

SOME RAPIDLY CONVERGING SERIES FOR $\zeta(2n + 1)$

H. M. SRIVASTAVA

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ABSTRACT. For a natural number n , the author derives several families of series representations for the Riemann Zeta function $\zeta(2n + 1)$. Each of these series representing $\zeta(2n + 1)$ converges remarkably rapidly with its general term having the order estimate:

$$O(k^{-2n-1} \cdot m^{-2k}) \quad (k \rightarrow \infty; \quad m = 2, 3, 4, 6).$$

Relevant connections of the results presented here with many other known series representations for $\zeta(2n + 1)$ are also pointed out.

1. INTRODUCTION AND PRELIMINARIES

The Riemann Zeta function $\zeta(s)$ and the (Hurwitz's) generalized Zeta function $\zeta(s, a)$, defined usually by (see, *e.g.*, Titchmarsh [26])

$$(1.1) \quad \zeta(s) := \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1), \\ \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; \quad s \neq 1) \end{cases}$$

and

$$(1.2) \quad \zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad (\Re(s) > 1; \quad a \neq 0, -1, -2, \dots),$$

so that

$$(1.3) \quad \zeta(s, 1) = \zeta(s) \quad \text{and} \quad \zeta(s, 2) = \zeta(s) - 1,$$

are known to be meromorphic (that is, analytic everywhere in the complex s -plane except for a simple pole at $s = 1$ with residue 1). Making use of the binomial theorem *and* the Pochhammer symbol $(\lambda)_n$ defined by

$$(1.4) \quad (\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N} := \{1, 2, 3, \dots\}), \end{cases}$$

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it is easily seen from the definition (1.2) that

$$(1.5) \quad \sum_{k=0}^{\infty} \frac{(s)_k}{k!} \zeta(s+k, a) t^k = \zeta(s, a-t) \quad (|t| < |a|),$$

which immediately yields the familiar identities (*cf.*, *e.g.*, Hansen [16, p. 359] where other references are also cited):

$$(1.6) \quad \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \zeta(s+2k, a) t^{2k} = \frac{1}{2} [\zeta(s, a-t) + \zeta(s, a+t)] \quad (|t| < |a|)$$

and

$$(1.7) \quad \sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)!} \zeta(s+2k+1, a) t^{2k+1} = \frac{1}{2} [\zeta(s, a-t) - \zeta(s, a+t)] \quad (|t| < |a|)$$

or, equivalently,

$$(1.8) \quad \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k+1)!} \zeta(s+2k, a) t^{2k+1} = \frac{1}{2(s-1)} [\zeta(s-1, a-t) - \zeta(s-1, a+t)] \quad (|t| < |a|).$$

Since

$$(1.9) \quad \zeta(s) = \frac{1}{m^s - 1} \sum_{j=1}^{m-1} \zeta\left(s, \frac{j}{m}\right) \quad (m \in \mathbb{N} \setminus \{1\}),$$

which follows readily from the definitions (1.1) and (1.2), the special case of the identity (1.6) when $a = 1$ and $t = 1/m$ can be rewritten in the form:

$$(1.10) \quad \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{m^{2k}} = \begin{cases} (2^s - 1) \zeta(s) - 2^{s-1} & (m = 2), \\ \frac{1}{2} \left[(m^s - 1) \zeta(s) - m^s - \sum_{j=2}^{m-2} \zeta\left(s, \frac{j}{m}\right) \right] & (m \in \mathbb{N} \setminus \{1, 2\}), \end{cases}$$

where (and *throughout this paper*) an empty sum is to be interpreted as nil. In addition to the case $m = 2$, the formula (1.10) simplifies also in the cases when $m = 3, 4$, and 6 , and we thus obtain the identities:

$$(1.11) \quad \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{3^{2k}} = \frac{1}{2} [(3^s - 1) \zeta(s) - 3^s],$$

$$(1.12) \quad \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{4^{2k}} = \frac{1}{2} [(4^s - 2^s) \zeta(s) - 4^s],$$

and

$$(1.13) \quad \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{6^{2k}} = \frac{1}{2} [(6^s - 3^s - 2^s + 1)\zeta(s) - 6^s],$$

respectively.

Identities of this kind seem to have first appeared in the work of Ramaswami [22], who actually proved the cases $m = 2, 3$, and 6 of the general result in (1.10). Each of these three identities of Ramaswami [22] can also be found in the work of Hansen [16, p. 357], who referred to Apostol [1] as his source for the identities (1.11) and (1.13) only. As a matter of fact, Apostol [1] reproduced the identities (1.11) and (1.13) from Ramaswami's work [22] and then proved an interesting *arithmetical* generalization of these identities (see also Klusch [17, p. 520]).

