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A BOUND FOR $|G:\mathbf{O}_p(G)|_p$ IN TERMS OF THE LARGEST IRREDUCIBLE CHARACTER DEGREE OF A FINITE p-SOLVABLE GROUP G

DIANE BENJAMIN

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ABSTRACT. Let $b(G)$ denote the largest irreducible character degree of a finite group G, and let p be a prime. Two results are obtained. First, we show that, if G is a p-solvable group and if $b(G) < p^2$, then $p^2 \nmid |G:\mathbf{O}_p(G)|$. Next, we restrict attention to solvable groups and show that, if $b(G) \leq p^\alpha$ and if P is a Sylow p-subgroup of G, then $|P:\mathbf{O}_p(G)| \leq p^{2\alpha}$.

1. INTRODUCTION

Suppose G is a finite group. Let $\mathrm{cd}(G)$ denote the set $\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$, and let $b(G)$ denote the largest irreducible character degree of a group G. Theorem 12.29 of [1] states: *Let p be a prime and let $b(G) < p$. Then G has a normal abelian Sylow-p subgroup.* It is immediate from this result that if $b(G) < p$, then $p \nmid |G:\mathbf{O}_p(G)|$. Later, in the same chapter of [1], we have Theorem 12.32: *Suppose $b(G) < p^{3/2}$ for some prime p. Then $p^2 \nmid |G:\mathbf{O}_p(G)|$.* These facts raise the question: if $b(G) < p^\alpha$ for a real number α, what can be said about the p part of $|G:\mathbf{O}_p(G)|$? We address this question in the case when G is a p-solvable group and obtain the following two results:

Theorem A. *Let G be a p-solvable group and let p be a prime. If $b(G) < p^2$, then $p^2 \nmid |G:\mathbf{O}_p(G)|$.*

Theorem B. *Let G be a solvable group, let p be a prime, and let α be a real number. If $b(G) \leq p^\alpha$ and if P is a Sylow p-subgroup of G, then $|P:\mathbf{O}_p(G)| \leq p^{2\alpha}$. In addition, if $|G|$ is odd, then $|P:\mathbf{O}_p(G)| \leq p^\alpha$.*

2. PRELIMINARIES

In this section, we establish several facts regarding coprime actions that will be needed in the proofs of Theorem A and Theorem B.

Theorem (2.1) (Brodkey). *Let G be a finite group and assume that $S \in \mathrm{Syl}_p(G)$ is abelian. Then there exists $T \in \mathrm{Syl}_p(G)$ with $S \cap T = \mathbf{O}_p(G)$.*

Proof. This is Theorem 5.28 of [2]. □

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Lemma (2.2). *Suppose that a p-group P acts on a p′-group H. If P fixes every character in* $\mathrm{Irr}(H)$*, then the action of P on H is trivial.*

Proof. If P fixes every character of H, then, by Brauer's Theorem (6.32 of [1]), P fixes every conjugacy class of H. Since the size of a conjugacy class of H is a p'-number, it follows that each congugacy class contains a fixed point of P. Let $C = \mathbf{C}_H(P)$ be the subgroup of fixed points. Since C meets each class of H nontrivially, it follows that H is the (setwise) union of H-conjugates of C. This forces $H = C$, and thus the action of P on H is trivial. $\square$

Last, we prove a special case of Theorem A.

Proposition (2.3). *Suppose an abelian p-group P acts faithfully on an abelian p′-group H, and let* $G = H \rtimes P$*. If* $b(G) < p^2$*, then* $|P| \leq p$*.*

Proof. If $x \in \mathbf{C}_P(h)$, then $h = h^x$; thus $x = x^h$ and, since $x \in P$, it follows that $x \in P \cap P^h$. Thus, in the action of P on H, the stabilizer of a point h in H, which is $\mathbf{C}_P(h)$, is contained in $P \cap P^h$. Since the action of P on H is faithful and since H is abelian, Brauer's Theorem (6.32 of [1]) implies that the action of P on the abelian group $\mathrm{Irr}(H)$ is faithful. Since P is abelian, Brodkey's Theorem (2.1) together with the fact that point stabilizers are contained in Sylow intersections implies that there is a regular orbit of P on $\mathrm{Irr}(H)$. Thus, for some character $\lambda \in \mathrm{Irr}(H)$, we have $I_G(\lambda) = H$. It follows that $\lambda^G \in \mathrm{Irr}(G)$, and therefore $|P| = |G : H| = \lambda^G(1) \leq b(G)$. Since $b(G) < p^2$, the conclusion holds. $\square$

3. Proof of Theorem A

In this section we prove Theorem A. We begin with a technical lemma.

