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ON SUBSPACES OF PSEUDORADIAL SPACES

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Abstract. A topological space X is pseudoradial if each of its non closed subsets A has a sequence (not necessarily with countable length) convergent to outside of A. We prove the following results concerning pseudoradial spaces and the spaces $\omega \cup \{p\}$, where p is an ultrafilter on ω:

(i) CH implies that, for every ultrafilter p on ω, $\omega \cup \{p\}$ is a subspace of some regular pseudoradial space.

(ii) There is a model in which, for each P-point p, $\omega \cup \{p\}$ cannot be embedded in a regular pseudoradial space while there is a point q such that $\omega \cup \{q\}$ is a subspace of a zero-dimensional Hausdorff pseudoradial space.

1. Introduction and definitions

In [1] the authors asked if $\omega \cup \{p\}$ is a subspace of a pseudoradial space for $p \in \beta\omega \setminus \omega$, where $\omega \cup \{p\}$ takes the subspace topology in $\beta\omega$, i.e, p is the only non-isolated point and a neighbourhood of p is $A \cup \{p\}$ for $A \in p$. It is proved in [9] that under the assumption $\mathfrak{p} = \mathfrak{c}$, each space with countable tightness is a subspace of some pseudoradial space. It is essentially proved that, under the assumption $\mathfrak{p} = \mathfrak{c}$, $\omega \cup \{p\}$ is a subspace of a Hausdorff pseudoradial space for $p \in \beta\omega \setminus \omega$. However the pseudoradial space constructed in [9] is not regular. In a communication with the second author, J. Vaughan asked if it is possible to make the pseudoradial space in [9] regular. In [7] P. Nyikos asked whether each topological space can be embedded into a pseudoradial space (Problem 6.22). Our discussion will be focused on the class of regular spaces. We first prove, in section 2 , that, under CH, $\omega \cup \{p\}$ is a subspace of a regular pseudoradial space for each ultrafilter p on ω . However, the question of whether CH can be replaced by MA remains open. In section 3, we prove that if the ground model satisfies CH, then in the forcing extension obtained by adding $\aleph_2$ many Cohen reals, $\omega \cup \{p\}$, for any P-point p, is not a subspace of any regular pseudoradial space. In section 4, we show that, for a special ultrafilter p constructed in [5], if there exists a special stationary set and $\mathfrak{c} = \omega_2$, then $\omega \cup \{p\}$ can be embedded into a zero-dimensional Hausdorff pseudoradial space. We also prove that it is consistent that $\mathfrak{c} = \omega_2$ while there is such a stationary set. We conclude that there is a forcing extension in which, for every P-point p, $\omega \cup \{p\}$ cannot be embedded in any regular pseudoradial space while there is an ultrafilter q on ω such that $\omega \cup \{q\}$ is a subspace of a zero-dimensional Hausdorff pseudoradial space.

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Definition 1.1. A sequence $\{x_\alpha : \alpha < \lambda\}$ of points in X is said to be convergent to x provided that for each neighbourhood U of x, there exists $\alpha < \lambda$ such that $\{x_\beta : \beta > \alpha\} \subseteq U$. (We will call any transfinite sequence a sequence.)

A space X is *radial* if for each subset A of X and x in $\bar{A}$, there is a sequence in A convergent to x. The pseudoradial spaces have weaker properties. This is analogous to the relationship between sequential spaces and Frechét spaces.

Definition 1.2. A subset A of a space of X is said to be *radially closed* if no sequence in A is convergent to outside of A. A space X is said to be *pseudoradial* provided that any radially closed subset of X is closed in X.

In this paper, a topological space is always regular. The set theory notions are standard. For example, if A and B are two sets, we write $A =^* B$ when A and B are equal modulo finite.

2. Under CH

If p is a $\mathrm{P}_\mathfrak{c}$ point on ω , let $\{A_\alpha : \omega \le \alpha < \mathfrak{c}\}$ be a strictly decreasing modulo finite base for p and let B be the Boolean Algebra generated by $\{A_\alpha : \alpha \in \mathfrak{c}\} \cup [\omega]^{<\omega}$; then it is easy to see that the Stone space of B is just $\omega \cup [\omega, \mathfrak{c}+1]$ where $[\omega, \mathfrak{c}+1]$ has the usual ordinal topology and $\omega \cup \{\mathfrak{c}\}$ is homeomorphic to $\omega \cup \{p\}$. An immediate corollary is that , under CH, if p is a P-point, then $\omega \cup \{p\}$ can be embedded into a zero-dimensional Hausdorff compact pseudoradial space.

