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TOEPLITZ C^*-ALGEBRAS ON ORDERED GROUPS AND THEIR IDEALS OF FINITE ELEMENTS

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Abstract. Let G be a discrete abelian group and (G, G_+) an ordered group. Denote by (G, G_F) the minimal quasily ordered group containing (G, G_+). In this paper, we show that the ideal of finite elements is exactly the kernel of the natural morphism between these two Toeplitz C^*-algebras. When G is countable, we show that if the direct sum of K-groups $K_0(\mathcal{T}^{G_+}) \oplus K_1(\mathcal{T}^{G_+}) \cong \mathbb{Z}$, then $K_0(\mathcal{T}^{G_+}) \cong \mathbb{Z}$.

1. Preliminaries and the main result

Let G be a discrete abelian group and G_+ a subset of G, (G, G_+) is said to be a quasily ordered group, if $0 \in G_+, G_+ + G_+ \subseteq G_+$, and $G = G_+ \cup (-G_+)$. If in addition, $G_+^0 = G_+ \cap (-G_+) = \{0\}$, then (G, G_+) is referred to as an ordered group.

In this section, we always fix an ordered group (G, G_+). Let $\widehat{G}$ be the dual group of G, $\widehat{G}$ is compact and it is connected since G is torsion-free. When given the normal Haar measure, it is easy to show that $\{\, \varepsilon_x \,|\, x \in G \,\}$ is an orthonormal basis for $L^2(\widehat{G})$, where ε_x is defined by $\varepsilon_x(\gamma) = \gamma(x)$ for $\gamma \in \widehat{G}$.

For any $E \subseteq G$, let $H^2(E)$ be the closed subspace of $L^2(\widehat{G})$ generated by $\{\, \varepsilon_x \,|\, x \in E \,\}$; its projection is denoted by p^E. For any $\varphi \in C(\widehat{G})$, define T_φ^E on $H^2(E)$ by $T_\varphi^E(f) = p^E(\varphi f)$ for $f \in H^2(E)$. We say that T_φ^E is a Toeplitz operator (relative to E) with symbol φ. The C^*-algebra generated by $\{\, T_\varphi^E \,\big|\, \varphi \in C(\widehat{G}) \,\}$ is denoted by $\mathcal{T}_r^E$, and is called the Toeplitz C^*-algebra with respect to E. By the Stone-Weierstrass theorem, $\mathcal{T}_r^E$ is also generated by $\{\, T_{\varepsilon_x}^E \,\big|\, x \in G \,\}$.

Now let (G, G_+) be an ordered group, denote by $F(G) = F(G_+) \cup (-F(G_+))$, where

$$F(G_+) = \{\, x \in G_+ \,\big|\, \forall y \in G_+ \setminus \{0\}, \exists n \in N, \text{ such that } ny - x \in G_+ \}.$$

It is easy to show that $F(G)$ is a subgroup of G and its elements are usually called the finite elements. Denote by $K(F(G))$, the closed two-sided ideal of $\mathcal{T}_r^{G_+}$

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generated by

$$\{\,1 - T^{G_+}_{\varepsilon_x} T^{G_+}_{\varepsilon_{-x}} \,\big|\, x \in F(G_+)\,\}.$$

Let $G_F = G_+ \cup F(G)$, then obviously $G_F^0 = F(G)$, and it is easy to show that

$$G_F = \bigcap_{g \in G_+ \setminus \{0\}} (G_+ - Z_+ g) = G_+ - F(G_+).$$

So when $F(G) \neq \{0\}$, (G, G_F) is the minimal quasily ordered group containing (G, G_+).

A representation of an ordered group (G, G_+) by isometries on a Hilbert space H, is a map $V : G_+ \to \mathbb{B}(H)$ such that

$$V(0) = 1, V(x)^* V(x) = 1 \text{ and } V(x)V(y) = V(x+y) \text{ for all } x, y \in G_+.$$

