# INTEGRATION AND HOMOGENEOUS FUNCTIONS

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ABSTRACT. We show that integrating a (positively) homogeneous function $f$ on a compact domain $\Omega \subset R^n$ reduces to integrating a related function on the boundary $\partial\Omega$. The formula simplifies when the boundary $\partial\Omega$ is determined by homogeneous functions. Similar results are also presented for integration of exponentials and logarithms of homogeneous functions.

## 1. INTRODUCTION

We consider the integration of a continuous (positively) homogeneous function $f : R^n \to R$ on a compact domain $\Omega$ with boundary $\partial\Omega$. Using Euler's identity for homogeneous functions, Green's formula simplifies so that integrating $f$ on $\Omega$ reduces to integrating a (simply related) function on the boundary $\partial\Omega$. In the particular case where $\Omega := \{x \in R^n \mid g_i(x) \le a_i, \ i = 1, \ldots, m\}$ and where the functions $g_i$ are each (positively) homogeneous of degree $p_i$, the formula is even simpler. Actually, an alternative proof that uses only Euler's formula is also outlined.

We thus extend to more general domains a result in [4] for integrating a homogeneous function on a convex polytope.

A potential application is the integration of arbitrary continuous functions on a compact set $\Omega$. Indeed, as the polynomials are sums of homogeneous polynomials, and are dense in $C(\Omega)$, the space of continuous functions on $\Omega$ with the sup norm, or even in $L_1(\Omega)$, one could approximate $\int_\Omega f dx$ by $\sum_i \int_\Omega P_i dx$, where the $P_i$'s are homogeneous polynomials, and therefore use the previous result. This result could also be used in finite element methods for the integration of a poynomial on each individual volume element.

Finally, we also provide simple formulas for the integration of exponentials and logarithms of homogeneous functions. For instance, integrating $\log f$ on a convex polytope $\Omega$ reduces to integrating $\log f$ on $\partial\Omega$ and to computing the volume of $\Omega$.

## 2. INTEGRATION OF A HOMOGENEOUS FUNCTION

Let $f : R^n \to R$ be a real (positively) *homogeneous* function of degree $p$ (or in short, $f$ is $p$-homogeneous), i.e. $f(\lambda x) = \lambda^p f(x)$ for all $\lambda > 0$, $x \in R^n$. For a (positively) $p$-homogeneous function that is continuously differentiable, Euler's

formula states that

$$(2.1) \qquad pf(x) = \langle \nabla f(x), x \rangle \text{ for all } x.$$

2.1. **On the Riemann-Green formula.** We first treat the 2-dimensional case. Let $\Omega$ be a compact domain in $R^2$ with boundary $\partial\Omega$. Let $P(x,y)$ and $Q(x,y)$ be continuously differentiable on $\Omega$. Then, under some regularity conditions, the well-known Riemann-Green formula (cf. [5]) states that

$$(2.2) \qquad \iint_\Omega (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \, dx \, dy = \int_{\partial\Omega} P \, dx + Q \, dy.$$

This yields:

**Lemma 2.1.** *Let $f(x,y)$ be a continuously differentiable p-homogeneous function on $\Omega$. Then*

$$(2.3) \qquad (p+2) \iint_\Omega f \, dx \, dy = \int_{\partial\Omega} f(x,y)(x \, dy - y \, dx).$$

*Proof.* As $f$ is $p$-homogeneous, from (2.1)

$$pf(x,y) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \; \forall (x,y) \in \Omega.$$

Let $P(x,y) := -yf(x,y)$ and $Q(x,y) := xf(x,y)$ for every $(x,y) \in \Omega$.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2f(x,y) + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = (p+2)f(x,y) \; \forall (x,y) \in \Omega.$$

Hence, applying the Riemann-Green formula (2.2) to $P$ and $Q$ yields

$$(p+2) \iint_\Omega f \, dx \, dy = \int_{\partial\Omega} f(x,y)(x \, dy - y \, dx),$$

the desired result. $\qquad\square$

When $f \equiv 1$ (i.e. $p = 0$), one immediately retrieves the well-known formula

$$\text{area}(\Omega) = \frac{1}{2} \int_{\partial\Omega} x \, dy - y \, dx.$$

2.2. **The general case.** The previous result generalizes to $R^n$ as follows:

**Lemma 2.2.** *Let $f : R^n \to R$ be a continuously differentiable p-homogeneous function and $\Omega$ a compact domain $R^n$ with boundary $\partial\Omega$. Then*

$$(2.4) \qquad (n+p) \int_\Omega f \, d\omega = \int_{\partial\Omega} \langle \vec{M}, \vec{n} \rangle f(M) \, d\sigma,$$

*where $\vec{n}$ is the unit outward-pointing normal to $\partial\Omega$.*

*Proof.* With notation as in [6], let $X$ be the vector field $X := \sum_j x_j \frac{\partial}{\partial x_j}$. From Proposition 2.3 in [6], we have

$$(2.5) \qquad \int_\Omega \text{div}(X) f \, d\omega + \int_\Omega Xf \, d\omega = \int_{\partial\Omega} \langle X, \vec{n} \rangle f \, d\sigma.$$

Now, $\text{div}(X) = n$ and from (2.1), $Xf = pf$ so that the result follows. $\qquad\square$

Thus, integrating $f$ on $\Omega$ reduces to integrating $\langle \vec{M}, \vec{n} \rangle f$ on the boundary $\partial\Omega$. We next show that Lemma 2.2 further simplifies in the case where the boundary $\partial\Omega$ is determined by homogeneous functions.

## 3. Boundary determined by homogeneous functions

We now consider the special case where

(3.1) $$\Omega(\subset R^n) := \{x \in R^n \mid g_i(x) \le a_i, \ i = 1, \dots, m\},$$

and the functions $g_i$ are each continuously differentiable (positively) $p_i$-homogeneous functions, $i = 1, \dots, m$. We still assume that $\Omega$ is compact. Finally, let $\Omega_i := \{x \in \Omega \mid g_i(x) = a_i\}$ so that $\partial\Omega = \bigcup_{i=1}^m \Omega_i$.

**Theorem 3.1.** *Assume that $f$ is a continuously differentiable $q$-homogeneous function and the functions $g_i$ are each continuously differentiable and $p_i$-homogeneous, $i = 1, \dots, m$. Then*

(3.2) $$(n+q) \int_\Omega f d\omega = \sum_{i=1}^m p_i a_i \int_{\Omega_i} \frac{f}{\|\nabla g_i\|} d\sigma.$$

*In particular, with $f \equiv 1$, one obtains*

(3.3) $$n \times \mathrm{vol}(\Omega) = \sum_{i=1}^m p_i a_i \int_{\Omega_i} \|\nabla g_i\|^{-1} d\sigma.$$

*Proof.* Lemma 2.2 applies so that

(3.4) $$(n+q) \int_\Omega f d\omega = \sum_{i=1}^m \int_{\Omega_i} \langle X, \vec{n} \rangle f d\sigma.$$

Now, $\vec{n} = \|\nabla g_i\|^{-1} \nabla g_i$ on $\Omega_i$. Therefore, using (2.1), we have on $\Omega_i$

$$\langle X, \nabla g_i \rangle = \langle x, \nabla g_i(x) \rangle = p_i g_i(x) = p_i a_i,$$

which yields the desired result. $\qquad\square$

*Remark.* When $\Omega$ is a convex polytope, i.e. $g_i(x) = \langle A_i, x \rangle$ and $p_i = 1$, $i = 1, \dots, m$, then $\|\nabla g_i\| = \|A_i\|$ and thus (3.2) simplifies to

$$(n+q) \int_\Omega f d\omega = \sum_{i=1}^m \frac{a_i}{\|A_i\|} \int_{\Omega_i} f d\sigma.$$

In addition, one may iterate the process for $\int_{\Omega_i} f d\sigma$ (cf. [4] for more details).

*An alternative proof.* Interestingly enough, there is a direct proof that does not use Green's formula. It was used in [4] for the case where $\Omega$ is a convex polytope.

