A VARIANT OF THE DIAMOND PRINCIPLE FOR COMBINATORIAL IDEALS

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Abstract. We use a variant of the diamond principle to show many ideals on \( \kappa \) are not \( 2^{\kappa} \)-saturated if \( \kappa \) is large. For instance, the \( \Pi_1^1 \)-indescribable ideal is not \( 2^{\kappa} \)-saturated if \( \kappa \) is almost ineffable.

Kunen proved that the diamond principle for \( \kappa \), \( \diamondsuit(\kappa) \) holds if \( \kappa \) is subtle. A consequence of \( \diamondsuit(\kappa) \) is that the nonstationary ideal on \( \kappa \) is not \( 2^{\kappa} \)-saturated.

Meanwhile Baumgartner, Taylor and Wagon [2] proved that the ethereal ideal on \( \kappa \) is not \( \kappa^+ \)-saturated if \( \kappa \) is almost ineffable.

These two facts have a point in common. If \( \kappa \) has a strong property, then an ideal corresponding to a weaker property is less saturated.

For a regular uncountable cardinal \( \kappa \), \( \diamondsuit(\kappa) \) can be regarded as a property of the nonstationary ideal. We consider the following principle for an ideal \( I \) on \( \kappa \):

The Diamond Principle for \( I \), \( \diamondsuit(I) \). There is a sequence \( \langle S_\alpha \subset \alpha \mid \alpha < \kappa \rangle \) such that for every \( X \subset \kappa \),

\[
\{ \alpha < \kappa \mid X \cap \alpha = S_\alpha \} \notin I.
\]

We modify Kunen’s construction of a diamond sequence assuming \( \kappa \) has a sufficiently strong property so that \( \diamondsuit(I) \) holds. It is clear that no ideal \( J \subseteq I \) is \( 2^{\kappa} \)-saturated if \( \diamondsuit(I) \) holds. Specifically we prove the following.

Theorem. (1) If \( \kappa \) is almost ineffable, then any ideal extended by the \( \Pi_1^1 \)-indescribable ideal on \( \kappa \) is not \( 2^{\kappa} \)-saturated.

(2) If \( \kappa \) is completely ineffable, then any ideal extended by the ineffable ideal on \( \kappa \) is not \( 2^{\kappa} \)-saturated.

Before proving the theorem we state the definition of these ideals. Throughout the rest of this paper, \( \kappa \) is a regular uncountable cardinal and \( I \) is a \( \kappa \)-complete ideal on \( \kappa \). The filter dual to an ideal \( I \) is denoted by \( I^* \), and \( I^+ \) is the set \( \{ X \subset \kappa \mid X \notin I \} \).

Definition. Let \( X \subset \kappa \).

(i) \( X \) is \( \Pi_1^1 \)-indescribable if for any \( R \subset V_\kappa \) and \( \Pi_1^1 \) sentence \( \varphi \) such that \( \langle V_\kappa, \in, R \rangle \models \varphi \), there is \( \alpha \in X \) such that \( \langle V_\alpha, \in, R \cap V_\alpha \rangle \models \varphi \).

(ii) \( X \) is almost ineffable if for any sequence \( \langle S_\alpha \subset \alpha \mid \alpha < \kappa \rangle \) there is \( S \subset \kappa \) such that \( \{ \alpha \in X \mid S_\alpha = S \cap \alpha \} \) is unbounded in \( \kappa \).
(iii) $X$ is *ineffable* if for any sequence $(S_\alpha \subset \alpha \mid \alpha < \kappa)$ there is $S \subset \kappa$ such that \{\alpha \in X \mid S_\alpha = S \cap \alpha\} is stationary in $\kappa$.

(iv) The *completely ineffable ideal* on $\kappa$ is the minimal normal ideal $I$ such that for any $X \in I^+$ and any sequence $(S_\alpha \subset \alpha \mid \alpha < \kappa)$ there is $S \subset \kappa$ such that \{\alpha \in X \mid S_\alpha = S \cap \alpha\} \in I^+$. $X \in I^+$ is called *completely ineffable*.

(v) $X$ is *subtle* if for any sequence $(S_\alpha \subset \alpha \mid \alpha < \kappa)$ and $C$ closed unbounded in $\kappa$, there exist $\alpha < \beta$ both in $C \cap X$ such that $S_\alpha = S_\beta \cap \alpha$.

For each property $A$ stated above, we consider the set
\[\{X \subset \kappa \mid X \text{ does not have property } A\},\]
which is a normal ideal on $\kappa$. For instance the $\Pi^1_1$-indescribable ideal is the set
\[\{X \subset \kappa \mid X \text{ is not } \Pi^1_1\text{-indescribable}\}.

These ideals were studied in Baumgartner [1] and Johnson [4].

**Proof of the Theorem.** (1) Suppose that $\kappa$ is almost ineffable. Let $NAIn_\kappa$ denote the almost ineffable ideal on $\kappa$ and $P_\alpha$ the $\Pi^1_1$-indescribable ideal on $\alpha$ for $\alpha \leq \kappa$. We use the fact that $P_\kappa \subset NAIn_\kappa$ and for every $X \in P_\kappa^*$,
\[\{\alpha \in X \mid X \cap \alpha \in P_\alpha^*\} \in NAIn_\kappa^*.

We recursively define $(S_\alpha, C_\alpha)$ for $\alpha < \kappa$ such that $S_\alpha \subset \alpha$ and $C_\alpha \in P_\alpha^*$ as follows.

