j=1}^{\infty} a_j x_j \right\|_p \sim \left\| \left(\sum_{j=1}^{\infty} |a_j x_j|^2 \right)^{1/2} \right\|_p \quad \forall x = \sum_{j=1}^{\infty} a_j x_j \in L^p(\mu).$$

Proof. We only need to prove the necessity. In fact, $\forall x = \sum_{j=1}^{\infty} a_j x_j \in X$, by Khintchine's inequality (see [SYL, p.30]) we have that

$$\int_0^1 \left| \sum_{j=1}^m r_j(t) a_j x_j(\omega) \right|^p dt \sim \left(\sum_{j=1}^m |a_j x_j(\omega)|^2 \right)^{p/2} \quad \forall \omega \in \Omega, m \in \mathbf{N}$$

where $\{r_j(t)\}$ are the Rademacher functions, i.e.

$$r_j(t) := \text{signsin}(2^j \pi t) \quad (j = 1, 2, \dots).$$

Hence

$$\begin{aligned} \int_0^1 \left\| \sum_{j=1}^m r_j(t) a_j x_j \right\|_p^p dt &= \int_0^1 dt \int_{\Omega} \left| \sum_{j=1}^m r_j(t) a_j x_j(\omega) \right|^p d\mu \\ &\sim \int_{\Omega} \left(\sum_{j=1}^m |a_j x_j(\omega)|^2 \right)^{p/2} d\mu(\omega). \end{aligned}$$

Since $\{x_j\}$ is an unconditional basic sequence and $r_j(t) = \pm 1$ a.e., we have (see [SYL, p.22])

$$\begin{aligned} \left\| \sum_{j=1}^m a_j x_j \right\|_p^p &= \int_0^1 \left\| \sum_{j=1}^m a_j x_j \right\|_p^p dt \sim \int_0^1 \left\| \sum_{j=1}^m r_j(t) a_j x_j \right\|_p^p dt \\ &\sim \int_{\Omega} \left(\sum_{j=1}^m |a_j x_j(\omega)|^2 \right)^{p/2} d\mu(\omega). \end{aligned}$$

Letting $m \rightarrow \infty$ we deduce that

$$\left\| \sum_{j=1}^{\infty} a_j x_j \right\|_p \sim \left\| \left(\sum_{j=1}^{\infty} |a_j x_j|^2 \right)^{1/2} \right\|_p.$$

This completes the proof. \square

Remark. If X is the closed subspace of $L^p(\mu)$ such that $\{x_j\}_{j=1}^\infty$ is its basis, it is clear that

$$\left\| \sum_{j=1}^{\infty} a_j x_j \right\|_p := \left\| \left(\sum_{j=1}^{\infty} |a_j x_j|^2 \right)^{1/2} \right\|_p \quad (\forall x = \sum_{j=1}^{\infty} a_j x_j \in X)$$

is a norm on X .

3. l^p -STABILITY AND UNCONDITIONAL BASIC SEQUENCES IN $L^p(\mu)$

In this section, we study the equivalence between the l^p -stability and unconditionality of bounded basic sequences in $L^p(\mu)$. Our results show that, under some conditions, $\{x_j\}$ is a bounded unconditional basic sequence if and only if it is l^p -stable.

Theorem 3. Let $\{x_j\}_{j=1}^\infty$ be a l^p -stable ($1 \leq p \leq \infty$) sequence in a Banach space X . Then it is a bounded unconditional basic sequence in X .