Lemma (3.1). *Suppose that G is a group with subgroups H, P, A, and B such that* $G = HP$ *and* $P = AB$*. If* $B \leq \mathbf{N}_G([H, A])$*, then* $[H, A] \lhd G$ *and* $[H, A] \cdot [H, B] = [H, P]$*.*

Proof. Assume that $B \leq \mathbf{N}_G([H, A])$. Since H and A normalize $[H, A]$, we have that $[H, A] \lhd G$ and it follows that the product $[H, A] \cdot [H, B]$ is a subgroup.

Clearly $[H, A] \cdot [H, B] \leq [H, P]$. We will show that this is an equality. Let $[h, x]$ be a generator of $[H, P]$, with $h \in H$ and $x \in P$. Write $x = ab$, where $a \in A$ and $b \in B$. One can check that:

$$[h, x] = [h, ab] = [h, b][h, a]^b \in [H, B] \cdot [H, A]^b = [H, A] \cdot [H, B].$$

It follows that $[H, A] \cdot [H, B] = [H, P]$. $\square$

Proof of Theorem A. Let G be a counterexample of minimal order. We may assume that $\mathbf{O}_p(G) = 1$. Set $H = \mathbf{O}_{p'}(G)$. By the famous Lemma 1.2.3 of Hall and Higman, see Lemma 14.22 of [1], we have $\mathbf{C}_G(H) \leq H$.

First, we will prove a fact that will be used repeatedly: If K is a subgroup of G with $H \leq K$, then $\mathbf{O}_p(K) = 1$. Assume that $K \geq H$. Since H is a p'-group, H and $\mathbf{O}_p(K)$ are disjoint normal subgroups of K; thus $\mathbf{O}_p(K) \leq \mathbf{C}_K(H) \leq H$. Since H has p' order, this forces $\mathbf{O}_p(K) = 1$.

Now fix $P \in \mathrm{Syl}_p(G)$. As we have seen, $\mathbf{O}_p(HP) = 1$. Since $P \in \mathrm{Syl}_p(HP)$ and since G is a minimal counterexample, we have $G = HP$.

Next, we will show that P is abelian of order p^2. Let $P_1 \leq P$ with $|P : P_1| = p$. Since $H \leq HP_1$, we have $\mathbf{O}_p(HP_1) = 1$. Since $|HP_1| < |HP|$, the minimality of G implies that $|P_1| \leq p$. It follows that $|P| = p^2$ and that P is abelian.

Let $\psi \in \mathrm{Irr}(H)$ and $\chi \in \mathrm{Irr}(G|\psi)$. By Clifford's Theorem (6.1 of [1]), we have $\chi|_H = e\sum_{i=1}^{t} \psi_i$, where $\{\psi_i\}_{i=1}^{t}$ is the complete orbit of ψ in the conjugation action of G on $\mathrm{Irr}(H)$, labeled so that $\psi_1 = \psi$. Also et divides $|G : H| = |P| = p^2$, by Corollary 11.29 of [1]. Since $\chi(1) = et\psi(1)$ and since et divides p^2, the hypothesis on character degrees of G implies that $et \leq p$. It follows that et divides p. We claim that $e = 1$. Suppose, for a contradiction, that $e > 1$. Then $e = p$ and $t = 1$; consequently, ψ is G-invariant. Since H is a normal Hall subgroup of G, it follows that ψ extends to G (see Gallagher's Theorem 8.15 of [1]). Further, since P is abelian, every irreducible character of G that lies over ψ must be an extension (Corollary 6.17 of [1]); however, this contradicts $\chi_H = p\psi$. Thus, as claimed, $e = 1$, and it follows that $\chi|_H = \sum_{i=1}^{p} \psi_i$ or $\chi|_H = \psi$.

