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NONEXPANSIVE, $\mathcal{T}$–CONTINUOUS ANTIREPRESENTATIONS HAVE COMMON FIXED POINTS

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ABSTRACT. Let C be a closed convex subset of a Banach (dual Banach) space $\mathfrak{X}$. By $\mathcal{S}$ we denote an antirepresentation $\{T_s : s \in S\}$ of a semitopological semigroup S as nonexpansive mappings on C. Suppose that the mapping $S \times C \ni (s, x) \to T_s x \in C$ is jointly continuous when C has the weak (weak*) topology and the Banach space $RUC(S)$ of bounded right uniformly continuous functions on S has a right invariant mean. If C is weakly compact (for some $x \in C$ the set $\overline{\{T_s x : s \in S\}}^{\text{weak}^*}$ is weakly* compact) and norm separable, then $\{T_s : s \in S\}$ has a common fixed point in C.

Let S be a semitopological semigroup, i.e. S is a semigroup with Hausdorff topology such that for each $a \in S$ the mappings $s \to sa$ and $s \to as$ from S to S are continuous. Let C be a nonempty closed and convex subset of a Banach space $(\mathfrak{X}, \|\cdot\|)$. A family $\mathcal{S} = \{T_s : s \in S\}$ of self–maps of C is called *antirepresentation* of S if

$$T_{sr} = T_r \circ T_s \quad \text{for all } s, r \in S.$$

A set $K \subseteq C$ is said to be $\mathcal{S}$*–invariant* if

$$T_s x \in K \quad \text{for all } s \in S, \text{ whenever } x \text{ is from } K.$$

A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

We shall assume that besides the norm topology $\mathfrak{X}$ is equipped with a Hausdorff topology $\mathcal{T}$, that is weaker than $\|\cdot\|$. A standard example of such a triple $(\mathfrak{X}, \|\cdot\|, \mathcal{T})$ is when $\mathcal{T}$ is the corresponding weak topology on $(\mathfrak{X}, \|\cdot\|)$. Another example for $\mathcal{T}$ is the weak*–topology on $\mathfrak{X}$ if $(\mathfrak{X}, \|\cdot\|)$ is a dual Banach space.

The antirepresentation $\mathcal{S} = \{T_s : s \in S\}$ is said to be continuous if the mapping $S \times C \ni (s, x) \to T_s(x) \in C$ is jointly continuous when C has $\mathcal{T}$–topology. We will assume throughout the paper that all T_s are nonexpansive. Fixed points theorems for continuous antirepresentations have attracted broad attention (cf. [5], [7], [8], [9], [10]). Following them, we recall the concept of an invariant mean. For this, let $C_b(S)$ denote the Banach space of all bounded, real–valued functions on S with the supremum norm $\|\cdot\|_{\sup}$. Given $h \in C_b(S)$ and $s \in S$ we define $h_s(t) = h(ts)$. Clearly $h_s \in C_b(S)$. We say that $h \in C_b(S)$ is right uniformly continuous if the

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mapping $S \ni s \to h_s \in C_b(S)$ is continuous, and we let $RUC(S)$ denote the Banach subspace of all such functions h. A linear functional λ on $RUC(S)$ is called a *mean* if

$$\lambda(h) \geq 0 \quad \text{for nonnegative } h \text{ and } \lambda(\mathbf{1}) = 1\,.$$

A mean λ is said to be *right invariant* if

$$\lambda(h_s) = \lambda(h) \quad \text{for all } h \in RUC(S) \text{ and } s \in S.$$

If T_s are affine, then right invariant means on $RUC(S)$ guarantee common fixed points (see [5], [10] for more details). The question whether for a nonexpansive semigroup the affine condition can be replaced by joint continuity has appeared in [9]. In the papers ([7], [9]) the authors have obtained common fixed points for general nonexpansive semigroups when $\mathfrak{X}$ is uniformly convex. They also assume the norm continuity of $s \to T_s x$. Hence our results are loosely related to theirs. In the sequel we will use basic geometrical properties of compact subsets of Banach spaces. We recall that a convex and closed set K has normal structure if there exists $x \in K$ such that

$$r(x, K) = \sup\{\|x - y\| \,:\, y \in K\} < \operatorname{diam}(K) = \sup\{\|y - z\| \,:\, y, z \in K\}\ .$$