The Toeplitz C^*-algebra $\mathcal{T}_r^{G_+}$ has a universal property for isometric representations, that is, if (V, H) is any isometric representation of G_+, then there is a unique C^*-morphism $\pi_V : \mathcal{T}_r^{G_+} \to \mathbb{B}(H)$, such that $\pi_V(T^{G_+}_{\varepsilon_x}) = V(x)$ for all $x \in G_+$. Furthermore, if every $V(x), x \neq 0$ is non-unitary, then π_V is isometric (see [1], Theorem 1.3 and Theorem 2.9 or [5], Section 5.2). So for any quasily ordered group (G, E) with $G_+ \subseteq E$, there is a C^*-algebra morphism $\gamma^{E, G_+} : \mathcal{T}_r^{G_+} \to \mathcal{T}_r^E$ such that

$$\gamma^{E, G_+}(T^{G_+}_{\varepsilon_g}) = T^E_{\varepsilon_g} \text{ for all } g \in G_+.$$

Obviously, $K(F(G)) \subseteq Ker\, \gamma^{G_F, G_+}$. The main result of this paper is, we prove that $K(F(G))$ is equal to $Ker\, \gamma^{G_F, G_+}$.

Remark. By an isometric representation (V, H) of a quasily ordered group (G, G_F), we mean H is a Hilbert space and $V : G_F \to \mathbb{B}(H)$ is a map satisfying

$$V(0) = 1, V^*(x)V(x) = 1, V(x+y) = V(x)V(y) \text{ for all } x, y \in G_F;$$
$$V(x)V(x)^* = 1 \text{ for all } x \in F(G).$$

The following lemma is crucial to our proof of Theorem 1.2, we defer its proof to Section 3.

Lemma 1.1. *Any isometric representation (V, H) of G_F on a Hilbert space H can be lifted to be a C^*-algebra morphism $\pi_V : \mathcal{T}_r^{G_F} \to \mathbb{B}(H)$, such that $\pi_V(T^{G_F}_{\varepsilon_x}) = V(x)$ for all $x \in G_F$.*

Theorem 1.2. *Let (G, G_+) be an ordered group and $F(G)$ the subgroup of finite elements, then $K(F(G)) = Ker\, \gamma^{G_F, G_+}$.*

Proof. It needs only to prove "$Ker\, \gamma^{G_F, G_+} \subseteq K(F(G))$". Let ρ be the natural morphism induced by γ^{G_F, G_+} from $\mathcal{T}_r^{G_+}/K(F(G))$ to $\mathcal{T}_r^{G_F}$, then by Lemma 1.1 ρ has an inverse π_V lifted by the isometric representation of G_F on $\mathcal{T}_r^{G_+}/K(F(G))$ defined by

$$V(x) = [\,T^{G_+}_{\varepsilon_x}\,] \in \mathcal{T}_r^{G_+}/K(F(G)) \quad \text{for } x \in G_F,$$

therefore, $Ker\, \gamma^{G_F, G_+} \subseteq K(F(G))$. □

Let (G, G_+) be as above and let V be a representation of G_+ by isometries. If for every $x \in G_+$ with $x \neq 0$, $V(x)$ is non-unitary, then $\mathcal{T}_r^{G_+} \cong C^*(\{\,V(x) \,|\, x \in G_+\})$. On the other hand, if there does exist some $y \in G_+, y \neq 0$ such that $V(y)$ is a unitary, then it is easy to show that V is also an isometric representation of G_F (when $x \notin G_+$, we define $V(x) = V(-x)^*$). Indeed, for any $x \in F(G_+)$, by definition there is an $n \in N$ such that $ny - x \in G_+$. So

$$V(ny-x)V(x)V(x)^*V(ny-x)^* = V(ny)V(ny)^* = V(y)^n(V(y)^n)^* = 1.$$

Multiply with $V(ny-x)^*$ on the left and $V(ny-x)$ on the right, we obtain $V(x)V(x)^* = 1$. Therefore, we have the following

Corollary 1.3. *Let (G, G_+) be an ordered group with $F(G) \neq \{0\}$. Then every non-faithful representation π of $\mathcal{T}_r^{G_+} \to \mathbb{B}(H)$ can be factored through $\mathcal{T}_r^{G_F}$, in the sense that there exists a representation $\pi_F : \mathcal{T}_r^{G_F} \to \mathbb{B}(H)$ such that $\pi = \pi_F \circ \gamma^{G_F,G_+}$.*

Remark. (1) Let (G, G_+) be an ordered group and $F(G)$ the subgroup of finite elements. By Corollary 1.3 and Theorem 1.2, it is easy to show that $K(F(G))$ is contained in any closed two-sided ideal of $\mathcal{T}^{G_+}$. In particular, $K(F(G))$ is simple and $\mathcal{T}_r^{G_+}$ is prime. Such a property was also established by G. Murphy in a far different way (see [1], Theorem 2.10 and Theorem 2.11). Our proof here in some sense is sensibly simpler than the original one.