Proof. $\forall x \in \overline{\text{span}\{x_j\}}$ (the closure in X) and $n \in \mathbf{N}$, let $a^{(n)} \in l_0$ such that

$$\|x - \sum_{j=1}^{\infty} a_j^{(n)} x_j\| < \frac{1}{n}.$$

It is easily seen that

$$\|a^{(n)} - a^{(m)}\|_p \sim \left\| \sum_{j=1}^{\infty} (a_j^{(n)} - a_j^{(m)}) x_j \right\| \longrightarrow 0 \quad (n, m \rightarrow \infty).$$

Hence there exists $a \in l^p$ such that

$$\|a^{(n)} - a\|_p \longrightarrow 0 \quad (n \rightarrow \infty).$$

For any $n \in \mathbf{N}$, let $m \geq k$ be large enough such that $a_j^{(n)} = 0$ ($\forall j \geq k$). By the l^p -stability, we get

$$\left\| \sum_{j=k}^m a_j x_j \right\| = \left\| \sum_{j=k}^m (a_j - a_j^{(n)}) x_j \right\| \leq C_p \|a - a^{(n)}\|_p;$$

therefore, $\sum_{j=1}^{\infty} a_j x_j$ converges in X . It is also clear that

$$\left\| \sum_{j=1}^{\infty} a_j^{(n)} x_j - \sum_{j=1}^{\infty} a_j x_j \right\| \sim \|a^{(n)} - a\|_p \longrightarrow 0 \quad (n \rightarrow \infty).$$

Hence $x = \sum_{j=1}^{\infty} a_j x_j$. Again by the l^p -stability we conclude that such expression is unique, thus $\{x_j\}_{j=1}^{\infty}$ is a Schauder basis of $\overline{\text{span}\{x_j\}}$. At last, the l^p -stability implies that it is unconditional and bounded obviously. The proof of the theorem is complete. \square

Theorem 4. *Let $\{x_j\}_{j=1}^{\infty}$ be a bounded unconditional basic sequence in $L^p(\mu)$ ($1 \leq p \leq \infty$). Then there exist positive constants c_p, C_p such that*

(i) *if $1 \leq p \leq 2$, there holds*

$$c_p \|\{a_j\}\|_2 \leq \left\| \sum_{j=1}^{\infty} a_j x_j \right\|_p \leq C_p \|\{a_j\}\|_p \quad \forall \sum_{j=1}^{\infty} a_j x_j \in L^p(\mu);$$

(ii) *if $2 \leq p \leq \infty$, there holds*

$$c_p \|\{a_j\}\|_p \leq \left\| \sum_{j=1}^{\infty} a_j x_j \right\|_p \leq C_p \|\{a_j\}\|_2 \quad \forall \sum_{j=1}^{\infty} a_j x_j \in L^p(\mu).$$

Proof. (i). For $1 \leq p \leq 2$, we know that $L^p(\mu)$ is of Rademacher type p and Rademacher cotype 2 (see [SYL, pp.151–152]), i.e. there exist $c_p, C_p > 0$ such that

$$\left(\int_0^1 \left\| \sum_{j=1}^{\infty} r_j(t) z_j \right\|_p^2 dt \right)^{1/2} \leq C_p \left(\sum_{j=1}^{\infty} \|z_j\|_p^p \right)^{1/p}$$

and

$$\left(\int_0^1 \left\| \sum_{j=1}^{\infty} r_j(t) z_j \right\|_p^2 dt \right)^{1/2} \geq c_p \left(\sum_{j=1}^{\infty} \|z_j\|_p^2 \right)^{1/2}$$

for all $n \in \mathbf{N}$ and $\{z_j\}_{j=1}^\infty \subset L^p(\mu)$, where $\{r_j(t)\}$ are the Rademacher functions. Since $\{x_j\}_{j=1}^\infty$ is an unconditional basic sequence in $L^p(\mu)$, we conclude that

$$\begin{aligned} \left\| \sum_{j=1}^\infty a_j x_j \right\|_p &\sim \left(\int_0^1 \left\| \sum_{j=1}^\infty r_j(t) a_j x_j \right\|_p^2 dt \right)^{1/2} \\ &\leq C_p \left(\sum_{j=1}^\infty \|a_j x_j\|_p^p \right)^{1/p} \\ &\leq C_p \left(\sum_{j=1}^\infty |a_j|^p \right)^{1/p} \sup_{j \in \mathbf{N}} \|x_j\|_p \end{aligned}$$

and

$$\begin{aligned} \left\| \sum_{j=1}^\infty a_j x_j \right\|_p &\sim \left(\int_0^1 \left\| \sum_{j=1}^\infty r_j(t) a_j x_j \right\|_p^2 dt \right)^{1/2} \\ &\geq c_p \left(\sum_{j=1}^\infty \|a_j x_j\|_p^2 \right)^{1/2} \\ &\geq c_p \left(\sum_{j=1}^\infty |a_j|^2 \right)^{1/2} \inf_{j \in \mathbf{N}} \|x_j\|_p. \end{aligned}$$

This completes the proof of (i).

