MULTINOMIAL COEFFICIENTS MODULO A PRIME

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Abstract. We say that the multinomial coefficient (m.c.) \((j_1, \ldots, j_l) = n!/(j_1! \cdots j_l!)\) has order \(l\) and power \(n = j_1 + \cdots + j_l\). Let \(G(n, l, p)\) be the number of m.c. that are not divisible by \(p\) and have order \(l\) with powers which are not larger than \(n\). If \(\theta = \log_p(l, p-1)\) and

\[ q_{l, p}^{(r)} = \min_{p^r \leq n < p^{r+1}} G(n, l, p)/(n+1)^\theta, \]

then for any integer \(r = 1, 2, \ldots\)

\[ 0 < q_{l, p}^{(r)} - \liminf_{n \to \infty} G(n, l, p)/n^\theta \leq \frac{1}{\theta p^r} \left( 1 + \frac{1}{p^r} \right)^{\theta - 1}. \]

1. Introduction

We consider the \(n\)th row of multinomial coefficients of order \(l\):

\[ (j_1, j_2, \ldots, j_l) = \frac{n!}{j_1!j_2! \cdots j_l!}, \]

where \(j_i \geq 0, i = 1, \ldots, l,\) and \(j_1 + j_2 + \cdots + j_l = n\). Let \(g(n, l, p^N)\) be the number of multinomial coefficients in the \(n\)th row not divisible by \(p^N\), where \(p\) is a prime and \(N\) is a fixed integer. We also define

\[ G(n, l, p^N) = \sum_{k=1}^{n} g(k, l, p^N). \]

The properties and behaviour of functions \(g(n, l, p^N)\) and \(G(n, l, p^N)\) were studied in [1]–[14] where an interested person could find the history of the topic and references on earlier papers.

\(G(n, l, p)\) is the subject of this paper, hence we will report some results concerning this function. The function \(G(n, l, p)\) has been studied by K. B. Stolarsky [10], [11] and H. Harborth [3] for \(l = p = 2\); by A. H. Stein [9] for \(l = 2\) and arbitrary \(p\); and by Volodin [12], [13] for arbitrary \(l\) and \(p\).

Theorem (Volodin, 1989). If \(n+1 = a_0 + a_1p + \cdots + a_m p^m\), then

\[ G(n, l, p) = \sum_{k=0}^{m} (l, p-1)^k a_k \prod_{i=k}^{m} (l-1, a_i). \]
Theorem (Volodin, 1994). If \( \theta = \log_p(l, p - 1) \), then
\[
\limsup_{n \to \infty} G(n, l, p)/n^\theta = 1.
\]
For \( l = 2 \) the last theorem was proved by H. Harborth [3] for \( p = 2 \) and by A. H. Stein [9] for arbitrary \( p \).

There are fewer results for \( \beta_{l,p} = \liminf_{n \to \infty} G(n, l, p)/n^\theta \equiv 1 \).

Theorem (Harborth, 1977). \( \beta_{2,2} = 0.812556 \ldots \) and \( \beta_{2,2} \) is determined to six decimal places.

Theorem (Wilson, 1996a). \( \beta_{2,p} \) are calculated to six decimal places for \( p = 3, 5, 7, \ldots, 37 \) and
\[
\lim_{p \to \infty} \beta_{2,p} = 0.5.
\]

Theorem (Wilson, 1996b). \( \lim_{p \to \infty} \beta_{l,p} = \frac{1}{l} \).

In this paper we give a general approach to the calculation of \( \beta_{l,p} \) for arbitrary \( l \) and \( p \).

2. MAIN RESULT

Let \( q_{l,p}^{(r)} = \min_{\rho^r \leq n < \rho^{r+1}} G(n, l, p)/(n + 1)^\theta \).

Theorem. For any \( r = 1, 2, \ldots \)
\[
0 < q_{l,p}^{(r)} - \beta_{l,p} \leq \frac{1}{\theta \rho^r} \left( 1 + \frac{1}{\rho^r} \right)^{\theta - 1}.
\]

We give some preliminary lemmas.

Lemma 1. The sequence \( q_{l,p}^{(r)} \), \( r = 1, 2, \ldots \), is nonincreasing and \( \lim_{r \to \infty} q_{l,p}^{(r)} = \beta_{l,p} \).