Now let A be a subgroup of P that fixes every character in $\mathrm{Irr}(H)$. By Lemma (2.2), $A \leq \mathbf{C}_P(H) \leq H$, and, since A is a p-group, this implies that $A = 1$. Thus, the action of P on $\mathrm{Irr}(H)$ is faithful. Next we will deduce that P must be an elementary abelian p-group. Let $I_P(\psi)$ denote the stabilizer in P of ψ. As we have seen, for every character $\psi \in \mathrm{Irr}(H)$, either $|P : I_P(\psi)| = 1$ or p; as a consequence, $|I_P(\psi)| > 1$. If P is cyclic, then P has a unique subgroup of order p. This subgroup would have to be contained in $I_P(\psi)$, for every character $\psi \in \mathrm{Irr}(H)$. This contradicts the fact that the action on $\mathrm{Irr}(H)$ is faithful. Thus P is not cyclic, and, therefore, is elementary abelian of order p^2.

Next we will show that $H = [H, P]$. By properties of coprime actions, we have that $H = [H, P] \cdot \mathbf{C}_H(P)$. If $X \leq P$, then $[H, X] \lhd G$, since H normalizes $[H, X]$ and P normalizes both H and X. Also, if X is nontrivial, then $[H, X]$ is nontrivial, since $\mathbf{C}_G(H) \leq H$. In particular, $[H, P]$ is an nontrivial normal subgroup of G. Now consider $[H, P] \cdot P$. Observe that $\mathbf{O}_p([H, P] \cdot P)$ and $[H, P]$ are disjoint normal subgroups of $[H, P] \cdot P$, therefore they centralize each other. Of course, $\mathbf{O}_p([H, P] \cdot P)$ is contained in P and centralizes $\mathbf{C}_H(P)$, thus it follows that $\mathbf{O}_p([H, P] \cdot P)$ centralizes $H = [H, P] \cdot \mathbf{C}_H(P)$. Since the action of P on H is faithful, we have $\mathbf{O}_p([H, P] \cdot P) = 1$. Further, $P \in \mathrm{Syl}_p([H, P] \cdot P)$; therefore, by the minimality of G, we have $G = [H, P] \cdot P$. Thus, we may conclude that $H = [H, P]$.

Let $M \leq H$ be a minimal normal subgroup of G; we will show that $M = [H, X]$ for some nontrivial subgroup $X \leq P$. Consider the group G/M. Since G is a minimal counterexample and since a Sylow p-subgroup of G/M is isomorphic to P, we must have $\mathbf{O}_p(G/M) > 1$. Let $X \leq P$ such that $\mathbf{O}_p(G/M) = XM/M$. Then $X > 1$ and $[H, X] \leq M$. As we have seen, $1 < [H, X] \lhd G$ for every nontrivial subgroup $X \leq P$; thus $M = [H, X]$.

We now make several observations about an arbitrary subgroup A of P, with $|A| = p$. We have seen that A must move some character in $\mathrm{Irr}(H)$. Also, we know that $1 < [H, A] \lhd G$.

Assume now that $[H, A] < H$, and let $B = \mathbf{O}_p(P \cdot [H, A])$. We will show that A and B have the following dual relationship:

(i) $[H, B] < H$ and $A = \mathbf{O}_p(P \cdot [H, B])$;
(ii) $B = \mathbf{C}_P([H, A])$ and $A = \mathbf{C}_P([H, B])$;
(iii) $|B| = |A| = p$ and $A \cap B = 1$, thus $P = AB$.