(ii). For $2 \leq p < \infty$, the proof is similar. We consider the case of $p = \infty$ at last. By Theorem 2, it is easily seen that

$$\left\| \sum_{j=1}^\infty a_j x_j \right\|_\infty \sim \left\| \left(\sum_{j=1}^\infty |a_j x_j|^2 \right)^{1/2} \right\|_\infty \leq \left(\sum_{j=1}^\infty |a_j|^2 \right)^{1/2} \sup_{j \in \mathbf{N}} \|x_j\|_\infty$$

and

$$\left\| \sum_{j=1}^\infty a_j x_j \right\|_\infty \sim \left\| \left(\sum_{j=1}^\infty |a_j x_j|^2 \right)^{1/2} \right\|_\infty \geq \sup_{j \in \mathbf{N}} \|a_j x_j\|_\infty \geq \{a_j\}_\infty \inf_{j \in \mathbf{N}} \|x_j\|_\infty.$$

Hence the theorem also holds for $p = \infty$. The proof is complete. □

Theorem 5. Let $\{x_j\}_{j=1}^\infty$ be a bounded unconditional basic sequence in $L^p(\mu)$ ($1 \leq p \leq \infty$) and

$$\tilde{x}_j := |x_j|^{2/p} \left(\sum_{i=1}^\infty |x_i|^2 \right)^{\frac{1}{2} - \frac{1}{p}} \quad (1 \leq p \leq \infty)$$

where $\tilde{x}_j := 0$ if $\sum_{i=1}^\infty |x_i| = 0$ and $\frac{1}{p} := 0$ for $p = \infty$. If

$$0 < \inf_{j \in \mathbf{N}} \|\tilde{x}_j\|_p \leq \sup_{j \in \mathbf{N}} \|\tilde{x}_j\|_p < \infty$$

then $\{x_j\}_{j=1}^\infty$ is l^p -stable.

Proof. (1). If $1 \leq p < 2$, by Theorem 4 and Theorem 2 we have that, $\forall x = \sum_{j=1}^\infty a_j x_j \in L^p(\mu)$, there exists $C_p > 0$ such that

$$\|x\|_p \leq C_p \|\{a_j\}\|_p$$

and

$$\|x\|_p \sim \left(\int_{\Omega} \left(\sum_{j=1}^{\infty} |a_j x_j|^2 \right)^{p/2} d\mu \right)^{1/p}.$$

Using the inverse Hölder-inequality, we have that, for $\theta := 2/p$,

$$\begin{aligned} \left(\sum_{j=1}^{\infty} |a_j x_j|^2 \right)^{p/2} &= \left(\sum_{j=1, x_j \neq 0}^{\infty} |a_j|^2 |x_j|^{2\theta} |x_j|^{2(1-\theta)} \right)^{p/2} \\ &\geq \left[\left(\sum_{j=1, x_j \neq 0}^{\infty} (|a_j|^2 |x_j|^{2\theta})^{p/2} \right)^{2/p} \left(\sum_{j=1, x_j \neq 0}^{\infty} |x_j|^{2(1-\theta) \frac{p}{p-2}} \right)^{1-\frac{2}{p}} \right]^{p/2} \\ &= \left(\sum_{j=1, x_j \neq 0}^{\infty} |a_j|^p |x_j|^{\theta p} \right) \left(\sum_{j=1, x_j \neq 0}^{\infty} |x_j|^{2(1-\theta) \frac{p}{p-2}} \right)^{\frac{p}{2}-1} \\ &= \sum_{j=1, x_j \neq 0}^{\infty} |a_j|^p |x_j|^{\theta p} \left(\sum_{j=1, x_j \neq 0}^{\infty} |x_j|^{2(1-\theta) \frac{p}{p-2}} \right)^{\frac{p}{2}-1} \\ &= \sum_{j=1}^{\infty} |a_j|^p |\tilde{x}_j|^p. \end{aligned}$$