However, we have the following result about non P-points.

Lemma 2.1 (p=$\mathfrak{c}$)**.** *If p is not a P-point, then $\omega \cup \{p\}$ can be embedded in a zero-dimensional pseudo-radial space.*

Proof. Let $p = \{A_\alpha : \alpha < \mathfrak{c}\}$ such that $\{A_i : i < \omega\}$ witnesses that p is not a P-point. First of all, we will construct a base $\{B_\alpha : \alpha < \mathfrak{c}\}$ of p and an almost disjoint family $\{C_\alpha : \alpha < \mathfrak{c}\}$ such that, for $\xi, \alpha \in \mathfrak{c}$, the following are true:

$(i)_\alpha$ $\quad B_\alpha \subseteq A_\alpha$,
$(ii)_{\xi\alpha}$ $\quad \xi \le \alpha \to C_\alpha \subseteq^* B_\xi$,
$(iii)_{\xi\alpha}$ $\quad \alpha < \xi \to C_\alpha \cap B_\xi =^* \emptyset$.

Without loss of generality, we can assume that $\{A_n : n \in \omega\}$ is strictly decreasing with empty intersection. We also assume that $A_0 = \omega$. It is easy to construct $\{B_\xi, C_\xi : \xi \le \omega\}$ with $B_n = A_n$ for $n \in \omega$. Suppose we have constructed $\{B_\xi, C_\xi : \xi < \alpha\}$ with $\alpha \ge \omega$ and such that $(i)_\xi$, $(ii)_{\xi\eta}$ and $(iii)_{\xi\eta}$ are true for each $\xi, \eta < \alpha$. We have to define B_α and C_α. For $\xi \in \alpha$ and $n \in \omega$, if $\xi \ge \omega$, then $C_\xi \subseteq^* A_n$. By the fact that $\mathfrak{p} = \mathfrak{c}$, there is a $D \subseteq \omega$ such that D is a pseudo-intersection of $\{A_n : n \in \omega\}$ and, for $\omega \le \xi < \alpha$, $C_\xi \subseteq^* D$. Indeed, we can take, for $\omega \le \xi < \alpha$, an $f_\xi \in {}^\omega\omega$ such that for each n, $C_\xi - A_n \subseteq f_\xi(n)$. Since $\mathfrak{p} = \mathfrak{c}$, $\mathfrak{b} = \mathfrak{c}$, hence we can choose f which is an upper bound of $\{f_\xi : \omega \le \xi < \alpha\}$ in $({}^\omega\omega, <^*)$. We assume f is strictly increasing. Let $D = \bigcup_{n\in\omega} A_n \cap f(n)$. It is not difficult to see that D works. By the choice of D, we know that D is not a member of p. Let $B_\alpha = A_\alpha \cap B_\omega - D$. Let C_α be any pseudo-intersection of $\{B_\xi : \xi \le \alpha\}$. It is easy to check that B_α and C_α satisfies the requirements.

Next we will use the family $\{B_\alpha, C_\alpha : \alpha \in \mathfrak{c}\}$ to construct a topology τ on $X = \omega \cup ((\mathfrak{c}+1) \times \{0\})$ by assigning a neighbourhood system for each point. First of all, ω is an open discrete dense subset of $\langle X, \tau\rangle$. For $\alpha < \mathfrak{c}$, $\{(C_\alpha - n) \cup \{\langle \alpha, 0\rangle\} : n \in \omega\}$ is a neighbourhood base for $\langle \alpha, 0\rangle$. Let $\{(B_\alpha - n)\cup\{\langle \xi, 0\rangle : \alpha \le \xi \le \mathfrak{c}\} : \alpha \in \mathfrak{c}, n \in \omega\}$ be a neighbourhood base for $\langle \mathfrak{c}, 0\rangle$. It is easy to see that $\langle X, \tau\rangle$ is a zero-dimensional Hausdorff pseudoradial space and the subspace $\omega \cup \{\langle \mathfrak{c}, 0\rangle\}$ is homeomorphic to $\omega \cup \{p\}$. □

Thus we proved

Theorem 2.2 (CH)**.** *For every point p in ω^*, $\omega \cup \{p\}$ can be embedded in a zero-dimensional pseudoradial space.* □

By Theorem 3.1 in the next section, the above lemma is not always true in ZFC.