It is well known that norm compact and convex sets have the normal structure. In particular, for each norm relatively compact set $D \subset \mathfrak{X}$ there exists $x \in \overline{\operatorname{conv} D}$ such that

$$\sup\{\|x - d\| \,:\, d \in D\} < \operatorname{diam}(D)\ .$$

Lemma 1. *Let C be a $\mathcal{T}$–closed subset of a Banach space $(\mathfrak{X}, \|\cdot\|, \mathcal{T})$ and $\{T_s : s \in S\}$ be a nonexpansive and $\mathcal{T}$–continuous antirepresentation of S on C. Suppose that for each $y \in C$ the function $C \ni z \to \|z - y\|$ is $\mathcal{T}$ lower semicontinuous and $RUC(S)$ has a right invariant mean λ. Then the following conditions are equivalent:*

(i) *there exists a norm compact and $\mathcal{S}$–invariant subset of C, on which the mappings T_s are invertible isometries*;

(ii) *there exists $x \in C$ such that the closed orbit $\mathcal{O}_{\mathcal{T}}(x) = \overline{\{T_s x\; s \in S\}}^{\mathcal{T}}$ is $\mathcal{T}$–compact and norm separable.*

Proof. Only (ii) $\Rightarrow$ (i) needs to be proved. Let $x \in C$ be arbitrary. For a $\mathcal{T}$–continuous function $H : \mathcal{O}_{\mathcal{T}}(x) \to \mathbb{R}$ we define

$$S \ni s \to h(s) = H(T_s x).$$

Clearly $h \in RUC(S)$. In fact, if $s_\alpha \to s_0$, then

$$\begin{aligned}\|h_{s_\alpha} - h_{s_0}\|_{\sup} &= \sup_{s \in S} |H(T_{ss_\alpha}(x)) - H(T_{ss_0}(x))| \\ &= \sup_{s \in S} |H(T_{s_\alpha}(T_s(x))) - H(T_{s_0}(T_s(x)))| \\ &\leq \sup_{y \in \mathcal{O}_{\mathcal{T}}(x)} |H(T_{s_\alpha}(y)) - H(T_{s_0}(y))| \to 0\end{aligned}$$

as the antirepresentation is $\mathcal{T}$–continuous and $\mathcal{O}_{\mathcal{T}}(x)$ is $\mathcal{T}$–compact.

Now consider the functional $\mu(H) = \lambda(H(T_s(x)))$ which is defined on the Banach space $C_{\mathcal{T}}(\mathcal{O}_{\mathcal{T}}(x))$ of all $\mathcal{T}$–continuous functions on $\mathcal{O}_{\mathcal{T}}(x)$. It follows from the Riesz Representation Theorem that μ is a Radon probability measure on $\mathcal{O}_{\mathcal{T}}(x)$, which is well defined on the Borel σ–field $\mathcal{B}_{\mathcal{T}}$ generated by $\mathcal{T}$. Since the balls $B_{y,r} = \{z \in \mathcal{O}_{\mathcal{T}}(x) \,:\, \|z - y\| \leq r\}$ are $\mathcal{T}$–compact, thus $\mathcal{B}_{\mathcal{T}}$ coincides with $\mathcal{B}_{\|\cdot\|}$.