In its slightly variant form:

$$(1.14) \quad \sum_{k=1}^{\infty} \frac{(s+1)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{2^{2k}} = (2^s - 2)\zeta(s),$$

which can indeed be proven *directly* from the known special cases of (1.6) and (1.7) when $a = 1$ and $t = \frac{1}{2}$, the case $m = 2$ of the general result (1.10) was applied by Zhang and Williams [29] (and, more recently, by Cvijović and Klinowski [8]) with a view to finding two seemingly different series representations for $\zeta(2n + 1)$ ($n \in \mathbb{N}$). The main object of this paper is to obtain much more rapidly converging series representations for $\zeta(2n + 1)$ ($n \in \mathbb{N}$) chiefly by appealing appropriately to each of the aforementioned cases ($m = 2, 3, 4$, and 6) of the general result (1.10).

The following properties of the Riemann ζ -function will be required in our investigation:

$$\zeta(0) = -\frac{1}{2}; \quad \zeta(-2n) = 0 \quad (n \in \mathbb{N}); \quad \zeta'(0) = -\frac{1}{2} \log(2\pi),$$

and (in general)

$$(1.15) \quad \begin{aligned} \zeta'(-2n) &= \lim_{\epsilon \rightarrow 0} \frac{\zeta(-2n + \epsilon)}{\epsilon} \\ &= \frac{(-1)^n}{2(2\pi)^{2n}} (2n)! \zeta(2n + 1) \quad (n \in \mathbb{N}), \end{aligned}$$

where use is made of the familiar functional equation:

$$(1.16) \quad 2^s \Gamma(1-s) \zeta(1-s) \sin\left(\frac{1}{2} \pi s\right) = \pi^{1-s} \zeta(s).$$

Furthermore, by l'Hôpital's rule, we have

$$(1.17) \quad \lim_{s \rightarrow -2n} \left\{ \frac{\sin\left(\frac{1}{2} \pi s\right)}{s + 2n} \right\} = (-1)^n \frac{\pi}{2} \quad (n \in \mathbb{N})$$

and

$$(1.18) \quad \begin{aligned} \lim_{s \rightarrow -2n} \left\{ \frac{\zeta(s+2k)}{s+2n} \right\} &= \frac{(-1)^{n-k}}{2(2\pi)^{2(n-k)}} (2n-2k)! \zeta(2n-2k+1) \\ &\quad (k = 1, \dots, n-1; \quad n \in \mathbb{N} \setminus \{1\}). \end{aligned}$$

2. A SET OF SERIES REPRESENTATIONS

We begin with the case $m = 2$ of the general result (1.10). Upon separating the first $n + 1$ terms of the series occurring on the left-hand side, if we transpose the terms for $k = 0$ and $k = n$ to the right-hand side, we obtain

$$\begin{aligned}
 & \sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{2^{2k}} + \sum_{k=n+1}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{2^{2k}} \\
 (2.1) \qquad & = (2^s - 2)\zeta(s) - 2^{s-1} - \frac{(s)_{2n}}{(2n)!} \frac{\zeta(s+2n)}{2^{2n}},
 \end{aligned}$$

which readily yields the identity:

$$\begin{aligned}
 & \sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} 2^{2(n-k)} \zeta(s+2k) + \sum_{k=1}^{\infty} \frac{(s)_{2n+2k}}{(2n+2k)!} \frac{\zeta(s+2n+2k)}{2^{2k}} \\
 (2.2) \qquad & = 2^{2n}(2^s - 2)\zeta(s) - 2^{s+2n-1} - \frac{(s)_{2n}}{(2n)!} \zeta(s+2n) \quad (n \in \mathbb{N}),
 \end{aligned}$$

it being understood, as before, that an empty sum is to be interpreted as nil.