Suppose that $\alpha < \kappa$ and $(S_\beta, C_\beta)$ has been defined for $\beta < \alpha$. Set $(S_\alpha, C_\alpha) = (\emptyset, \alpha)$ except in the case that
\[(\triangledown) \text{ There exist } S \subset \alpha \text{ and } C \in P_\alpha^* \text{ such that } S \cap \beta \neq S_\beta \text{ for any } \beta \in C.

In this case, let $(S_\alpha, C_\alpha)$ be one such pair $(S, C)$.

We show that $(S_\alpha \mid \alpha < \kappa)$ is a diamond sequence for $P_\kappa$. Suppose to the contrary that there are $X \subset \kappa$ and $C \in P_\kappa^*$ such that $X \cap \alpha \neq S_\alpha$ for $\alpha \in C$. Let $D = \{\alpha \in C \mid C \cap \alpha \neq S_\alpha\}$. For $\alpha \in D$, $(S \cap \alpha, C \cap \alpha)$ satisfies the condition of $(\triangledown)$. Hence $C_\alpha \in P_\alpha^*$ and $S_\alpha \cap \beta \neq S_\beta$ for $\beta \in C_\alpha$. Since $D \in NAIn_\kappa^*$, $D$ is subtle. By Theorem 4.1 in Baumgartner [1],
\[\{\alpha \in D \mid \{\beta \in D \cap \alpha \mid S_\beta \neq S_\alpha \cap \beta\} \in P_\alpha\} \text{ is not subtle.}\]

Thus we have
\[E = \{\alpha \in D \mid \{\beta \in D \cap \alpha \mid S_\beta = S_\alpha \cap \beta\} \in P_\alpha^+\} \in NAIn_\kappa^*.

For any $\alpha \in E$, $C_\alpha \in P_\alpha^*$. Hence we can find $\beta \in C_\alpha$ such that $S_\beta = S_\alpha \cap \beta$ contradicting the definition of $(S_\alpha, C_\alpha)$.

(2) Suppose that $\kappa$ is completely ineffable. Let $NCIn_\kappa$ denote the completely ineffable ideal on $\kappa$ and $NIIn_\kappa$ the ineffable ideal on $\alpha$ for $\alpha \leq \kappa$. We need only replace $P_\alpha$ by $NIIn_\kappa^*$ in the definition of $(S_\alpha, C_\alpha)$ to get a diamond sequence for $NIIn_\kappa^*$.

Consider the notion of forcing $Q = (NCIn_\kappa^+, \subseteq)$ and let $G$ be a $V$ generic filter on $Q$ and $M = Ult_G(V)$ the generic ultrapower. Since $NCIn_\kappa$ is normal $(\kappa, \kappa)$ distributive, $V_{\kappa+1}^V = V_{\kappa+1}^M$. (See [3], [4].) Hence, $NIIn_\kappa^V = NIIn_\kappa^M$ and, for any $X \in NIIn_\kappa^*$,
\[\{\alpha \in X \mid X \cap \alpha \in NIIn_\kappa^*\} \in NCIn_\kappa^*.

If \( \langle S_\alpha \mid \alpha < \kappa \rangle \) is not a diamond sequence for \( N_{\text{In}} \kappa \), there is \( Y \in N_{\text{In}}^* \kappa \subset N_{\text{CIn}}^* \kappa \) such that, for any \( \alpha \in Y \),

\[
C_\alpha \in N_{\text{In}}^* \kappa \text{ and } S_\beta \neq S_\alpha \cap \beta \text{ for } \beta \in C_\alpha.
\]

By complete ineffability, there exist \( T, U \subset \kappa \) such that

\[
H = \{ \beta \in Y \mid S_\alpha = T \cap \alpha \text{ and } C_\alpha = U \cap \alpha \} \in N_{\text{CIn}}^+ \kappa.
\]

Since \( H \models U \in N_{\text{In}}^* \kappa \), \( U \cap H \in N_{\text{In}}^+ \kappa \). For any \( \beta < \alpha \) both in \( U \cap H \), \( \beta \in U \cap \alpha = C_\alpha \) and \( S_\beta = T \cap \beta = (T \cap \alpha) \cap \beta = S_\alpha \cap \beta \), which contradicts the fact that \( \alpha \in Y \).

There are several facts which can be proved by the same argument. For instance:

- If \( \kappa \) is ineffable, then the \( \Pi_1^1 \)-indescribable ideal on \( \kappa \) is not \( 2^\kappa \)-saturated.
- If \( \kappa \) is 2-subtle, then the ineffable ideal on \( \kappa \) is not \( 2^\kappa \)-saturated.
- If \( \kappa \) is measurable, then the completely ineffable ideal on \( \kappa \) is not \( 2^\kappa \)-saturated.

Such an argument can be carried out for ideals on \( P_{\kappa \lambda} \) as well.

Johnson proved in [4] that the completely ineffable ideal is not precipitous if \( \kappa \) is completely ineffable. Thus it seems natural to ask:

**Question.** (1) Can it be proved that these combinatorial ideals mentioned above are not precipitous?

(2) Is it possible to prove the ideal corresponding to property \( A \) is not \( 2^\kappa \)-saturated just assuming \( \kappa \) has property \( A \)? For instance, in order to prove the ineffable ideal on \( \kappa \) is not \( 2^\kappa \)-saturated, does it suffice to assume \( \kappa \) is ineffable?

**References**