By properties of coprime actions, $H = [H, A] \cdot \mathbf{C}_H(A)$. Now consider the group $P \cdot [H, A]$. This is a proper subgroup of G, since $[H, A]$ is proper in H; also, $P \in \mathrm{Syl}_p(P \cdot [H, A])$. Since G is a minimal counterexample, it follows that $B > 1$. Next observe that $[H, A]$ and B are disjoint normal subgroups of $P \cdot [H, A]$, hence $B \leq \mathbf{C}_P([H,A])$. Using properties of coprime actions again, we have that $[[H,A],A] = [H,A]$, which is nontrivial. It follows that A does not centralize $[H,A]$, however, B does, and therefore $A \not\leq B$. Since B is nontrivial and $|P| = p^2$, it follows that $|B| = p$. Also, since A is nontrivial, we have that $P = AB$ and $A \cap B = 1$; thus statement (iii) is proved. Further, since $A \not\leq \mathbf{C}_P([H,A])$, we have $\mathbf{C}_P([H,A]) < P$. Thus $1 < B \leq \mathbf{C}_P([H,A]) < P$, and it follows that $B = \mathbf{C}_P([H,A])$. Thus the first statement in (ii) is proved.

Observe that, since B centralizes $[H,A]$ and since P is abelian, we have $[[A,H],B] = 1 = [[B,A],H]$. By the Three Subgroups Theorem, it follows that $[[H,B],A] = 1$, and thus $A \leq \mathbf{C}_P([H,B])$. Since A is not centralized by H, we have that $[H,B] < H$, and thus the first statement in (i) holds. Further, since $\mathbf{C}_P([H,B])$ is contained in the abelian group P, we have $\mathbf{C}_P([H,B]) \leq \mathbf{O}_p(P \cdot [H,B])$, and, since $[H,B] < H$, the same reasoning that we used to show that $\mathbf{O}_p(P \cdot [H,A])$ is proper in P yields that $\mathbf{O}_p(P \cdot [H,B])$ is proper in P. Since A is nontrivial, it follows that $A = \mathbf{C}_P([H,B]) = \mathbf{O}_p(P \cdot [H,B])$, and the rest of the assertion has been proved.

When the situation arises that $[H,A] < H$, we will call the group B thus identified the *dual* of A.

Now suppose that A is a subgroup of P of order p, and assume that $[H,A]$ is proper in H. We claim that $[H,A]$ is a minimal normal subgroup of G. Since $1 < [H,A] \triangleleft G$, we may fix a minimal normal subgroup M of G with $M \leq [H,A]$. As we have seen, $M = [H,X]$ for some nontrivial subgroup $X \leq P$. If $X = A$, then $M = [H,A]$ and the claim holds. Otherwise, $X \neq A$, in which case $P = XA$, and Lemma (3.1) yields

$$H = [H,X] \cdot [H,A] \leq M \cdot [H,A] \leq [H,A].$$

This contradicts the fact that $[H,A]$ is proper in H. Therefore $X = A$, and hence $[H,A]$ is minimal normal.

Continue to assume that A is a subgroup of P of order p with $[H,A]$ proper in H, and let B be the dual of A. We will show that $H = [H,A] \times [H,B]$, where $[H,A]$ and $[H,B]$ are minimal normal subgroups of G. Since $[H,A]$ is proper in H, the duality of A and B implies that $[H,B]$ is proper in H, and thus both $[H,A]$ and $[H,B]$ are minimal normal subgroups of G. Since $P = AB$, Lemma (3.1) yields $[H,A] \cdot [H,B] = H$. If $[H,A] \cap [H,B] > 1$, then, by the minimality of the factors, we have $[H,A] = [H,B]$, and hence $H = [H,A]$, which is a contradiction. It follows that $[H,A] \cap [H,B] = 1$, and thus $H = [H,A] \times [H,B]$.

We will now show that, in fact, there are no subgroups $A \leq P$ of order p with $[H,A]$ proper in H. Assume, for a contradiction, that such a subgroup A does exist, and let B be the dual of A. Then $P = A \times B$, and $H = [H,A] \times [H,B]$. We claim that

$$G = ([H,A] \cdot A) \times ([H,B] \cdot B).$$

Each of $[H,A] \cdot A$ and $[H,B] \cdot B$ is a subgroup of G. By duality, A centralizes $[H,B]$, and thus A centralizes $[H,B] \cdot B$. Also $[H,A]$ centralizes $[H,B]$, since these are disjoint normal subgroups, and $[H,A]$ centralizes B, by the duality of A and B. Thus $[H,A] \cdot A$ centralizes $[H,B] \cdot B$. Since $H = [H,A] \cdot [H,B]$ and $P = AB$, we have that $G = ([H,A] \cdot A) \cdot ([H,B] \cdot B)$. To see that this is a direct product, we consider the orders of the factors. Since H and P are direct products, we have that $|H| = |[H, A]| \cdot |[H, B]|$ and $|P| = |A| \cdot |B|$. Since $G = HP$, it follows that