Write $a_i = b_i^{p_i}$, $i = 1, \dots, m$, and let

(3.5) $$h(b) := \int_\Omega f d\omega = \int_{\{g_i(x) \le b_i^{p_i}, \ i=1,\dots,m.\}} f dx.$$

It is immediate that $h(\lambda b) = \lambda^{n+q} h(b)$, i.e. $h$ is $(n+q)$-homogeneous and continuously differentiable. In addition, one may show that

$$\frac{\partial h}{\partial b_i} = p_i b_i^{p_i-1} \int_{\Omega_i} \frac{f}{\|\nabla g_i\|} d\sigma.$$

Hence, applying Euler's formula to $h$ yields

(3.6) $$(n+q)h(b) = \langle \nabla h(b), b \rangle = \sum_{i=1}^m p_i a_i \int_{\Omega_i} \frac{f}{\|\nabla g_i\|} d\sigma,$$

the desired result.

**Example.** In $R^2$, consider the domain

$$\Omega := \{(x,y) \in R^2 \,|\, xy \le 1; \ x^2 + y^2 \le R^2; \ x, y \ge 0\}$$

and its volume $\int_\Omega d\omega$ (i.e. $f \equiv 1$ is 0-homogeneous).

Note that in (3.3), the coordinate axis boundaries of $\Omega$ do not contribute since the boundary values $a_i$ are zero in this case.

Both $g_i(x,y) := xy$ and $g_2(x,y) := x^2 + y^2$ are 2-homogeneous. $\Omega_1 := \{(x,y) \in \Omega \,|\, xy = 1\}$ and $\Omega_2 := \{(x,y) \in \Omega \,|\, x^2 + y^2 = R^2\}$. $\|\nabla g_1(M)\| = \sqrt{x^2 + y^2}$ so that on $\Omega_1$ we have $\|\nabla g_1(M)\| = \sqrt{(x^4 + 1)/x^2}$ and $d\sigma = \sqrt{(x^4 + 1)/x^4} dx$, which yields

$$(3.7) \qquad \int_{\Omega_1} \|\nabla g_1\|^{-1} d\sigma = \int_{a^{-1}}^a x^{-1} dx = 2\log(a),$$

where $x = a$ is the solution to $x^2 + y^2 = R^2$, $xy = 1$.

Similarly, $\|\nabla g_2\| = 2R$ on $\Omega_2$, so that

$$\int_{\Omega_2} \|\nabla g_2\|^{-1} d\sigma = 2 \times (1/2R) \times (R\alpha) = \alpha,$$

where $\alpha = \arctan(a^{-2})$. Hence, Theorem 3.1 yields

$$2\,\mathrm{vol}(\Omega) = 4\log(a) + R^2 \arctan(a^{-2}),$$

i.e. $\mathrm{vol}(\Omega) = 2\log(a) + R^2 \arctan(a^{-2})/2$. This can be retrieved directly.

If we now take $\Omega := \{(x,y) \,|\, xy \le 1; b \le y \le a; x \ge 0\}$ with $a \ge b > 0$, we get $\Omega_2 := \{(x,y) \in \Omega \,|\, y = a\}$ and $\Omega_3 := \{(x,y) \in \Omega \,|\, -y = -b\}$. Therefore,

$$\int_{\Omega_1} \|\nabla g_1\|^{-1} d\sigma = \log\left(\frac{a}{b}\right); \ \int_{\Omega_2} \|\nabla g_2\|^{-1} d\sigma = \frac{1}{a}; \ \int_{\Omega_3} \|\nabla g_3\|^{-1} d\sigma = \frac{1}{b},$$

so that Theorem 3.1 yields

$$2\mathrm{vol}(\Omega) = 2\log\left(\frac{a}{b}\right) + a\frac{1}{a} - b\frac{1}{b} = 2\log\left(\frac{a}{b}\right),$$

i.e. $\mathrm{vol}(\Omega) = \log(a/b)$, which is immediate to check.