(2) Suppose $F(G) \neq \{0\}$. If G admits a least positive element, equivalently, $\mathcal{T}_r^{G_+}$ contains a Fredholm operator of non-zero index, or $K(F(G))$ is the ideal of compact operators on $H^2(G_+)$ ([2], Lemma 2.2), then by Theorem 1.2 we know that

$$\forall T \in \mathcal{T}_r^{G_+},\ T \text{ is Fredholm if and only if } \gamma^{G_F,G_+}(T) \text{ is invertible in } \mathcal{T}_r^{G_F}.$$

When $(G, G_+) = (\mathbb{Z}, \mathbb{Z}_+)$, it reduces obviously to the classical case. A precise character of a single Toeplitz operator with continuous symbol to be Fredholm can be found in ([2], Theorem 2.5 and Theorem 4.2), and a generalized version of [2] has been given in [8].

2. A CHARACTER OF K-GROUPS OF TOEPLITZ C^*-ALGEBRAS ON ORDERED GROUPS

Let A be a C^*-algebra, denote by $K_0(A)$ and $K_1(A)$ the K_0-group and K_1-group of A respectively. If $K_0(A) \oplus K_1(A) \cong \mathbb{Z}$, then obviously either $K_0(A) \cong \mathbb{Z}, K_1(A) = 0$ or $K_1(A) \cong \mathbb{Z}, K_0(A) = 0$. Let $\mathcal{T}$ be the classical Toeplitz C^*-algebra, it is well-known that $K_0(\mathcal{T}) \cong \mathbb{Z}$ and $K_1(\mathcal{T}) = 0$. On the other hand, there are many C^*-algebras, such as $\mathbb{B}/\mathbb{K}, C_0(\mathbb{R}^{2n+1})$, their K_1-groups are all isomorphic to $\mathbb{Z}$ and their K_0-groups are all equal to zero. Quite surprisingly, in this section, we show that when G is a countably infinite discrete abelian group and (G, G_+) is an ordered group, if the direct sum of the K-groups $K_0(\mathcal{T}_r^{G_+}) \oplus K_1(\mathcal{T}_r^{G_+})$ is isomorphic to $\mathbb{Z}$, then $K_0(\mathcal{T}_r^{G_+})$ must be isomorphic to $\mathbb{Z}$ (of course in this case $K_1(\mathcal{T}_r^{G_+})$ must be equal to zero).

In this section, we always assume G is a countably infinite discrete abelian group. Let (G, G_+) be an ordered group, a subgroup I of G is said to be an ideal, if $0 \leq x \leq y \in I \cap G_+$ implies $x \in I$ for all $x \in G$, where "$\leq$" is the usual order on G induced by G_+. For any $x \in G$, set $V_x = T_{\varepsilon_x}^{G_+} \in \mathcal{T}_r^{G_+}$. Let I be an ideal of G, we define $\mathcal{T}(G, I)$ to be the closed ideal in $\mathcal{T}_r^{G_+}$ generated by $\{\, 1 - V_xV_x^* \mid x \in I \cap G_+ \,\}$. If $(G_1, (G_1)_+), \ldots, (G_n, (G_n)_+)$ are ordered groups, we denote by $(G_1 \times \cdots \times G_n, G_1 * \cdots * G_n)$ the *lexicographic-ordered* group.

Lemma 2.1 ([3]). *Let $(G_1,(G_1)_+)$ and $(G_2,(G_2)_+)$ be two ordered groups. Then there is a C^*-morphism $\alpha : \mathcal{T}_r^{G_1 * G_2} \to C(\widehat{G_2}) \otimes \mathcal{T}_r^{G_1}$ such that for all $(x,y) \in G_1 * G_2$, we have $\alpha(V_{(x,y)}) = \varepsilon_y \otimes V_x$. Moreover Ker $\alpha = \mathcal{T}(G_1 \times G_2, G_2)$.*

Theorem 2.2. *Let (G, G_+) be an ordered group. If $K_0(\mathcal{T}_r^{G_+}) \oplus K_1(\mathcal{T}_r^{G_+}) \cong \mathbb{Z}$, then $K_0(\mathcal{T}_r^{G_+}) \cong \mathbb{Z}$ and $K_1(\mathcal{T}_r^{G_+}) = 0$.*