Hence

$$\begin{aligned} \|x\|_p &\sim \left(\int_{\Omega} \left(\sum_{j=1}^{\infty} |a_j x_j|^2 \right)^{p/2} d\mu \right)^{1/p} \\ &\geq \left(\int_{\Omega} \sum_{j=1}^{\infty} |a_j|^p |\tilde{x}_j|^p d\mu \right)^{1/p} \geq \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{1/p} \inf_{j \in \mathbf{N}} \|\tilde{x}_j\|_p, \end{aligned}$$

i.e. $\|x\|_p \sim \|\{a_j\}\|_p$.

(2). If $2 < p < \infty$, we can conclude by the same way that, $\forall x = \sum_{j=1}^{\infty} a_j x_j \in L^p(\mu)$, there exists $c_p > 0$ such that

$$\|x\|_p \geq c_p \|\{a_j\}\|_p$$

and, for $\theta := 2/p$,

$$\begin{aligned} \|x\|_p &\sim \left(\int_{\Omega} \left(\sum_{j=1}^{\infty} |a_j x_j|^2 \right)^{p/2} d\mu \right)^{1/p} \\ &= \left[\int_{\Omega} \left(\sum_{j=1}^{\infty} |a_j|^2 |x_j|^{2\theta} |x_j|^{2(1-\theta)} \right)^{p/2} d\mu \right]^{1/p} \\ &\leq \left[\int_{\Omega} \left(\sum_{j=1}^{\infty} |a_j|^p |x_j|^{\theta p} \right) \left(\sum_{j=1}^{\infty} |x_j|^{2(1-\theta) \frac{p}{p-2}} \right)^{\frac{p}{2}-1} d\mu \right]^{1/p} \\ &= \left[\sum_{j=1}^{\infty} |a_j|^p \left(\int_{\Omega} |\tilde{x}_j|^p d\mu \right) \right]^{1/p} \\ &\leq \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{1/p} \sup_{j \in \mathbf{N}} \|\tilde{x}_j\|_p. \end{aligned}$$

Hence $\|x\|_p \sim \|\{a_j\}\|_p$.

- (3). If $p = 2$, the proof is trivial.
- (4). At last, if $p = \infty$, it is easily seen by Theorem 4 that

$$\|x\| \geq c_p \|\{a_j\}\|_\infty$$

and

$$\|x\|_p \sim \left\| \left(\sum_{j=1}^\infty |a_j x_j|^2 \right)^{1/2} \right\|_\infty \leq \|\{a_j\}\|_\infty \left\| \left(\sum_{j=1}^\infty |x_j|^2 \right)^{1/2} \right\|_\infty$$

i.e. $\|x\|_p \sim \|\{a_j\}\|_p$. The proof of the theorem is complete. □

Remark. From the proof above, we can see easily that, for $1 \leq p < 2$ and $2 < p < \infty$, Theorem 5 holds still if \tilde{x}_j is replaced by the following function:

$$g_{j,\theta} := |x_j|^\theta \left(\sum_{i=1}^\infty |x_i|^{\frac{2p(1-\theta)}{p-2}} \right)^{\frac{1}{2} - \frac{1}{p}}$$

where θ is any positive constant independent of j satisfying $\frac{2p(1-\theta)}{p-2} > 0$. It is clear that $g_{j,\theta} \sim \tilde{x}_j$ if $\theta := 2/p$.