## 4. INTEGRATING EXPONENTIALS AND LOGARITHMS

In this section, we consider exponentials and logarithms of homogeneous functions. Indeed, we show that integrating such functions on $\Omega$ reduces to integrating a related function on the boundary $\partial\Omega$.

**4.1. Exponentials.** We first consider the case of an exponential $e^{h(x)}$ where $h : R^n \to R$ is $q$-homogeneous. The particular case where $h = \langle c, x \rangle$ was considered in [1] and [2] and was used to compute the number of integral points in a convex polytope. We have the following result.

**Theorem 4.1.** *Let $\Omega$ be the convex polytope $\{x \in R^n \,|\, \langle A_i, x \rangle \le a_i, i = 1, \ldots, m\}$.*
*(a) Let $h : R^n \to R$ be $q$-homogeneous and continuously differentiable. Then, for every $k = 0, 1, \ldots,$*

$$(4.1) \qquad (n + qk) \int_\Omega h^k e^h \, d\omega + q \int_\Omega h^{k+1} e^h \, d\omega = \sum_{i=1}^m \frac{a_i}{\|A_i\|} \int_{\Omega_i} h^k e^h \, d\sigma.$$

*(b) Let $h : R^n \to R$ be $q$-homogeneous, continuously differentiable and strictly positive on $\Omega$. Then, for every $k = 1, \ldots,$*

$$(4.2) \qquad (n - qk) \int_\Omega h^{-k} e^h \, d\omega + q \int_\Omega h^{-k+1} e^h \, d\omega = \sum_{i=1}^m \frac{a_i}{\|A_i\|} \int_{\Omega_i} h^{-k} e^h \, d\sigma.$$

*(c) If $p := n/q$ is an integer, then*

$$(4.3) \qquad q \int_\Omega h^{-p+1} e^h \, d\omega = \sum_{i=1}^m \frac{a_i}{\|A_i\|} \int_{\Omega_i} h^{-p} e^h \, d\sigma$$

*and $\int_\Omega h^k e^h \, d\omega$, $k = -p+1, \ldots, -1, 0, 1, \ldots,$ can all be expressed as integrals on $\partial\Omega$. In particular,*

$$(4.4) \quad q \int_\Omega e^h \, d\omega = \sum_{i=1}^m \frac{a_i}{\|A_i\|} \int_{\Omega_i} e^h [h^{-1} - (p-1)h^{-2} + (p-1)(p-2)h^{-3}$$
$$+ \cdots + (-1)^{p+1}(p-1)! h^{-p}] d\sigma.$$

*(d) With $h := \langle c, x \rangle$, and for every $\xi \in R^n$, we get*

$$(4.5) \qquad \langle c, \xi \rangle \int_\Omega e^{\langle c, x \rangle} \, d\omega = \sum_{i=1}^m \frac{\langle A_i, \xi \rangle}{\|A_i\|} \int_{\Omega_i} e^{\langle c, x \rangle} \, d\sigma.$$

*Proof.* Again, with $f := h^k e^h$ and with the vector field $X := \sum_i x_i \frac{\partial}{\partial x_i}$, Green's formula yields

$$(4.6) \qquad n \int_\Omega f \, d\omega + \int_\Omega Xf \, d\omega = \int_{\partial\Omega} \langle X, \vec{n} \rangle f \, d\sigma.$$

Using

$$Xf = (kh^{k-1} + h^k) e^h \sum_{i=1}^n x_i \frac{\partial h}{\partial x_i} = q(kh^k + h^{k+1}) e^h, \text{ and } \langle X, \vec{n} \rangle = \frac{a_i}{\|A_i\|} \text{ on } \Omega_i,$$

we obtain

$$(4.7) \qquad (n + qk) \int_\Omega h^k e^h \, d\omega + q \int_\Omega h^{k+1} e^h \, d\omega = \sum_{i=1}^m \frac{a_i}{\|A_i\|} \int_{\Omega_i} h^k e^h \, d\sigma,$$

i.e. (4.1). Similarly, with $f := h^{-k} e^h$ and similar arguments we obtain (4.2).