Proof. We identify $\{0\} \times \mathbb{Z}$ with $\mathbb{Z}$. By the above lemma, we have the following short exact sequence of C^*-algebras:

$$0 \longrightarrow \mathcal{T}(G \times \mathbb{Z}, \mathbb{Z}) \xrightarrow{\ i\ } \mathcal{T}_r^{G*\mathbb{Z}} \xrightarrow{\ \alpha\ } C(T) \otimes \mathcal{T}_r^{G_+} \longrightarrow 0$$

where i is the inclusion map and $\alpha(V_{(x,n)}) = \varepsilon_n \otimes V_x$ for any $(x,n) \in G * \mathbb{Z}$. Since the subgroup of finite elements in $G \times \mathbb{Z}$ is just $\mathbb{Z}$, by ([2], Lemma 2.2) we know that $\mathcal{T}(G \times \mathbb{Z}, \mathbb{Z}) = \mathbb{K}(H^2(G * \mathbb{Z}))$, where $\mathbb{K}(H^2(G * \mathbb{Z}))$ is the ideal of compact operators on $H^2(G * \mathbb{Z})$. So we have the following periodic six-term exact sequence of K-theory:

$$\begin{tikzcd}
\mathbb{Z} \cong K_0(\mathbb{K}) \arrow[r, "i_*"] & K_0(\mathcal{T}_r^{G*\mathbb{Z}}) \arrow[r, "\alpha_*"] & K_0(C(T) \otimes \mathcal{T}_r^{G_+}) \arrow[d, "\delta"'] \\
K_1(C(T) \otimes \mathcal{T}_r^{G_+}) \arrow[u, "\text{index map}"'] & K_1(\mathcal{T}_r^{G*\mathbb{Z}}) \arrow[l, "\alpha_*"] & K_1(\mathbb{K}) = 0 \arrow[l, "i_*"]
\end{tikzcd}$$

Since $C(T) \otimes \mathcal{T}_r^{G_+} \cong C(T \to \mathcal{T}_r^{G_+})$, by ([7], Exercise 8.B) (or simply use the Künneth theorem) we know that $K_0(C(T) \otimes \mathcal{T}_r^{G_+}) \cong K_1(C(T) \otimes \mathcal{T}_r^{G_+}) \cong K_0(\mathcal{T}_r^{G_+}) \oplus K_1(\mathcal{T}_r^{G_+}) \cong \mathbb{Z}$. If we also denote the index map by δ, then since $V_{(0,1)}$ is an isometry, by the definition of index map ([7], Definition 8.1.1) and ([7], Exercise 8.C) we know that $\delta[z \otimes 1_{\mathcal{T}_r^{G_+}}] = -[1 - V_{(0,1)}V_{(0,1)}^*] \in K_0(\mathbb{K})$, where $z \in C(T)$ is the inclusion of T into $\mathbb{R}$. Note that any homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ is injective iff it is non-zero, and it is surjective iff 1 or -1 is in its range. Obviously, the rank of $1 - V_{(0,1)}V_{(0,1)}^*$ is 1, so the index map in the exact sequence is an isomorphism; therefore,

$$K_0(\mathcal{T}_r^{G*\mathbb{Z}}) \cong \mathbb{Z}, \qquad K_1(\mathcal{T}_r^{G*\mathbb{Z}}) = 0.$$

The map

$$G * \mathbb{Z} \to \mathcal{T}_r^{G_+}, \quad (x,n) \to V_x,$$

is an isometric homomorphism, and therefore induces a C^*-morphism $\rho : \mathcal{T}_r^{G*\mathbb{Z}} \to \mathcal{T}_r^{G_+}$ such that $\rho(V_{(x,n)}) = V_x$ for all $(x,n) \in G * \mathbb{Z}$. Similarly, there is a C^*-morphism $\theta : \mathcal{T}_r^{G_+} \to \mathcal{T}_r^{G*\mathbb{Z}}$ such that $\theta(V_x) = V_{(x,0)}$ for all $x \in G_+$. Since $\mathcal{T}_r^{G_+}$ is generated by $\{\, V_x \,|\, x \in G_+ \,\}$, we know that $\rho \circ \theta = id_{\mathcal{T}_r^{G_+}}$, so $\rho_* \circ \theta_* = id_{K_1(\mathcal{T}_r^{G_+})}$, it follows that the map $\rho_* : K_1(\mathcal{T}_r^{G*\mathbb{Z}}) \to K_1(\mathcal{T}_r^{G_+})$ is onto, so $K_1(\mathcal{T}_r^{G_+})$=0. ☐