4. SHIFT-INVARIANT SUBSPACE OF $L^p(R^s)$

Let $L^p(\mu) = L^p(R^s)$ ($s \in \mathbf{N}, 1 \leq p \leq \infty$) be the space of p -power Lebesgue integrable functions. For a finite subset $\Phi \subset L^p(R^s)$, the set generated by the shifts of Φ is defined by

$$S(\Phi) := \{\phi(\cdot - j) \mid j \in \mathbf{Z}^s, \phi \in \Phi\}$$

where \mathbf{Z} is the set of all integers. The l^p -stability and unconditionality of $S(\Phi)$ are important topics in wavelet analysis. In this field, many interesting results have been established (see [MEYER, p.30], [DAU, p.298], [JM], etc.). In this section, we will use Theorem 5 to show that $S(\Phi)$ is l^p -stable if and only if it is a bounded unconditional basic sequence under some weak conditions.

We recall the space \mathcal{L}^p defined in [JM] by

$$\mathcal{L}^p := \mathcal{L}^p(R^s) := \{f \mid f \text{ is Lebesgue measurable and } \|f\|_p := \|f^0\|_{L^p([0,1]^s)} < \infty\}$$

where

$$f^0 := \sum_{\alpha \in \mathbf{Z}^s} |f(\cdot - \alpha)|.$$

For convenience, we define another space as follows:

$$\mathcal{L}_*^p := \mathcal{L}_*^p(R^s) := \{f \mid f \text{ is Lebesgue measurable and } \|f\|_p^* := \|f^*\|_{L^p([0,1]^s)} < \infty\}$$

where

$$f^* := \left(\sum_{\alpha \in \mathbf{Z}^s} |f(\cdot - \alpha)|^2 \right)^{1/2}.$$

It is easy to verify that

$$\begin{cases} \mathcal{L}^p \subset L^p \subset \mathcal{L}_*^p & (1 \leq p \leq 2), \\ \mathcal{L}^p \subset \mathcal{L}_*^p \subset L^p & (2 \leq p \leq \infty). \end{cases}$$

Theorem 6. *Let $S(\Phi)$ be generated by the shifts of finite subset $\Phi \subset L^p(R^s) \cap \mathcal{L}_*^p(R^s)$ ($1 \leq p \leq \infty, s \in \mathbf{N}$). Then $S(\Phi)$ is an unconditional basic sequence in $L^p(R^s)$ if and only if it is l^p -stable, i.e.*

$$\left\| \sum_{j \in \mathbf{Z}^s, \phi \in \Phi} a_j \phi(\cdot - j) \right\| \sim \|\{a_j\}\|_p \quad \forall \sum_{j \in \mathbf{Z}^s, \phi \in \Phi} a_j \phi(\cdot - j) \in L^p(R^s).$$

Proof. We only need to prove the necessity. Let $S(\Phi)$ be an unconditional basic sequence in $L^p(\mu)$. $\forall \phi \in \Phi$ and $j \in \mathbf{Z}^s$, let

$$\tilde{\phi}(\cdot - j) := |\phi(\cdot - j)|^{2/p} \left(\sum_{j \in \mathbf{Z}^s, \psi \in \Phi} |\psi(\cdot - j)|^2 \right)^{\frac{1}{2} - \frac{1}{p}}.$$

One has

$$\|\phi(\cdot - j)\|_p = \|\phi\|_p$$

and

$$\begin{aligned} \|\tilde{\phi}(\cdot - j)\|_p &= \left(\int_{R^s} |\phi(x - j)|^2 \left(\sum_{i \in \mathbf{Z}^s, \psi \in \Phi} |\psi(x - i)|^2 \right)^{\frac{p}{2} - 1} dx \right)^{1/p} \\ &= \left(\int_{R^s} |\phi(x)|^2 \left(\sum_{i \in \mathbf{Z}^s, \psi \in \Phi} |\psi(x - i)|^2 \right)^{\frac{p}{2} - 1} dx \right)^{1/p} \\ &= \left(\int_{[0,1]^s} \left(\sum_{i \in \mathbf{Z}^s} |\phi(x - i)|^2 \right) \left(\sum_{i \in \mathbf{Z}^s, \psi \in \Phi} |\psi(x - i)|^2 \right)^{\frac{p}{2} - 1} dx \right)^{1/p}. \end{aligned}$$