To get (c), just notice from (4.2) that $\int_\Omega h^{-n/q+1} e^h d\omega$ is expressed directly as an integral over $\partial\Omega$, which yields (4.3). Using (4.2) with $k := n/q - 1$ and (4.3), one obtains $\int h^{-n/q+2} e^h d\omega$ as an integral on $\partial\Omega$. Therefore, iterating and using (4.2) and (4.1), all the $\int_\Omega h^k e^h d\omega$, for $k := -n/q + 1, \ldots, -1, 0, 1, \ldots,$ are expressed as integrals on $\partial\Omega$.

To get (d), apply Green's formula (4.6), but now with $X = \sum_i \xi_i \partial/\partial x_i$, so that $\text{div}(X) = 0$. $\qquad\square$

The formula (4.1) remains valid even with $h \equiv 0$ and $k = 0$. With the convention $0^0 = 1$, (4.1) reduces to the volume formula (3.3) of $\Omega$.

Note that (4.5) was obtained in [1] using Stokes' formula, and a similar argument was used to show that it suffices to consider the vertices of the polyhedron. The same can be done using the above argument.

Indeed, let $\mathcal{H}_i$ be the $(n-1)$-dimensional affine variety that contains $\Omega_i$, and $\{u_1, \ldots, u_{n-1}\}$ an orthonormal basis of the associated vector space. $x \in \Omega_i$ can be

written $x_0 + \sum_{i=1}^{n-1} y_i u_i$ with $x_0 \in \Omega_i$, arbitrary. On $\Omega_i$, consider the vector field $\sum_{1}^{n-1} \xi_i \partial/\partial y_i$. Green's formula yields

$$\langle c, \sum_{i=1}^{n-1} \xi_i u_i \rangle \int_{\Omega_i} e^{\langle c,x \rangle} d\sigma \,=\, \sum_{j \neq i} \langle \xi, \vec{n}_i \rangle \int_{\Omega_{ij}} e^{\langle c,x \rangle} d\nu,$$

where $\Omega_{ij} := \Omega_i \cap \Omega_j$ and $\vec{n}_i$ (in the basis $\{u_1, \ldots, u_{n-1}\}$) is the unit outward-pointing normal to $\Omega_{ij}$. Obviously, the process can be repeated up to the 0-dimensional faces, i.e. the vertices of $\Omega$.

4.2. **Logarithms.** Consider now the function $\log f$ where $f : R^n \to R$ is continuously differentiable, $q$-homogeneous and strictly positive on $\Omega$.

**Lemma 4.2.** *Let $\Omega \subset R^n$ be a compact domain with boundary $\partial\Omega$, and let $f : R^n \to R$ be continuously differentiable, $q$-homogeneous and strictly positive on $\Omega$. Then*

$$(4.8) \qquad n \int_\Omega \log f \, d\omega \,+\, q \times \mathrm{vol}(\Omega) \,=\, \int_{\partial\Omega} \langle \vec{M}, \vec{n} \rangle \log f(M) \, d\sigma.$$

*In addition, if $\Omega := \{x \in R^n \,|\, \langle A_i, x \rangle \leq a_i, \, i = 1, \ldots, m\}$, and $\Omega_i := \{x \in \Omega \,|\, \langle A_i, x \rangle = a_i\}$, then*

$$(4.9) \qquad n \int_\Omega \log f \, d\omega \,+\, q \times \mathrm{vol}(\Omega) \,=\, \sum_{i=1}^m \frac{a_i}{\|A_i\|} \int_{\Omega_i} \log f \, d\sigma.$$

*Proof.* The proof is again the same as for the exponential. Note that with the vector field $X := \sum_j x_j \partial/\partial x_j$, we have $X \log f = q$ and $\mathrm{div}(X) = n$, so that (2.5) reduces to (4.9). □

Hence, integrating $\log f$ on a convex polytope $\Omega$ reduces to integrating $\log f$ on the boundary $\partial\Omega$ and to computing the volume of the polyhedron.

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