Now we apply the functional equation (1.16) in the first term on the right-hand side of (2.2) and divide both sides by $s + 2n$. We thus find that

$$\begin{aligned}
 & \sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} 2^{2(n-k)} \left\{ \frac{\zeta(s+2k)}{s+2n} \right\} + \sum_{k=1}^{\infty} \frac{(s)_{2n}(s+2n+1)_{2k-1}}{(2n+2k)!} \frac{\zeta(s+2n+2k)}{2^{2k}} \\
 & = 2^{s+2n}(2^s - 2)\pi^{s-1}\Gamma(1-s)\zeta(1-s) \left\{ \frac{\sin(\frac{1}{2}\pi s)}{s+2n} \right\} \\
 (2.3) \qquad & - \left\{ \frac{2^{s+2n-1} + \frac{(s)_{2n}}{(2n)!} \zeta(s+2n)}{s+2n} \right\} \quad (s \neq -2n; \ n \in \mathbb{N}).
 \end{aligned}$$

Since

$$(-n)_k = (-1)^k \frac{n!}{(n-k)!} \quad (k = 0, 1, \dots, n; \ n \in \mathbb{N}),$$

so that, obviously,

$$(2.4) \qquad (-n)_n = (-1)^n n! \quad (n \in \mathbb{N}),$$

in view of the definition (1.4), it is easily seen by logarithmic differentiation that

$$(2.5) \qquad \frac{d}{ds} \{(s)_n\} = (s)_n \sum_{j=0}^{n-1} \frac{1}{s+j} \quad (n \in \mathbb{N}),$$

so that

$$(2.6) \qquad \frac{d}{ds} \{(s)_{2n}\} \Big|_{s=-2n} = -(2n)! H_{2n} \quad (n \in \mathbb{N}),$$

where H_n denotes the familiar harmonic numbers defined by

$$(2.7) \quad H_n := \sum_{j=1}^n \frac{1}{j} \quad (n \in \mathbb{N}).$$

We observe also that the limit formula (1.18) is needed in the first sum on the left-hand side of (2.3) only when this sum is nonzero (that is, only when $n \in \mathbb{N} \setminus \{1\}$). Furthermore, by l'Hôpital's rule, we have

$$(2.8) \quad \begin{aligned} & \lim_{s \rightarrow -2n} \left\{ \frac{2^{s+2n-1} + \frac{(s)_{2n}}{(2n)!} \zeta(s+2n)}{s+2n} \right\} \\ &= \left[2^{s+2n-1} \log 2 + \frac{d}{ds} \{(s)_{2n}\} \cdot \frac{\zeta(s+2n)}{(2n)!} + \frac{(s)_{2n}}{(2n)!} \zeta'(s+2n) \right] \Big|_{s=-2n} \\ &= \frac{1}{2} (H_{2n} - \log \pi) \quad (n \in \mathbb{N}). \end{aligned}$$

Finally, letting $s \rightarrow -2n$ in (2.3), and making use of the limit relationships (1.17), (1.18), and (2.8), we obtain our first series representation for $\zeta(2n + 1)$:

$$(2.9) \quad \begin{aligned} \zeta(2n + 1) &= (-1)^{n-1} \frac{(2\pi)^{2n}}{2^{2n+1} - 1} \left[\frac{H_{2n} - \log \pi}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n - 2k)!} \frac{\zeta(2k + 1)}{\pi^{2k}} \right. \\ &\quad \left. + 2 \sum_{k=1}^{\infty} \frac{(2k - 1)!}{(2n + 2k)!} \frac{\zeta(2k)}{2^{2k}} \right] \quad (n \in \mathbb{N}). \end{aligned}$$

In precisely the same manner, we can apply the identities (1.11), (1.12), and (1.13) in order to prove the following *additional* series representations for $\zeta(2n + 1)$:

$$(2.10) \quad \begin{aligned} \zeta(2n + 1) &= (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n+1} - 1} \left[\frac{H_{2n} - \log(\frac{2}{3}\pi)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n - 2k)!} \frac{\zeta(2k + 1)}{(\frac{2}{3}\pi)^{2k}} \right. \\ &\quad \left. + 2 \sum_{k=1}^{\infty} \frac{(2k - 1)!}{(2n + 2k)!} \frac{\zeta(2k)}{3^{2k}} \right] \quad (n \in \mathbb{N}); \end{aligned}$$

$$(2.11) \quad \begin{aligned} \zeta(2n + 1) &= (-1)^{n-1} \frac{2(2\pi)^{2n}}{2^{4n+1} + 2^{2n} - 1} \left[\frac{H_{2n} - \log(\frac{1}{2}\pi)}{(2n)!} \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n - 2k)!} \frac{\zeta(2k + 1)}{(\frac{1}{2}\pi)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k - 1)!}{(2n + 2k)!} \frac{\zeta(2k)}{4^{2k}} \right] \quad (n \in \mathbb{N}); \end{aligned}$$

$$(2.12) \quad \begin{aligned} \zeta(2n + 1) &= (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n}(2^{2n} + 1) + 2^{2n} - 1} \left[\frac{H_{2n} - \log(\frac{1}{3}\pi)}{(2n)!} \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n - 2k)!} \frac{\zeta(2k + 1)}{(\frac{1}{3}\pi)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k - 1)!}{(2n + 2k)!} \frac{\zeta(2k)}{6^{2k}} \right] \quad (n \in \mathbb{N}). \end{aligned}$$