3. In Cohen model

We assume CH is true in this section.

Theorem 3.1 (CH)**.** *If $\mathbb{P} = Fn(\omega_2, 2)$, then in $V^{\mathbb{P}}$, for every P-point p , the space $\omega \cup \{p\}$ cannot be embedded in any regular pseudoradial space.*

Let us just remark that the theorem remains valid if CH is dropped and κ many Cohen reals are added for any regular $\kappa > \mathfrak{c}$.

Proof. Suppose

$q \Vdash$ "$\dot p$ is an ultrafilter on ω and $\omega \cup \{\dot p\}$

is embedded into a regular pseudoradial space $\langle X, \dot\tau\rangle$" .

We have to prove that $q \not\Vdash$ "$\dot p$ is a P-point ". We will work in $V^{\mathrm{Fn}(\lambda,2)}$, where λ is $M \cap \omega_2$ for some elementary submodel of H_θ for a certain regular cardinal θ. We prove some preliminary facts. We will assume CH is true in the remainder of the section. Without loss of generality, we assume that X is in the ground model and we identify $\omega \cup \{\dot p\}$ with its copy in X. We furthermore assume that $\dot p$ has the form $\{\{\pi\} \times A_\pi : \pi \in \mathrm{dom}(\dot p)\}$, where π is a nice name for a subset of ω. Let $\dot{\mathcal{I}}$ be a nice $\mathbb{P}$-name such that

$$\mathbf{1} \Vdash \text{“}\dot{\mathcal{I}} = \{I \in [\omega]^\omega : I \text{ converges in } \langle X, \dot\tau\rangle\}\text{”}.$$

We also fix a sufficiently large regular cardinal θ and an elementary submodel M of H_θ such that the following are true:

(i) $M^\omega \subseteq M$,
(ii) $\{\mathbb{P}, X, \dot p, \dot\tau, \dot{\mathcal{I}}\} \subset M$ and
(iii) M has cardinality $\aleph_1$.

Let $\mathbb{P}_M = \mathbb{P} \cap M$, $X_M = X \cap M$ and $\lambda = M \cap \omega_2$. For each $\mathbb{P}$-name $\dot Y \in M$, if $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\dot Y \subseteq X$", we can define a $\mathbb{P}_M$-name $\dot Y_M$ as follows. For each $x \in X_M$, let A_x be a maximal antichain in $\{q \in \mathbb{P}_M : q \Vdash_{\mathbb{P}} \text{“}x \in \dot Y\text{”}\}$. $\mathbb{P}_M$ is actually $\mathrm{Fn}(\lambda, 2)$; therefore $\mathbb{P}_M$ is completely embedded in $\mathbb{P}$. Hence A_x is also a maximal antichain in $\{q \in \mathbb{P} : q \Vdash_{\mathbb{P}} \text{“}x \in \dot Y\text{”}\}$. Since $M^\omega \subseteq M$ and $\mathbb{P}$ is *ccc*, we can choose A_x from M. Let $\dot Y_M$ be $\{\{x\} \times A_x : x \in X_M\}$. If, on the other hand, $\dot{\mathcal{Y}}$ is a $\mathbb{P}$-name such that $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\dot{\mathcal{Y}} \subseteq \mathcal{P}X$", we define a $\mathbb{P}_M$-name $\dot{\mathcal{Y}}_M$ to be

$$\{\langle \dot Y_M, \mathbf{1}\rangle : \dot Y \in M \text{ and } \mathbf{1} \Vdash_{\mathbb{P}} \text{“}\dot Y \in \dot{\mathcal{Y}}\text{”}\}.$$

Fact 3.2. *Suppose G is a $\mathbb{P}$-generic over V, and $\sigma \in M$ such that $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\sigma \subseteq X$", then the following hold:*

(i) *if $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\sigma \subseteq \omega$", then $\sigma_M \in M$; moreover, if σ is a nice name, we can define σ_M to be σ;*

(ii) $V[G] \models$ "$\mathrm{val}(\sigma_M, G) = \mathrm{val}(\sigma_M, M \cap G) = \mathrm{val}(\sigma, G) \cap M$"; *and*

(iii) *if* $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\dot{p}$ is a P-point ", *then,* $\mathbf{1} \Vdash_{\mathbb{P}_M}$ "$\dot{p}_M$ is an P-point ".