Remark. It is easy to show by induction that for any $n \in N$, $K_0(\mathcal{T}^{\mathbb{Z}*\cdots*\mathbb{Z}}) \cong \mathbb{Z}$ and $K_1(\mathcal{T}^{\mathbb{Z}*\cdots*\mathbb{Z}}) = 0$, and this was established in [4]. Next we give another Toeplitz C^*-algebra, whose K_0-group is also isomorphic to $\mathbb{Z}$ and whose K_1-group is trivial.

Example. Let

$$G = \mathbb{Z}^2, G_+ = \{\, (m,n) \in \mathbb{Z}^2 \,\big|\, m + n > 0, \text{ or } m + n = 0 \text{ and } m \geq 0 \,\}.$$

Clearly, (G, G_+) is an ordered group, $(1, -1)$ is the least positive element of G and

$$F(G) = \{\, (n, -n) \,\big|\, n \in \mathbb{Z} \,\}.$$

So

$$G_F = \{\, (m, n) \in \mathbb{Z}^2 \,\big|\, m + n \geq 0 \,\}.$$

By Theorem 1.2, we have the following short exact sequence of C^*-algebras:

$$0 \longrightarrow \mathbb{K}(H^2(G_+)) \xrightarrow{\ i\ } \mathcal{T}_r^{G_+} \xrightarrow{\gamma^{G_F, G_+}} \mathcal{T}_r^{G_F} \longrightarrow 0.$$

Therefore, we have the following six-term exact sequence of K-theory:

$$\begin{tikzcd}
\mathbb{Z} \cong K_0(\mathbb{K}) \arrow[r, "i_*"] & K_0(\mathcal{T}_r^{G_+}) \arrow[r, "(\gamma^{G_F, G_+})_*"] & K_0(\mathcal{T}_r^{G_F}) \arrow[d, "\delta"'] \\
K_1(\mathcal{T}_r^{G_F}) \arrow[u, "\text{index map}"'] & K_1(\mathcal{T}_r^{G_+}) \arrow[l, "(\gamma^{G_F, G_+})_*"] & K_1(\mathbb{K}) = 0 \arrow[l, "i_*"]
\end{tikzcd}$$

By ([6], Section 3 (set $\alpha = -1$)), we know that $\mathcal{T}_r^{G_F} \cong C(T) \otimes \mathcal{T}$, where $\mathcal{T}$ is the classical Toeplitz C^*-algebra, so $K_0(\mathcal{T}_r^{G_F}) \cong \mathbb{Z} \cong K_1(\mathcal{T}_r^{G_F})$. As before we know that the index map is an isomorphism, therefore we know that

$$K_0(\mathcal{T}_r^{G_+}) \cong \mathbb{Z}, \quad K_1(\mathcal{T}_r^{G_+}) = 0.$$

3. The proof of Lemma 1.1

The ideals of this section are mostly contained in [5]. Roughly speak, the quasi-lattice ordered groups discussed in [5] are "ordered groups". What we consider here are the quasily ordered groups; in other words, the unit space may be not trivial.

We deal with Toeplitz C^*-algebras in another context.

Let G be a discrete (not necessary to be abelian) group and P a subsemigroup of G. Then (G, P) is said to be a quasily ordered group, if

$$e \in P,\ P \cdot P \subseteq P, \text{ and } G = P \cup P^{-1},$$

where e is the unit of G and $P^{-1} = \{\, x^{-1} \,\big|\, x \in P \,\}$.