3. REMARKS AND OBSERVATIONS

Our series representation (2.9) is markedly different from each of the series representations for $\zeta(2n+1)$, which were given earlier by Zhang and Williams [29, p. 1591, Equation (3.16)] and (more recently) by Cvijović and Klinowski [8, p. 1265, Theorem A]. Since $\zeta(2k) \rightarrow 1$ as $k \rightarrow \infty$, the general term in our series representation (2.9) has the order estimate:

$$O(2^{-2k} \cdot k^{-2n-1}) \quad (k \rightarrow \infty; \quad n \in \mathbb{N}),$$

whereas the general term in each of these earlier series representations has the order estimate:

$$O(2^{-2k} \cdot k^{-2n}) \quad (k \rightarrow \infty; \quad n \in \mathbb{N}).$$

By suitably combining (2.9) and (2.11), it is fairly straightforward to obtain the series representation:

$$\begin{aligned} \zeta(2n+1) = & (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)} \left[\frac{\log 2}{(2n)!} \right. \\ & + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k}-1)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\ & \left. - 2 \sum_{k=1}^{\infty} \frac{(2k-1)!(2^{2k}-1)}{(2n+2k)!} \frac{\zeta(2k)}{2^{4k}} \right] \quad (n \in \mathbb{N}). \end{aligned} \quad (3.1)$$

Now, in terms of the Bernoulli numbers B_n and the Euler polynomials $E_n(x)$ defined by the generating functions:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (|z| < 2\pi) \quad (3.2)$$

and

$$\frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} \quad (|z| < \pi), \quad (3.3)$$

respectively, it is known that (*cf.*, *e.g.*, Magnus *et al.* [20, p. 29])

$$E_n(0) = (-1)^n E_n(1) = \frac{2(1-2^{n+1})}{n+1} B_{n+1} \quad (n \in \mathbb{N}) \quad (3.4)$$

and [20, p. 19]

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \quad (n \in \mathbb{N}), \quad (3.5)$$

which, together, imply that

$$E_{2n-1}(0) = \frac{4(-1)^n}{(2\pi)^{2n}} (2n-1)! (2^{2n}-1) \zeta(2n) \quad (n \in \mathbb{N}). \quad (3.6)$$

Making use of this last relationship (3.6), the series representation (3.1) can immediately be put in the form:

$$\begin{aligned}
 \zeta(2n + 1) = & (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n} - 1)(2^{2n+1} - 1)} \left[\frac{\log 2}{(2n)!} \right. \\
 & + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k} - 1)}{(2n - 2k)!} \frac{\zeta(2k + 1)}{\pi^{2k}} \\
 & \left. + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2n + 2k)!} \left(\frac{\pi}{2}\right)^{2k} E_{2k-1}(0) \right] \quad (n \in \mathbb{N}),
 \end{aligned}
 \tag{3.7}$$

which is a slightly modified (and corrected) version of a result proven in a significantly different way by Tsumura [27, p. 383, Theorem B].

Another interesting combination of our series representations (2.9) and (2.11) leads us to the following variant of Tsumura’s result (3.1) or (3.7):

$$\begin{aligned}
 \zeta(2n + 1) = & (-1)^{n-1} \frac{\pi^{2n}}{2^{2n+1} - 1} \left[\frac{H_{2n} - \log\left(\frac{1}{4}\pi\right)}{(2n)!} \right. \\
 & + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k+1} - 1)}{(2n - 2k)!} \frac{\zeta(2k + 1)}{\pi^{2k}} \\
 & \left. - 4 \sum_{k=1}^{\infty} \frac{(2k - 1)! (2^{2k-1} - 1)}{(2n + 2k)!} \frac{\zeta(2k)}{2^{4k}} \right] \quad (n \in \mathbb{N}),
 \end{aligned}
 \tag{3.8}$$

which is essentially the same as the *determinantal* expression for $\zeta(2n + 1)$ derived recently by Ewell [12, p. 1010, Corollary 3] by employing an entirely different technique from ours.