Proof of Fact 3.2. (i) and (ii) are straightforward. For (iii), suppose that $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\dot{p}$ is a P-point ". It is easy to see that $\mathbf{1} \Vdash_{\mathbb{P}_M}$ "$\dot{p}_M$ is an ultrafilter on ω". We prove that $\mathbf{1} \Vdash_{\mathbb{P}_M}$ "$\dot{p}_M$ is countably complete ". Suppose $\dot{f}$ is a $\mathbb{P}_M$-name and $q \in \mathbb{P}_M$ such that q forces $\dot{f}$ is a function from ω onto $\dot{p}_M$. For each $n \in \omega$, we can choose an antichain $B_n = \{q_i^n : i \in \omega\}$ which is maximal in $\{r \in \mathbb{P}_M : r \leq q\}$ and an associated set $\{\pi_i^n : i \in \omega\} \subseteq \mathrm{dom}(\dot{p}_M)$ such that q_i^n forces $\dot{f}(n) = \pi_i^n$ for each $i \in \omega$. By (i), $\mathrm{dom}(\dot{p}_M) \subseteq M$. Therefore the $\mathbb{P}_M$-name $\rho = \{\langle op\langle \check{n}, \pi_i^n\rangle, q_i^n\rangle : n,\ i \in \omega\}$ is in M, where $op\langle \check{n}, \pi_i^n\rangle$ is a $\mathbb{P}_M$-name such that $\mathrm{val}(op\langle \check{n}, \pi_i^n\rangle, G)$ is $\langle n, \mathrm{val}(\pi_i^n, G)\rangle$ (see [6]). It is obvious that $q \Vdash$ "$\dot{f} = \rho$".

Since $q \Vdash_{\mathbb{P}}$ "$\dot{p}$ is countably complete ", we can choose a nice $\mathbb{P}$-name $\dot{Y} \in M$ such that $q \Vdash_{\mathbb{P}}$ "$\dot{Y}$ is a pseudo-intersection of $\mathrm{ran}(\rho)$" and $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\dot{Y} \in \dot{p}$". By (i) , we can define $\dot{Y}_M$ to be $\dot{Y}$. It is clear that $q \Vdash_{\mathbb{P}_M}$ "$\dot{Y}_M = \dot{Y}$"; hence $q \Vdash_{\mathbb{P}_M}$ " $\mathrm{ran}(\dot{f}) = \mathrm{ran}(\rho) > \dot{Y}_M \in \dot{p}_M$".

It is obvious that $\mathbb{P}_M$ is actually $\mathrm{Fn}(\lambda, 2)$. Let $\mathbb{P}_1$ be $\mathbb{P}_M$ and $\mathbb{P}_2$ be $\mathrm{Fn}(\omega_2 - \lambda, 2)$, then $\mathbb{P}$ is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_2$. For a $\mathbb{P}$-generic filter G, we will also refer to $G \cap M$ as G_M or as G_1 and G_2 will be $G \cap \mathbb{P}_2$ while V_1 will be $V[G_1]$. To further simplify the notation, we will use Y to denote $\mathrm{val}(\dot{Y}, G)$ for a $\mathbb{P}$-name $\dot{Y}$. If $\dot{\mathcal{Y}}$ and $\dot{Y}$ are in M, we will use Y_M and $\mathcal{Y}_M$ to denote $\mathrm{val}(\dot{Y}_M, G_M)$ and $\mathrm{val}(\dot{\mathcal{Y}}_M, G_M)$ respectively. Thus p, τ, p_M, τ_M, etc. are well-defined when it is clear from context which $\mathbb{P}$-generic filter G is being used.

Fact 3.3. *The following are true:*

(i) $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\dot{p}_M = \dot{p} \cap V_1,\ \dot{\mathcal{I}}_M = \dot{I} \cap V_1$";

(ii) *if* $\rho \in M$ *is a* $\mathbb{P}$*-name such that* $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\rho \subseteq \mathcal{P}(\omega)$", *then* $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\rho_M = \rho \cap V_1$"; *and*

(iii) $\mathbf{1} \Vdash_{\mathbb{P}_M}$ "$\dot{\mathcal{I}}_M = \{a \in [\omega]^\omega : a$ converges in $\langle X_M, \dot{\tau}_M\rangle\}$".