Hence

$$\begin{aligned} \|\tilde{\phi}(\cdot - j)\|_p &\leq \left(\int_{[0,1]^s} \left(\sum_{i \in \mathbf{Z}^s, \psi \in \Phi} |\psi(x - i)|^2 \right)^{p/2} dx \right)^{1/p} \\ &\leq \left(\int_{[0,1]^s} \left(\sum_{\psi \in \Phi} \left(\sum_{i \in \mathbf{Z}^s} |\psi(x - i)|^2 \right)^{1/2} \right)^p dx \right)^{1/p} \\ &\leq \sum_{\psi \in \Phi} \left(\int_{[0,1]^s} \left(\sum_{i \in \mathbf{Z}^s} |\psi(x - i)|^2 \right)^{p/2} dx \right)^{1/p} \\ &= \sum_{\psi \in \Phi} \|\psi\|_p^* \end{aligned}$$

and

$$\begin{aligned} \|\tilde{\phi}(\cdot - j)\|_p &\geq \left(\int_{[0,1]^s} \left(\sum_{i \in \mathbf{Z}^s} |\phi(x - i)|^2 \right)^{p/2} dx \right)^{1/p} \\ &= \|\phi\|_p^*. \end{aligned}$$

Therefore

$$\min_{\phi \in \Phi} \|\phi\|_p = \inf_{j \in \mathbf{Z}^s, \phi \in \Phi} \|\phi(\cdot - j)\|_p \leq \sup_{j \in \mathbf{Z}^s, \phi \in \Phi} \|\phi(\cdot - j)\|_p = \max_{\phi \in \Phi} \|\phi\|_p$$

and

$$\min_{\phi \in \Phi} \|\phi\|_p^* \leq \inf_{j \in \mathbf{Z}^s, \phi \in \Phi} \|\tilde{\phi}(\cdot - j)\|_p \leq \sup_{j \in \mathbf{Z}^s, \phi \in \Phi} \|\tilde{\phi}(\cdot - j)\|_p \leq \sum_{\phi \in \Phi} \|\phi\|_p^*.$$

Since $S(\Phi)$ is a basic sequence in $L^p(R^s)$, we conclude that $\min_{\phi \in \Phi} \|\phi\|_p > 0$ and $\min_{\phi \in \Phi} \|\phi\|_p^* > 0$. Moreover, $\Phi \subset L^p(R^s) \cap \mathcal{L}_*^p(R^s)$ implies $\max_{\phi \in \Phi} \|\phi\|_p < \infty$ and $\sum_{\phi \in \Phi} \|\phi\|_p^* < \infty$. Then, the conclusion follows from Theorem 5. \square

By [JIA, Theorem 1.1–1.2] we further obtain the following theorem.

Theorem 7. *Let $S(\Phi)$ be generated by the finite subset $\Phi \subset \mathcal{L}^p(\mathbb{R}^s)$ ($1 \leq p \leq \infty$, $s \in \mathbf{N}$). Then the following conditions are equivalent:*

- (1) $S(\Phi)$ constitutes an unconditional basic sequence in $L^p(\mathbb{R}^s)$,
- (2) $S(\Phi)$ is l^p -stable,
- (3) $\{\hat{\phi}(\xi + 2\pi\beta)\}_{\phi \in \Pi, \beta \in \mathbf{Z}^s}$ are linearly independent,

where $\hat{\phi}$ is the Fourier transform of ϕ defined in [JIA].

For the definition of “linear independent”, we refer to [JIA].

ACKNOWLEDGEMENT

The author is grateful to Professor J. Marshall Ash and the referee for their helpful suggestions.

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DEPARTMENT OF SCIENTIFIC COMPUTING AND COMPUTER APPLICATIONS, ZHONGSHAN UNIVERSITY, 510275, PEOPLE'S REPUBLIC OF CHINA

INSTITUTE OF MATHEMATICS, ACADEMY SINICA, BEIJING, 100080, PEOPLE'S REPUBLIC OF CHINA

E-mail address: yang@comp.hkbu.edu.hk

E-mail address: ylh@math03.math.ac.cn