In particular, μ has a support $\operatorname{supp}(\mu)$ which is the smallest norm closed subset of $\mathcal{O}_{\mathcal{T}}(x)$ of full measure. For each $s \in S$ the measure μ is T_s–invariant. This easily follows from the right invariance of λ. By the classical Poincaré recurrence theorem each $y \in \operatorname{supp}(\mu)$ is T_s–recurrent. Moreover, it follows from nonexpansiveness of T_s that the orbit $\overline{\{T_s^n(y) \,:\, n \geq 0\}}^{\|\cdot\|}$ is T_s minimal and norm compact, and T_s is an invertible isometry on $\operatorname{supp}(\mu)$ (see [2], [3], [4] for all details). The argument used in the proof below is a modification of one from [2]. We strengthen the above and show that the full norm closed orbit

$$\mathcal{O}_n(y) = \overline{\{T_s(y) \,:\, s \in S\}}^{\|\cdot\|} \qquad \text{is norm compact.}$$

For this let us fix $\varepsilon > 0$ and consider $y_j = T_{s_j}(y)$ such that $\|y_j - y_k\| \geq 2\varepsilon$ if $1 \leq j \neq k \leq L$. Note that

$$\mu(\{z \in \operatorname{supp}(\mu) \,:\, \|z - y\| < \varepsilon\}) = \delta > 0\ .$$

Clearly

$$\{z \in \operatorname{supp}(\mu) \,:\, \|z - y\| < \varepsilon\} = T_{s_j}^{-1}\{z \in \operatorname{supp}(\mu) \,:\, \|z - y_j\| < \varepsilon\}$$

as T_s are invertible isometries on $\operatorname{supp}(\mu)$. By invariance of μ we get

$$\mu(\{z \in \operatorname{supp}(\mu) \,:\, \|z - y_j\| < \varepsilon\}) = \delta \qquad \text{for all } 1 \leq j \leq L\ .$$

Since all the sets $\{z \in \operatorname{supp}(\mu) \,:\, \|z - y_j\| < \varepsilon\}$ are pairwise disjoint, we get $L \leq \frac{1}{\delta}$ and $\mathcal{O}_n(y)$ is norm compact. □

Now we are in a position to formulate the main results of the paper.

Theorem 1. *Let C be a closed, convex and norm separable subset of a Banach space $(\mathfrak{X}, \|\cdot\|, \mathcal{T})$ and $\{T_s \,:\, s \in S\}$ be a nonexpansive and $\mathcal{T}$–continuous antirepresentation of S on C. Suppose that for each $y \in C$ the function $C \ni z \to \|z - y\|$ is $\mathcal{T}$ lower semicontinuous and $RUC(S)$ has a right invariant mean λ. If C is $\mathcal{T}$–compact, then there exists $u \in C$ such that $T_s u = u$ for all $s \in S$.*

Proof. Denote by $\mathcal{K}$ the (nonempty) family of all convex and $\mathcal{T}$–compact subsets of C which are $\mathcal{S}$–invariant. It follows from Kuratowski–Zorn's Lemma that there exists a minimal element $K_0 \in \mathcal{K}$. Applying Lemma 1 to K_0 we obtain a norm compact set $\mathcal{O}_n(y) \subseteq K_0$ which is $\mathcal{S}$–invariant. Suppose that $\mathcal{O}_n(y)$ is not a singleton. Then there exists $y_0 \in \overline{\operatorname{conv} \mathcal{O}_n(y)}^{\|\cdot\|} \subseteq K_0$ and $0 < r < \operatorname{diam} \mathcal{O}_n(y)$ such that

$$\|z - y_0\| \leq r \qquad \text{for all } z \in \mathcal{O}_n(y).$$

Define

$$K(y) = \bigcap_{z \in \mathcal{O}_n(y)} \{v \in K_0 \,:\, \|v - z\| \leq r\}\ .$$

It is straightforward to verify that $K(y)$ is convex, nonempty ($y_0 \in K(y)$) and $\mathcal{T}$–compact. Let $v \in K(y)$, $z \in \mathcal{O}_n(y)$ and $s \in S$ be arbitrary. Then

$$(*) \qquad \|T_s(v) - z\| = \|T_s(v) - T_s(T_s^{-1}z)\| \leq \|v - T_s^{-1}z\| \leq r\ ,$$

where $T_s^{-1}z$ is chosen from $\mathcal{O}_n(y)$. This proves $\mathcal{S}$–invariance of $K(y)$. Note that the diametral points of $\mathcal{O}_n(y)$ do not belong to $K(y)$. Hence

$$\mathcal{K} \ni K(y) \subsetneq K_0$$

what contradicts minimality of K_0. □

We remark that for each y the mapping $z \to \|z - y\|$ is weak l.s.c. or weak* l.s.c. if $\mathfrak{X}$ is a dual space. So, directly from our Theorem 1 we get the proposition which refers to some results from [7] and [9]. There is a version of Theorem 1 for weak* topologies. We also see that the assumption on the weak* compactness of C may be weakened. Namely we have:

Proposition 1. *Let $\mathfrak{X} = (\mathfrak{X}_*)^*$ be a dual Banach space and C a weak* closed, convex and norm separable subset of $\mathfrak{X}$. Assume that $\mathcal{S} = \{T_s : s \in S\}$ is a weak* continuous antirepresentation of a semitopological semigroup S as nonexpansive mappings of C into itself, and $RUC(S)$ has a right invariant mean. Then the following conditions are equivalent:*

(i) *there exists $u \in C$ such that $T_s u = u$ for all $s \in S$;*

(ii) *there exists $x \in C$ such that $\overline{\{T_s x\ s \in S\}}^{weak^*} = \mathcal{O}_*(x)$ is bounded.*

Proof. Similarly as in Lemma 1 we only need to prove (ii) $\Rightarrow$ (i). By the Banach–Alaoglu Theorem the set $\mathcal{O}_*(x)$ is weak* compact. It follows from Lemma 1 that $\mathcal{O}_n(y) \subseteq \mathcal{O}_*(x)$ for some $y \in \mathcal{O}_*(x)$, where $\mathcal{O}_n(y)$ is norm compact and $\mathcal{S}$–invariant. From now we repeat the same arguments as in the proof of Theorem 1. Namely the set

$$K(y) = \bigcap_{z \in \mathcal{O}_n(y)} \{v \in C : \|v - z\| \leq r\}$$

is weak* compact, $\mathcal{S}$–invariant, convex and nonempty. In particular a minimal, weak* compact, $\mathcal{S}$–invariant and convex (nonempty) subset of $K(y)$ is a singleton. □

Problem. Is the assumption on norm separability essential in Proposition 1?

In the last paragraphs of the paper we briefly discuss the case of the weak topology. Here the separability assumption can be dropped as long as the semigroup S has the property that every finitely generated subsemigroup is right amenable. This obviously holds if S is commutative. It has been pointed out by the referee that in the light of [6] this extra condition may be very restrictive.

Given a subset $S' \subseteq S$ by $S(S')$ we denote a subsemigroup generated by S'. The corresponding family of transformations $\{T_s : s \in S(S')\}$ is denoted by $\mathcal{S}'$. Now let C be a convex and weakly compact subset of a Banach space $\mathfrak{X}$, and $\{T_s : s \in S\}$ be an antirepresentation of a semitopological semigroup S as nonexpansive mappings on C. If for every finite $S' \subseteq S$ the mapping

$$S(S') \times C \ni (s, x) \to T_s x \in C$$

is jointly continuous when C is considered with the weak topology and $RUC(S(S'))$ has a right invariant mean, then there exists $u \in C$ such that $T_s u = u$ for all $s \in S$. To show this let $C_x(S')$ denote the smallest convex and weakly closed set which is $\mathcal{S}(S')$–invariant, where $S' \subseteq S$ is finite and $x \in C$. Clearly $C_x(S')$ are norm separable. By Theorem 1 for every finite $S' \subseteq S$ there exists $u \in C_x(S') \subseteq C$ which is T_s–invariant for every $s \in S(S')$. The rest follows from the finite intersection property the set C enjoys.

If in addition the Banach space $\mathfrak{X}$ is a reflexive, then the assumption on weak compactness of C may be replaced by the weaker one. Namely it is sufficient that there exists $x \in C$ such that the trajectory $\{T_s x : s \in S\}$ is norm bounded. To get this we apply Proposition 1.

Remark. It follows from the example due to Alspach [1] that $\mathcal{T}$–continuity of the semigroup $\mathcal{S}$ is essential even if T_s are isometries.

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