Proof of Fact 3.3. (i) is a corollary of (ii). To prove (ii) , let ρ satisfy the conditions of (ii) and let G be a $\mathbb{P}$-generic filter over V. In $V[G]$, take arbitrary $a \in \mathrm{val}(\rho, G) \cap V_1$. Let $\pi \in \mathrm{dom}(\rho)$ and σ be a nice $\mathbb{P}_M$-name such that $a = \mathrm{val}(\pi, G) = \mathrm{val}(\sigma, G)$. Take a q in G which forces that $\pi = \sigma \in \rho$. Since σ, ρ are in M, $q \cap M \Vdash_{\mathbb{P}}$ "$\sigma \in \rho$". Let $\dot{Y} \in M$ such that $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\dot{Y} \in \rho$ and $(\sigma \in \rho \to \dot{Y} = \sigma)$"; then $\mathrm{val}(\dot{Y}_M, G) \in \mathrm{val}(\rho_M, G)$. But $q \Vdash_{\mathbb{P}}$ "$\pi = \sigma = \dot{Y}$". Hence $a = \mathrm{val}(\sigma, G) = \mathrm{val}(\dot{Y}, G) = \mathrm{val}(\dot{Y}_M, G) \in \mathrm{val}(\rho_M, G)$. Thus we proved $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\rho_M \supseteq \rho \cap V_1$". By the definition of ρ_M , it is obvious that

$$\mathbf{1} \Vdash_{\mathbb{P}} \text{"}\rho_M \subseteq \rho \cap V_1\text{"}.$$

Therefore $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\rho_M = \rho \cap V_1$". We are left to prove (iii)

Let H be a $\mathbb{P}_M$-generic filter over V; there exists a $\mathbb{P}$-generic filter G over V such that $H = G \cap \mathbb{P}_M$. We have to prove that, in $V_1 = V[G_M]$, $\mathcal{I}_M = \{a \in [\omega]^\omega : a$ converges in $\langle X_M, \tau_M\rangle\}$.

" $\subseteq$ ": Let $a \in \mathcal{I}_M$. We have to prove that a converges in $\langle X_M, \tau_M\rangle$. There exists $\dot{Y}_M \in \mathrm{dom}(\dot{\mathcal{I}}_M)$, such that $a = Y_M$. Since $a \subseteq \omega$, by (ii) of Fact 3.2, $Y_M = Y$; hence $a \in M[G]$. In $V[G]$, there exists $x \in X$, such that a converges to x in $\langle X, \tau\rangle$. By elementarity, we can assume x in X_M. Since $a \subseteq \omega \subseteq X_M$ and $\langle X_M, \tau_M\rangle$ is a weaker topology than $\langle X_M, \tau \restriction X_M\rangle$, a obviously converges to x in $\langle X_M, \tau_M\rangle$.

" $\supseteq$ ": Let $a \in V_1 \cap [\omega]^\omega$ and for some $x \in X_M$, a converges to x in $\langle X_M, \tau_M\rangle$. We have to show that a is in $\mathcal{I}_M$. By (i), we only have to prove that a is in $\mathcal{I}$, i.e.

$$V[G] \models \text{“}a \text{ converges in } \langle X, \tau\rangle\text{”}.$$

By the facts that $M^\omega \subseteq M$ and that $\mathbb{P}$ is *ccc* , any nice $\mathbb{P}_M$-name for a subset of ω is in M. In particular, a is in $M[G]$. By the choice of θ and the elementarity, it is sufficient to prove that

$$M[G] \models \text{“}a \to x \text{ in } \langle X, \tau\rangle\text{”} .$$

Let U be in $\tau \cap M[G]$ containing x. We have to show that a is almost contained in U. There exists $\dot{Y} \in M$ such that $U = \mathrm{val}(\dot{Y}, G)$, we can assume $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\dot{Y} \in \tau$"; hence $\dot{Y}_M \in \mathrm{dom}(\dot{\tau}_M)$. Notice that $U \cap X_M = U \cap M = Y_M$ is in τ_M. Therefore a is almost contained in $U \cap X_M$ therefore in U.