Other similar combinations of our series representations (2.9) to (2.12) would yield the following (presumably new) companions of Ewell’s result (3.8):

$$\begin{aligned}
 \zeta(2n + 1) = & (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n+1} - 1)(3^{2n} + 1)} \left[\frac{H_{2n} - \log\left(\frac{1}{6}\pi\right)}{(2n)!} \right. \\
 & + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k+1} - 1)}{(2n - 2k)!} \frac{\zeta(2k + 1)}{\left(\frac{2}{3}\pi\right)^{2k}} \\
 & \left. - 4 \sum_{k=1}^{\infty} \frac{(2k - 1)! (2^{2k-1} - 1)}{(2n + 2k)!} \frac{\zeta(2k)}{6^{2k}} \right] \quad (n \in \mathbb{N});
 \end{aligned}
 \tag{3.9}$$

$$\begin{aligned}
 \zeta(2n + 1) = & (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n} + 1)(3^{2n+1} - 1)} \left[\frac{2H_{2n} - \log\left(\frac{\pi^2}{27}\right)}{(2n)!} \right. \\
 & + \sum_{k=1}^{n-1} \frac{(-1)^k (3^{2k+1} - 1)}{(2n - 2k)!} \frac{\zeta(2k + 1)}{\pi^{2k}} \\
 & \left. - 6 \sum_{k=1}^{\infty} \frac{(2k - 1)! (3^{2k-1} - 1)}{(2n + 2k)!} \frac{\zeta(2k)}{6^{2k}} \right] \quad (n \in \mathbb{N});
 \end{aligned}
 \tag{3.10}$$

$$\begin{aligned}
\zeta(2n+1) &= (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n+2} - 2^{2n+3} + 1} \left[\frac{H_{2n} - \log\left(\frac{8\pi}{27}\right)}{(2n)!} \right. \\
&\quad + \sum_{k=1}^{n-1} \frac{(-1)^k (3^{2k+1} - 2^{2k+1})}{(2n-2k)!} \frac{\zeta(2k+1)}{(2\pi)^{2k}} \\
&\quad \left. - 12 \sum_{k=1}^{\infty} \frac{(2k-1)! (3^{2k-1} - 2^{2k-1})}{(2n+2k)!} \frac{\zeta(2k)}{6^{2k}} \right] \quad (n \in \mathbb{N});
\end{aligned}
\tag{3.11}$$

$$\begin{aligned}
\zeta(2n+1) &= (-1)^{n-1} \frac{2(2\pi)^{2n}}{2^{4n+3} + 2^{2n+2} - 3^{2n+2} - 1} \left[\frac{H_{2n} - \log\left(\frac{27\pi}{128}\right)}{(2n)!} \right. \\
&\quad + \sum_{k=1}^{n-1} \frac{(-1)^k (4^{2k+1} - 3^{2k+1})}{(2n-2k)!} \frac{\zeta(2k+1)}{(2\pi)^{2k}} \\
&\quad \left. - 24 \sum_{k=1}^{\infty} \frac{(2k-1)! (4^{2k-1} - 3^{2k-1})}{(2n+2k)!} \frac{\zeta(2k)}{12^{2k}} \right] \quad (n \in \mathbb{N}),
\end{aligned}
\tag{3.12}$$

and

$$\begin{aligned}
\zeta(2n+1) &= (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n+1}(2^{2n}+1) - 2^{4n+2} + 2^{2n} - 1} \left[\frac{H_{2n} - \log\left(\frac{4\pi}{27}\right)}{(2n)!} \right. \\
&\quad + \sum_{k=1}^{n-1} \frac{(-1)^k (3^{2k+1} - 2^{2k+1})}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\
&\quad \left. - 12 \sum_{k=1}^{\infty} \frac{(2k-1)! (3^{2k-1} - 2^{2k-1})}{(2n+2k)!} \frac{\zeta(2k)}{12^{2k}} \right] \quad (n \in \mathbb{N}).
\end{aligned}
\tag{3.13}$$

Next we turn to the identity (1.7). By setting $t = 1/m$ and differentiating both sides with respect to s , we find from (1.7) that

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)! m^{2k}} \left[\zeta'(s+2k+1, a) + \zeta(s+2k+1, a) \sum_{j=0}^{2k} \frac{1}{s+j} \right] \\
&= \frac{m}{2} \frac{d}{ds} \left\{ \zeta\left(s, a - \frac{1}{m}\right) - \zeta\left(s, a + \frac{1}{m}\right) \right\} \quad (m \in \mathbb{N} \setminus \{1\}),
\end{aligned}
\tag{3.14}$$

where we have made use of the derivative formula (2.5). In particular, when $m = 2$, (3.14) immediately yields