Now let (G, P) be a quasily ordered group, denote by $G_+^0 = P \cap P^{-1}$, it is a subgroup of G. Since $(P \setminus G_+^0) \cdot P = (P \setminus G_+^0) = P \cdot (P \setminus G_+^0)$, if we let $G_+ = (P \setminus G_+^0) \cup \{e\}$, then $G = G_+ \cup G_+^0 \cup (G_+)^{-1}$ and (G, G_+) is a partially ordered group in the sense that

$$e \in G_+, G_+ \cdot G_+ \subseteq G_+, G_+ \cap G_+^{-1} = \{e\}, \text{ and } G = G_+ \cdot (G_+)^{-1}.$$

For any $x, y \in G$, we say

$$x \leq y \text{ (resp. } x \ll y,\ x \sim y \text{) if } x^{-1}y \in G_+ \text{ (resp. } x^{-1}y \in P,\ x^{-1}y \in G_+^0 \text{)}.$$

Let $\{\, \delta_g \,|\, g \in G\}$ be the usual orthonormal basis for $\ell^2(G)$, where

$$\delta_g(h) = \begin{cases} 1, & \text{if } g = h, \\ 0, & \text{otherwise.} \end{cases} \qquad \text{for } g, h \in G.$$

For any quasily ordered group (G, P), another Toeplitz C^*-algebra $W(G, P)$ can be defined as the C^*-subalgebra of $\mathbb{B}(\ell^2(P))$ generated by $\{J^* L_t J \,|\, t \in G\}$ with $L : G \to \mathbb{B}(\ell^2(G))$ the left regular representation and $J : \ell^2(P) \to \ell^2(G)$ the inclusion operator. For any $t \in P$, the operator $J^* L_t J$, which will now be denoted by $w(t)$ and let $w(t)w(t)^* = m(t)$ as in [5] for any $t \in P$.

The formulas listed in Proposition 3.1 below are clear:

Proposition 3.1. *(1)* $w(s)w(t) = w(st), w(t)^*w(t) = 1, w(s) = w(t) \Leftrightarrow s = t$, $\forall s, t \in P$.

(2) $w(s_1)w(t_1)^* = w(s_2)w(t_2)^* \Leftrightarrow t_1 \sim t_2$ *and* $s_1t_1^{-1} = s_2t_2^{-1}$.

(3) $w(s)^*w(t) = \begin{cases} w(s^{-1}t), & \text{if } s \ll t, \\ w(t^{-1}s)^*, & \text{if } t \ll s. \end{cases} \quad \forall s, t \in P.$

(4) $m(s)m(t) = \begin{cases} m(s), & \text{if } t \ll s, \\ m(t), & \text{if } s \ll t. \end{cases} \quad m(s) = m(t) \Leftrightarrow s \sim t, \forall s, t \in P.$

Proposition 3.2. *(1) Let* $\mathcal{A} = sp\{\, w(s)w(t)^* \mid s, t \in P\}$, *then* $\mathcal{A}$ *is a dense unital* $*$*-subalgebra of* $W(G, P)$.

(2) The operators $\{\, w(s)w(t)^* \mid s, t \in P\}$ *are linear independent in the sense that, if* $\sum_j \lambda_j w(s_j)w(t_j)^* = 0$ *with* $w(s_{j_1})w(t_{j_1})^* \neq w(s_{j_2})w(t_{j_2})^*$ *when* $j_1 \neq j_2$, *then* $\lambda_j = 0$ *for all* j.

Proof. (1) It suffices to show for any $s_1, s_2, t_1, t_2 \in P$, $w(s_1)w(t_1)^*w(s_2)w(t_2)^* \in \mathcal{A}$, and this is clear by Proposition 3.1.

(2) If $T = \sum_j \lambda_j w(s_j)w(t_j)^* = 0$ with $w(s_{j_1})w(t_{j_1})^* \neq w(s_{j_2})w(t_{j_2})^*$ whenever $j_1 \neq j_2$, we show $\lambda_j = 0$ for all j. Without loss of generality, we may assume $t_1 \ll t_2 \ll \cdots \ll t_n$.

Suppose $t_1 \sim t_2 \sim \cdots \sim t_{k_1} < t_{k_1+1}$, then by $T\delta_{t_1} = 0$, we know that

$$\sum_{j=1}^{k_1} \lambda_j \delta_{s_j t_j^{-1} t_1} = 0.$$

If $\lambda_1 \neq 0$, then there exists $j \in \{2, 3, \ldots, k_1\}$ such that $s_j t_j^{-1} t_1 = s_1$, so $s_j t_j^{-1} = s_1 t_1^{-1}$, which is a contradiction since $w(s_j)w(t_j)^* \neq w(s_1)w(t_1)^*$. Similarly, we have $t_2, t_3, \ldots, t_{k_1} = 0$. Continue the above process, eventually we have $\lambda_j = 0$ for all j. ☐

Denote by $\mathcal{D} = \text{clos sp}\{m(t) \mid t \in P\}$. Clearly it is a unital abelian C^*-subalgebra of $W(G, P)$. Denote by $\mathcal{D}_0$ the C^*-subalgebra of $B(\ell^2(P))$ consisting of all the operators having diagonal matrix relative to the canonical basis. It is well-known that there exists a linear and contractive map $E_0 : B(\ell^2(P)) \to \mathcal{D}_0$ determined by the following rule: the matrix of $E_0(T)$ (relative to the canonical basis) is obtained from the one of T by replacing with zero all the entries which are not situated on the principal diagonal.