$$|G| = |H| \cdot |P| = |A| \cdot |[H, A]| \cdot |B| \cdot |[H, B]|.$$

Finally, since $G = ([H, A] \cdot A) \cdot ([H, B] \cdot B)$, we can deduce that $([H, A] \cdot A) \cap ([H, B] \cdot B) = 1$, and thus $G = ([H, A] \cdot A) \times ([H, B] \cdot B)$.

Next, observe that A must move some character in $\mathrm{Irr}([H, A])$. If not, then by Lemma (2.2), A acts trivially on $[H, A]$. However, by properties of coprime actions, $[[H, A], A] = [H, A]$, which contradicts the fact that $[H, A] > 1$. Thus, the direct factor $[H, A] \cdot A$ has some irreducible character of degree divisible p. The same is true for $[H, B] \cdot B$, hence G must have an irreducible character of degree at least p^2 and this contradicts the hypothesis on $b(G)$. Therefore, for every subgroup A of order p, we must have $[H, A] = H$.

For our last observation before returning to character theory, we will show that H is the direct product of isomorphic nonabelian simple groups. First, we will show that H is a minimal normal subgroup of G. Let M be minimal normal in G with $M \leq H$. Then $M = [H, X]$ for a nontrivial subgroup $X \leq P$. Since $X > 1$, we have $H = [H, X]$, and therefore $H = M$. It now follows that H is the direct product of isomorphic simple groups. Further, if one of these direct factors of H is abelian, then H is abelian and Proposition (2.3) leads to a contradiction. It follows that H has the claimed structure.

Let $\psi \in \mathrm{Irr}(H)$, and assume that ψ is moved by P. Let $A = I_P(\psi)$, and let $\chi \in \mathrm{Irr}(G|\psi)$. We have seen that $1 < A < P$, and that χ_H is the sum of p distinct conjugates of ψ. Since $\chi(1) < p^2$, it follows that $\psi(1) \leq p - 1$. Now, let $C = \mathbf{C}_H(A)$; notice that $H = [H, A] \cdot C$ and that $P \leq \mathbf{N}_G(C)$, since P centralizes A which uniquely determines C. Also note that $\ker(\psi) < H$, since the trivial character is, of course, invariant.

We now consider ψ_C. By Theorem 13.14 of [1], which follows from the Glauberman correspondence, we have that $\psi_C = a\alpha + p\Phi$ where $\alpha \in \mathrm{Irr}(C)$, $a \equiv \pm 1 \pmod{p}$, and Φ is a, possibly zero, character of C. Since $\psi(1) \leq p - 1$, we have $\Phi = 0$, and $a = 1$ or $a = p - 1$. Thus, either $\psi_C = \alpha$ or $\psi_C = (p - 1)\alpha$, where $\alpha \in \mathrm{Irr}(C)$, and if the latter case holds, then α is linear. The character α is the Glauberman correspondent of ψ. Note that, since ψ is nontrivial, α is nontrivial as well. Next we will see that, in fact, the case $\psi_C = \alpha$ never occurs.

Suppose that $\psi_C = \alpha$. Let $\hat{\psi}$ be the canonical extension of ψ to the inertial subgroup $I = I_G(\psi)$; thus $\hat{\psi}$ is the unique extension of ψ to I with $o(\hat{\psi})$ a p'-number. The existence of $\hat{\psi}$ is guaranteed by Gallagher's Theorem (Corollary 6.28 of [1]). Let $\Re$ be an irreducible representation that affords $\hat{\psi}$. Then $[\Re(C), \Re(A)] = 1$, since C is centralized by A. Also, $\Re_C$ is irreducible, since it affords the irreducible character α. By Schur's Lemma, we have that $\Re_A$ is a scalar representation. Thus $[\Re(H), \Re(A)] = 1$, and it follows that $[H, A] \leq \ker(\hat{\psi})$. Further, since $[H, A] \leq H$, we have $[H, A] \leq \ker(\hat{\psi}) \cap H = \ker(\psi) < H$. However, we have shown that $[H, A] = H$ for every subgroup $A \leq P$ with $|A| = p$. This contradiction implies that, for every non-P-invariant character $\psi \in \mathrm{Irr}(H)$, we have $\psi_C = (p - 1)\alpha$ for some nontrivial linear character $\alpha \in \mathrm{Irr}(C)$.