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)! 2^{2k}} \left[\zeta'(s+2k+1, a) + \zeta(s+2k+1, a) \sum_{j=0}^{2k} \frac{1}{s+j} \right] \\
&= - \left(a - \frac{1}{2}\right)^{-s} \log\left(a - \frac{1}{2}\right).
\end{aligned}
\tag{3.15}$$

By letting $s \rightarrow -2n - 1$ ($n \in \mathbb{N}$) in the further special of this last identity (3.15) when $a = 1$, Wilton [28, p. 92] obtained the following series representation for

$\zeta(2n + 1)$ (see also Hansen [16, p. 357, Entry (54.6.9)]):

$$(3.16) \quad \zeta(2n + 1) = (-1)^{n-1} \pi^{2n} \left[\frac{H_{2n+1} - \log \pi}{(2n + 1)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n - 2k + 1)!} \frac{\zeta(2k + 1)}{\pi^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k - 1)!}{(2n + 2k + 1)!} \frac{\zeta(2k)}{2^{2k}} \right] \quad (n \in \mathbb{N}),$$

which may be compared with our first series representation (2.9). As a matter of fact, since

$$(3.17) \quad \frac{(2k)!}{(2n + 2k)!} = \frac{(2k - 1)!}{(2n + 2k - 1)!} - 2n \frac{(2k - 1)!}{(2n + 2k)!} \quad (n, k \in \mathbb{N}),$$

it is not difficult to deduce from (2.9) and (3.16) (with n replaced by $n - 1$) that

$$(3.18) \quad \zeta(2n + 1) = (-1)^n \frac{(2\pi)^{2n}}{n(2^{2n+1} - 1)} \left[\sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n - 2k)!} \frac{\zeta(2k)}{\pi^{2k}} + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n + 2k)!} \frac{\zeta(2k)}{2^{2k}} \right] \quad (n \in \mathbb{N}),$$

which is precisely the aforementioned *main* result of Cvijović and Klinowski [8, p. 1265, Theorem A] (see also Zhang and Williams [29, p. 1591, Equation (3.16)] where an obviously more complicated version of (3.18) was proven by applying the same identity (1.14) above).

Observing also that

$$(3.19) \quad \frac{(2k)!}{(2n + 2k + 1)!} = \frac{(2k - 1)!}{(2n + 2k)!} - (2n + 1) \frac{(2k - 1)!}{(2n + 2k + 1)!} \quad (n, k \in \mathbb{N}),$$

we obtain yet another series representation for $\zeta(2n + 1)$ by applying (2.9) and (3.16):

$$(3.20) \quad \zeta(2n + 1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n - 1) 2^{2n} + 1} \left[\sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n - 2k + 1)!} \frac{\zeta(2k + 1)}{\pi^{2k}} + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n + 2k + 1)!} \frac{\zeta(2k)}{2^{2k}} \right] \quad (n \in \mathbb{N}),$$

which provides a significantly simpler (and much more rapidly convergent) version of the other *main* result of Cvijović and Klinowski [8, p. 1265, Theorem B]:

$$(3.21) \quad \zeta(2n + 1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n)!} \sum_{k=0}^{\infty} \Omega_{n,k} \frac{\zeta(2k)}{2^{2k}} \quad (n \in \mathbb{N}),$$

where the coefficients $\Omega_{n,k}$ are given explicitly by

$$(3.22) \quad \Omega_{n,k} := \sum_{j=0}^{2n} \binom{2n}{j} \frac{B_{2n-j}}{(j + 2k + 1)(j + 1) 2^j} \quad (n \in \mathbb{N}; \quad k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

in terms of the Bernoulli numbers defined by (3.2). Since [20, pp. 27 and 28]

$$B_1 = -\frac{1}{2} \quad \text{and} \quad B_{2n+1} = 0 \quad (n \in \mathbb{N}),$$

the definition (3.22) can be rewritten at once in the form:

$$(3.23) \quad \Omega_{n,k} = \sum_{j=0}^n \binom{2n}{2j} \frac{B_{2n-2j}}{(2j+2k+1)(2j+1)2^{2j}} - \frac{1}{(2n+2k)2^{2n}} \quad (n \in \mathbb{N}; k \in \mathbb{N}_0),$$