Proposition 3.3 (cf. [5], Section 3.3 and Section 3.6)**.** *(1)* $\mathcal{D} = \{T \in W(G, P) \mid T$ *has diagonal matrix relative to the canonical basis of* $\ell^2(P)\}$.

(2) Let $E = E_0|_{W(G,P)}$, *then* E *is a faithful bounded linear map from* $W(G, P)$ *to* $\mathcal{D}$ *such that for any* s, t *in* P:

$$E(w(s)w(t)^*) = \begin{cases} w(s)w(t)^*, & \text{if } s = t, \\ 0, & \text{if } s \neq t. \end{cases}$$

Proposition 3.4. *Let* $\{\, L(t) \mid t \in P\,\}$ *be a family of non-zero projections of the unital* C^**-algebra* $\mathcal{B}$ *with* $L(e) = 1$. *Then there exists a* C^**-algebra morphism* ρ :

$\mathcal{D} \to \mathcal{B}$ such that $\rho(m(t)) = L(t), \forall t \in P$, if and only if for any $x, y, s, t \in P$:

$$(*) \qquad L(x) = L(y) \ \text{ whenever } x \sim y, \quad \text{and } L(s)L(t) = \begin{cases} L(s), & \text{if } t \ll s, \\ L(t), & \text{if } s \ll t. \end{cases}$$

For L as above, ρ is faithful if and only if, $L(s) = L(t) \Leftrightarrow s \sim t$ for any $s, t \in P$.

Proof. "$\Longrightarrow$" clearly follows from Proposition 3.1. To prove "$\Longleftarrow$" it suffices to show that, for any $t_1, t_2, \ldots, t_n \in P$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in C$:

$$\Big\| \sum_{j=1}^{n} \lambda_j L(t_j) \Big\| \le \Big\| \sum_{j=1}^{n} \lambda_j m(t_j) \Big\|.$$

Without loss of generality, we may assume $t_1 \ll t_2 \ll \cdots \ll t_n$, more explicitly,

$$t_1 \sim \cdots \sim t_{k_1} < t_{k_1+1} \sim \cdots \sim t_{k_2} < \cdots < t_{k_{m-1}+1} \sim \cdots \sim t_{k_m} = t_n.$$

Let $s_i = \lambda_1 + \cdots + \lambda_{k_i}$ for $i = 1, \ldots, m$, and $L_{i,i+1} = L(t_{k_i}) - L(t_{k_{i+1}})$ for $i = 1, \ldots, m-1$. Since $T = \sum_{j=1}^{n} \lambda_j m(t_j)$ has diagonal matrix,

$$\begin{aligned} \Big\| \sum_{j=1}^{n} \lambda_j m(t_j) \Big\| &= \sup_{a \in P} \big| \langle T\delta_a, \delta_a \rangle \big| = \sup_{a \in P} \Big| \sum_{j} \lambda_j \big\langle m(t_j)\, \delta_a, \delta_a \big\rangle \Big| \\ &= \max\{ |s_1|, |s_2|, \ldots, |s_m| \}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{j=1}^{n} \lambda_j L(t_j) &= (\lambda_1 + \cdots + \lambda_{k_1}) L(t_{k_1}) + \cdots + (\lambda_{k_{m-1}+1} + \cdots + \lambda_{k_m}) L(t_{k_m}) \\ &= s_1 L_{1,2} + s_2 L_{2,3} + \cdots + s_{m-1} L_{m-1,m} + s_m L(t_{k_m}). \end{aligned}$$

$$\text{So} \qquad \Big\| \sum_{j=1}^{n} \lambda_j L(t_j) \Big\| = \max\{ \max_{1 \le i \le m-1} \{ |s_i| \, \big| L(t_{k_i}) \ne L(t_{k_{i+1}}) \}, |s_m| \}.$$