For the final contradiction, we consider what is known about ψ. Since ψ is not P-invariant, we have $\psi \neq 1_H$, and thus its Glauberman correspondent α is a nontrivial linear character of C. Since α is nontrivial, C is not contained in $\ker(\psi)$, and since α is linear, $\mathbf{Z}(\psi)$ is contained in C; thus $\mathbf{Z}(\psi) > \ker(\psi)$. Since each of $\mathbf{Z}(\psi)$ and $\ker(\psi)$ is normal in H, it follows that H has a nontrivial abelian (in fact, cyclic) section $\mathbf{Z}(\psi)/\ker(\psi)$. This contradicts the fact that H is the direct product of nonabelian simple groups. Thus, our minimal counterexample G cannot exist. $\square$

3. Proof of Theorem B

In this section, we consider what can be said if the character degree hypothesis of Theorem A is weakened to $b(G) \leq p^\alpha$. If attention is restricted to solvable groups, then, as a direct consequence of a theorem of D. Passman, Theorem B is gained.

Proof of Theorem B. Without loss of generality, we may assume that $1 = \mathbf{O}_p(G) < G$. For $P \in \mathrm{Syl}_p(G)$, our aim is to show that $|P| \leq p^{2\alpha}$. Set $H = \mathbf{O}_{p'}(G)$, and note that H is nontrivial, since G is solvable with $\mathbf{O}_p(G) = 1$. As in the proof of Theorem A, we may assume that $G = HP$.

Next we show that we may assume that H is nilpotent, or equivalently, that $H = \mathbf{F}(G)$. Since $\mathbf{O}_p(G) = 1$, we have $\mathbf{F}(G) \leq H$. Further, since G is solvable, $\mathbf{C}_G(\mathbf{F}(G)) \leq \mathbf{F}(G)$. It follows that $\mathbf{O}_p(\mathbf{F}(G) \cdot P) = 1$, and, since $b(\mathbf{F}(G) \cdot P) \leq b(G) \leq p^\alpha$, the hypotheses hold in the group $\mathbf{F}(G) \cdot P$. Since $P \in \mathrm{Syl}_p(\mathbf{F}(G) \cdot P)$, we may hence assume that H is nilpotent.

Now, we consider the coprime action of P on the abelian group $\mathrm{Irr}(H/\Phi(H))$. Since the action of P on the nilpotent group H is coprime and faithful, it follows that the action of P on the abelian group $H/\Phi(H)$ is faithful, and thus, by Lemma (2.2), the action of P on $\mathrm{Irr}(H/\Phi(H))$ is faithful. By Corollary 2.4 of [3], there exists $\lambda \in \mathrm{Irr}(H/\Phi(H))$, such that the P-orbit of λ has size at least $\sqrt{|P|}$, and thus for any character $\chi \in \mathrm{Irr}(G|\lambda)$, we have $\chi(1) \geq \sqrt{|P|}$. Since $p^\alpha \geq b(G) \geq \chi(1)$, it follows that $p^\alpha \geq \sqrt{|P|}$, and therefore $|P| \leq p^{2\alpha}$.

Finally, we observe that, if $|G|$ is odd, then Corollary 2.4 of [3] asserts that there exists $\lambda \in \mathrm{Irr}(H/\Phi(H))$, such that the P-orbit of λ has size at least $|P|$. In this case, it follows that $|P| \leq p^\alpha$. $\square$

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Department of Mathematics, University of Wisconsin – Platteville, Platteville, Wisconsin, 53818

E-mail address: benjamin@uwplatt.edu