Proof. At the beginning we make a note that \( \min_{\rho^r \leq n < \rho^{r+1}} G(n, l, p)/(n + 1)^\theta \) cannot be reached for \( n = \rho^{r+1} - 1 \) because
\[
G(\rho^{r+1} - 1, l, p)/\rho^{r\theta} = 1 \equiv \limsup_{n \to \infty} G(n, l, p)/(n + 1)^\theta
\]
and, because of that,
\[
\min_{\rho^r \leq n < \rho^{r+1}} G(n, l, p)/(n + 1)^\theta = \min_{\rho^r \leq n < \rho^{r+1} - 1} G(n, l, p)/(n + 1)^\theta.
\]

Now, let \( n_r \) be chosen such that \( G(n_r, l, p)/(n_r + 1)^\theta = q_{l,p}^{(r)} \) and \( \rho^r < n_r + 1 < \rho^{r+1}, n_r + 1 = a_0 + a_1 p + \cdots + a_r p^r \). It is clear that \( \rho^{r+1} < p(n_r + 1) < \rho^{r+2} \) and
\[
q_{l,p}^{(r+1)} \leq G(n_r p + p - 1, l, p)/(n_r + 1)^\theta = (l, p - 1)G(n_r, l, p)/(p^\theta (n_r + 1)^\theta) = G(n_r, l, p)/(n_r + 1)^\theta = q_{l,p}^{(r)}.
\]
This implies
\[(3) \quad q_{l,p}^{(r)} = \min_{l \leq n \leq l+p} G(n, l, p)/(n+1)^\theta\]
and
\[\lim_{r \to \infty} q_{l,p}^{(r)} = \lim_{r \to \infty} \min_{l \leq n \leq l+p} G(n, l, p)/(n+1)^\theta = \lim_{n \to \infty} G(n, l, p)/(n+1)^\theta = \beta_{l,p}.
\]
Lemma 1 is proved completely.

**Lemma 2.** For any \(r = 1, 2, \ldots\) and any positive integer \(m\)
\[(4) \quad 0 < q_{l,p}^{(r)} - q_{l,p}^{(m+r)} \leq \frac{1}{\theta p^r} \left(1 + \frac{1}{p^r}\right)^{\theta - 1}.
\]

**Proof.** Let \(n_{m+r} + 1 = a_0 + a_1 p + \cdots + a_m p^m + a_{m+1} p^{m+1} + \cdots + a_{m+r} p^{m+r}\) and \(n_{m+r}\) provide a minimum for \(q_{l,p}^{(m+r)}\). In other words
\[G(n_{m+r}, l, p)/(n_{m+r} + 1)^\theta = q_{l,p}^{(m+r)}.
\]
Now we consider a number \(\tilde{n}_r\) such that \(\tilde{n}_r + 1 = a_m + a_{m+1} p + \cdots + a_{m+r} p^r\). As \(\tilde{n}_r + 1 < p^{r+1}\), then \(G(\tilde{n}_r, l, p)/(\tilde{n}_r + 1)^\theta \geq q_{l,p}^{(r)}\) and \(n_{m+r} + 1 = (\tilde{n}_r + 1)p^m + x\), where \(x < p^m\). Using this notation and (1) we can write
\[0 < \Delta_r \equiv q_{l,p}^{(r)} - q_{l,p}^{(m+r)} = q_{l,p}^{(r)} - G(\tilde{n}_r, l, p)/(\tilde{n}_r + 1)^\theta + G(\tilde{n}_r p^m + p^m - 1, l, p)/(\tilde{n}_r + 1)^\theta - q_{l,p}^{(m+r)} \leq G((\tilde{n}_r + 1)p^m + x, l, p)/(\tilde{n}_r + 1)^\theta - G((\tilde{n}_r + 1)p^m - 1, l, p)/(\tilde{n}_r + 1)^\theta \leq (G((\tilde{n}_r + 1)p^m - 1, l, p) - G((\tilde{n}_r + 1)p^m + x, l, p))/(\tilde{n}_r + 1)^\theta \leq G((\tilde{n}_r + 1)p^m + x, l, p)/(\tilde{n}_r + 1)^\theta \leq \frac{((\tilde{n}_r + 1)p^m + x)^\theta - ((\tilde{n}_r + 1)p^m)^\theta}{\theta((\tilde{n}_r + 1)p^m)^{\theta - 1}} \leq \frac{1}{\theta(\tilde{n}_r + 1)} \left(1 + \frac{1}{\tilde{n}_r + 1}\right)^{\theta - 1} < \frac{1}{\theta p^r} \left(1 + \frac{1}{p^r}\right)^{\theta - 1},
\]
which proves Lemma 2.

The proof of the Theorem follows from (3) and (4).

**3. Some numerical results**

The approximation stated in the theorem allows us to calculate \(\beta_{l,p}\) with accuracy that is only restricted by the speed and float precision of the computer used for the calculations. In Tables 1 and 2 we give some approximations for \(\beta_{l,p}\) for different \(l\) and \(p\). In these tables we follow H. Harborth’s [3] presentation where he did not round approximations but gave the first six correct digits for \(\beta_{2,2}\).

**Example.** For \(l = 4\) and \(p = 5\) we give the number 0.4547060. It means that \(\beta_{l,p} = 0.4547060\ldots\) even when \(\beta_{l,p} = 0.454706096\ldots\).
Table 1. Values of $\beta_{l,p}$ with seven correct digits for $p = 2, \ldots, 29$ and $l = 2, \ldots, 6$.

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Table 2. Values of $\beta_{l,p}$ with seven correct digits for $p = 2, 3, 5, 7$ and $l = 7, \ldots, 32$.

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REFERENCES


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