Thus we obtain

$$\Big\| \sum_{j=1}^{n} \lambda_j L(t_j) \Big\| \le \Big\| \sum_{j=1}^{n} \lambda_j m(t_j) \Big\|.$$

The proof above shows also that, for L as above, ρ is faithful if and only if $L(s) = L(t) \Leftrightarrow s \sim t$ for any $s, t \in P$. □

Remark. Let V be a representation by isometries of P on some Hilbert space H, i.e.

$$\begin{aligned} &V(t)^* V(t) = 1, \forall t \in P; V(s)V(t) = V(st), \forall s, t \in P; \\ &V(e) = 1; V(s)V(s)^* = 1, \forall s \in G_+^0, \end{aligned}$$

then V is always *covariant* in the sense that the condition $(*)$ stated in Proposition 3.4 holds with $L(t) = V(t)V(t)^*, \forall t \in P$, and if $w(s_1)w(t_1)^* = w(s_2)w(t_2)^*$ for any $s_1, s_2, t_1, t_2 \in P$, then $V(s_1)V(t_1)^* = V(s_2)V(t_2)^*$. In fact, by Proposition 3.1, we know that $t_1 \sim t_2$ and $s_1 t_1^{-1} = s_2 t_2^{-1}$. Let $a = t_1^{-1} t_2 \in G_+^0$, then

$$\begin{aligned} V(s_2)V(t_2)^* &= V(s_1 a)V(t_1 a)^* = V(s_1)V(a)(V(t_1)V(a))^* \\ &= V(s_1)V(a)V(a)^* V(t_1)^* = V(s_1)V(t_1)^*. \end{aligned}$$

It follows by Proposition 3.2 that there is a unique linear operator $\pi_V : \mathcal{A} \to \mathbb{B}(H)$ such that

$$\pi_V(w(s)w(t)^*) = V(s)V(t)^* \text{ for all } s, t \in P,$$

which directly shows that for all $s_1, s_2, t_1, t_2 \in P$:

$$\pi_V(w(s_1)w(t_1)^*w(s_2)w(t_2)^*) = \pi_V(w(s_1)w(t_1)^*)\pi_V(w(s_2)w(t_2)^*).$$

So π_V is actually a *-representation.

For any $T \in \mathcal{A}$, let $T = \sum_j \lambda_j w(s_j)w(t_j)^*$, define

$$\|T\| = \sup\{\|\pi(T)\| \,\big|\, \pi \text{ is a unital } *\text{-representation of } \mathcal{A}\}.$$

$\|T\|$ is finite and actually not greater than $\sum_j |\lambda_j|$, because each $w(s_j)w(t_j)^*$ is a partial isometry. On the other hand, the canonical identification $id : \mathcal{A} \to \mathcal{A} \subseteq W(G, P)$ gives an injective unital $*$-representation, hence $\|T\| > 0$ if $T \neq 0$. It follows immediately that $\| \cdot \|$ is a C^*-norm on $\mathcal{A}$.

Definition. The completion of $\mathcal{A}$ with respect to $\|\cdot\|$ will be denoted by $C^*(G, P)$ and will be called the universal C^*-algebra of (G, P). (G, P) is said to be *amenable* if $\pi_{id} : C^*(G, P) \to W(G, P)$ is one-to-one.

Remark. (G, P) is amenable if and only if , for every isometric representation $V : P \to \mathbb{B}(H)$, there is a C^*-algebra morphism $\pi_V : W(G, P) \to \mathbb{B}(H)$ such that $\pi_V(w(t)) = V(t)$ for all $t \in P$.

The proof of the following theorem is essentially the same as that of [5], the details can be found in ([5], Section 4.3, Section 4.4 and Section 4.5).

Theorem 3.5. *If G is amenable, then (G, P) is amenable.*

Now suppose G is a discrete abelian group and (G, G_+) is an ordered group. Let $P = G_F$ and define a unitary $U : \ell^2(G) \to L^2(\widehat{G})$ by $U\delta_g = \varepsilon_g$ for $g \in G$, then $U^* \circ T^{G_+}_{\varepsilon_g} \circ U = w(g)$ for all $g \in G_+$, so the two Toeplitz C^*-algebras $\mathcal{T}^{G_+}_r$ and $W(G, G_+)$ are unitarily equivalent. Since in this case G is amenable, by Theorem 3.5 we know that the Lemma 1.1 stated in Section 1 now holds.

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