or, equivalently,

$$(3.24) \quad \Omega_{n,k} = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{j=0}^n \frac{(-\pi^2)^j \zeta(2n-2j)}{(2j+1)!(2j+2k+1)} - \frac{1}{(2n+2k)2^{2n}} \quad (n \in \mathbb{N}; k \in \mathbb{N}_0),$$

by virtue of the relationship (3.5). Combining the partial fractions occurring in (3.23) or (3.24), it is easily seen that

$$(3.25) \quad \Omega_{n,k} = \frac{\prod_{\ell=0}^n \{(2k+2\ell+1)^{-1}\}}{(2n+2k)2^{2n}} \left[\sum_{j=0}^n \binom{2n}{2j} \frac{2n+2k}{2j+1} 2^{2n-2j} B_{2n-2j} \cdot \prod_{\substack{\ell=0 \\ (\ell \neq j)}}^n (2k+2\ell+1) - \prod_{\ell=0}^n (2k+2\ell+1) \right] \quad (n \in \mathbb{N}; k \in \mathbb{N}_0).$$

In view of the identity:

$$(3.26) \quad \sum_{j=0}^n \binom{2n}{2j} \frac{2^{2n-2j} B_{2n-2j}}{2j+1} = 1 = \sum_{j=0}^n \binom{2n}{2j} \frac{2^{2j} B_{2j}}{2n-2j+1},$$

which is due essentially to Euler (*cf.*, *e.g.*, Riordan [23, p. 123, Problem 12]), the expression inside brackets in (3.25) is a polynomial in k of degree n (not $n+1$), and therefore

$$(3.27) \quad \Omega_{n,k} = O(k^{-2}) \quad (k \rightarrow \infty; n \in \mathbb{N}).$$

It follows from (3.27) that the general term in (3.21) has the order estimate:

$$(3.28) \quad O(2^{-2k} \cdot k^{-2}) \quad (k \rightarrow \infty),$$

whereas the general term in our series representation (3.20) has precisely the same order estimate:

$$(3.29) \quad O(2^{-2k} \cdot k^{-2n-1}) \quad (k \rightarrow \infty; n \in \mathbb{N}),$$

as that in (2.9). Thus, even in the special case when $n=1$, the series representing $\zeta(3)$ converges faster in (3.20) than in (3.21).

Various known series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$) of other types include those given (for example) by Ramanujan [21] (see also Berndt [3]), Glaisher [13] (see also Hansen [16, p. 359]), Koshliakov [18], Leshchiner [19], Grosswald ([14] and [15]), Terras [25], Cohen [7], Butzer *et al.* ([5] and [6]), Dąbrowski [9], and others (see, *e.g.*, Berndt [4, pp. 275 and 276]).

We conclude this paper by remarking that a particular case of our series representation (2.12) when $n=1$ was proven, by an entirely different method, by Zhang

and Williams [30, p. 707, Theorem 9]. Furthermore, the following particular case of (3.18) when $n = 1$:

$$(3.30) \quad \zeta(3) = -\frac{4\pi^2}{7} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}},$$

which is contained in a 1772 paper entitled *Exercitationes Analyticae* by Euler (see, e.g., Ayoub [2, pp. 1084–1085]), was rediscovered by Ramaswami [22] and (more recently) by Ewell [10]. In fact, Euler's formula (3.30) was reproduced by Srivastava [24, p. 7, Equation (2.23)] from the work of Ramaswami [22]. In the *current* mathematical literature, however, Euler's formula (3.30) is being attributed to Ewell [10].

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REFERENCES

- [1] T.M. Apostol, Some series involving the Riemann Zeta function, *Proc. Amer. Math. Soc.* **5**(1954), 239–243. MR **15**:685c
- [2] R. Ayoub, Euler and the Zeta functions, *Amer. Math. Monthly* **81**(1974), 1067–1086. MR **50**:12566
- [3] B.C. Berndt, Modular transformations and generalizations of several formulae of Ramanujan, *Rocky Mountain J. Math.* **7**(1977), 147–189. MR **55**:2714
- [4] B.C. Berndt, *Ramanujan's Notebooks*, Part **II**, Springer-Verlag, New York, 1989. MR **90b**:01039
- [5] P.L. Butzer and M. Hauss, Riemann Zeta function: Rapidly converging series and integral representations, *Appl. Math. Lett.* **5**(2) (1992), 83–88. MR **93b**:11106
- [6] P.L. Butzer, C. Markett, and M. Schmidt, Stirling numbers, central factorial numbers, and representations of the Riemann Zeta function, *Resultate Math.* **19**(1991), 257–274. MR **92a**:11095
- [7] H. Cohen, Généralisation d'une construction de R. Apéry, *Bull. Soc. Math. France* **109**(1981), 269–281. MR **84a**:10036
- [8] D. Cvijović and J. Klinowski, New rapidly convergent series representations for $\zeta(2n + 1)$, *Proc. Amer. Math. Soc.* **125**(1997), 1263–1271. MR **97g**:11090
- [9] A. Dąbrowski, A note on the values of the Riemann Zeta function at positive odd integers, *Nieuw Arch. Wisk.* (4) **14**(1996), 199–207. MR **97g**:11142
- [10] J.A. Ewell, A new series representation for $\zeta(3)$, *Amer. Math. Monthly* **97**(1990), 219–220. MR **91d**:11103
- [11] J.A. Ewell, On values of the Riemann Zeta function at integral arguments, *Canad. Math. Bull.* **34**(1991), 60–66. MR **92c**:11087
- [12] J.A. Ewell, On the Zeta function values $\zeta(2k + 1)$, $k = 1, 2, \dots$, *Rocky Mountain J. Math.* **25**(1995), 1003–1012. MR **97i**:11093
- [13] J.W.L. Glaisher, Summations of certain numerical series, *Messenger Math.* **42** (1912/1913), 19–34.
- [14] E. Grosswald, Die Werte der Riemannschen Zetafunktion an ungeraden Argumenstellen, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* **1970**(1970), 9–13. MR **42**:7606
- [15] E. Grosswald, Comments on some formulae of Ramanujan, *Acta Arith.* **21**(1972), 25–34. MR **46**:1744
- [16] E.R. Hansen, *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, New Jersey, 1975. CMP 19:01
- [17] D. Klusch, On the Taylor expansion of the Lerch Zeta-function, *J. Math. Anal. Appl.* **170**(1992), 513–523. MR **93k**:11080

- [18] N.S. Koshliakov, Application of the theory of sum-formulae to the investigation of a class of one-valued analytical functions in the theory of numbers, *Messenger Math.* **58**(1928/1929), 1–23.
- [19] D. Leshchiner, Some new identities for $\zeta(k)$, *J. Number Theory* **13**(1981), 355–362. MR **83k**:10072
- [20] W. Magnus, F. Oberhettinger, and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Bd. **52**, Third Enlarged Edition, Springer-Verlag, New York, 1966. MR **38**:1291
- [21] S. Ramanujan, *Notebooks of Srinivasa Ramanujan*, Vols. **I** and **II**, Tata Institute of Fundamental Research, Bombay, 1957. MR **20**:6340
- [22] V. Ramaswami, Notes on Riemann's ζ -function, *J. London Math. Soc.* **9**(1934), 165–169.
- [23] J. Riordan, *Combinatorial Identities*, John Wiley and Sons, New York, London, and Sydney, 1968. MR **38**:53
- [24] H.M. Srivastava, A unified presentation of certain classes of series of the Riemann Zeta function, *Riv. Mat. Univ. Parma* (4) **14**(1988), 1–23. MR **90h**:11080
- [25] A. Terras, Some formulas for the Riemann Zeta function at odd integer argument resulting from Fourier expansions of the Epstein Zeta function, *Acta Arith.* **29**(1976), 181–189. MR **53**:299
- [26] E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford University (Clarendon) Press, Oxford and London, 1951; Second Edition (Revised by D.R. Heath-Brown), 1986. MR **88c**:11049
- [27] H. Tsumura, On evaluation of the Dirichlet series at positive integers by q -calculation, *J. Number Theory* **48**(1994), 383–391. MR **96a**:11083
- [28] J.R. Wilton, A proof of Burnside's formula for $\log \Gamma(x + 1)$ and certain allied properties of Riemann's ζ -function, *Messenger Math.* **52**(1922/1923), 90–93.
- [29] N.-Y. Zhang and K.S. Williams, Some series representations of $\zeta(2n + 1)$, *Rocky Mountain J. Math.* **23**(1993), 1581–1592. MR **94m**:11099
- [30] N.-Y. Zhang and K.S. Williams, Some infinite series involving the Riemann Zeta function, in *Analysis, Geometry and Groups: A Riemann Legacy Volume* (H.M. Srivastava and Th. M. Rassias, Editors), Part **II**, pp. 691–712, Hadronic Press, Palm Harbor, Florida, 1993. MR **96a**:11084

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF VICTORIA, VICTORIA, BRITISH COLUMBIA, CANADA V8W 3P4

E-mail address: HMSRI@UVVM.UVIC.CA