Now we are ready to finish the proof. Assume that q does force $\dot{p}$ to be a P-point. We can assume that q is the empty function , i.e. $q = \mathbf{1}$ (otherwise we can work instead with $\mathbb{P}_q = \{r \in \mathbb{P} : r \leq q\}$ which is isomorphic to $\mathbb{P}$). Suppose θ, M, and $\mathcal{I}$ are obtained as in the previous discussions and G is a $\mathbb{P}$-generic filter over V. Let $a \in V[G]$ be a Cohen real over V_1 such that a is in p. Since $\langle X, \tau\rangle$ is regular, we can take two open neighbourhoods of p , say W and U, such that $\mathrm{cl}(W) \subseteq U$ and $a = U \cap \omega$. Let $b = W \cap \omega$. Given any I in $\mathcal{I}_M$, let x be in X_M such that I converges to x. By (i) of the Fact 3.3, I also converges to x in $\langle X, \tau\rangle$. If $b \cap I$ is infinite , then x is in $cl(W) \subseteq U$. Therefore I is almost contained in a. This contradicts the fact that a is a Cohen real over V_1. Thus we proved that for each I in $\mathcal{I}_M$, $b \cap I$ is finite. Let $\dot{\tau}, \dot{\mathcal{I}}, \dot{a}, \dot{U}, \dot{W}, \dot{b}$ and $\dot{p}$ denote both the $\mathbb{P}$-names and $\mathbb{P}_2$-names for the corresponding objects in $V[G]$. Choose an $r \in G_2$, such that r forces the above fact over V_1.

Now work in V_1, let κ be a sufficiently large regular cardinal, and let N be a countable submodel of H_κ such that N contains $\{\dot{\tau}, \dot{\mathcal{I}}, \dot{a}, \dot{U}, \dot{W}, \dot{b}, \dot{p}, \lambda\}$. By (iii) of Fact 3.2, p_M is a P-point. We can take A in p_M such that for each $B \in N \cap p_M$, $A \subseteq^* B$. Since in $V[G]$, A is in p and $\langle X, \tau\rangle$ is pseudoradial, there exists $I \in \mathcal{I}$ such that $I \subset A$. Notice that $A \in M[G]$, and , according to [8], $M[G]$ is an elementary submodel of $H_\theta^{V[G]}$. By elementarity, we can assume $I \in M[G]$. Hence $I \in V_1$. Therefore , by (i) of Fact 3.3, $I \in \mathcal{I}_M$. By the choice of r, $r \Vdash_{\mathbb{P}_2}$ "$\dot{b} \cap I$ is finite ", let $s \in \mathbb{P}_2$ and $m \in \omega$, such that $r \geq s \Vdash_{\mathbb{P}_2}$ "$\dot{b} \cap I \subseteq m$".

Claim. $s \cap N \Vdash_{\mathbb{P}_2}$ "$\dot{b} \cap I \subseteq m$". Otherwise, let $k \in I - m$, such that there is $t \leq s \cap N$, $t \Vdash_{\mathbb{P}_2}$ "$k \in \dot{b}$". Since k and $\dot{b}$ are in N, we can assume $t \in N$. But then t and s are compatible while they force contradicting statements. Therefore we can assume $s \in N$. Let $Z = \{n : \exists t \leq s, t \Vdash_{\mathbb{P}_2} \text{“}n \in \dot{b}\text{”}\} \in N$. Since $Z \supseteq b$, $Z \in p_M$. Now $I \subseteq A \subseteq^* Z$, and take $k \in I \cap Z - m$ and $t \leq s$ such that $t \Vdash_{\mathbb{P}_2}$ "$k \in \dot{b}$". This contradicts the fact that $s \Vdash_{\mathbb{P}_2}$ "$\dot{b} \cap I \subseteq m$". □

Remark. In a similar way, we can prove that in $V^{\mathrm{Fn}(\omega_2, 2)}$, for every ultrafilter p on ω, if p is a P-point limit of a sequence of P-points, then the space $\omega \cup \{p\}$ cannot be embedded in any regular pseudoradial space.

4. A stationary set in $[\omega_2]^\omega$ and a special ultrafilter

In this section, we show that there is an ultrafilter p such that $\omega \cup \{p\}$ can be embedded into a zero-dimensional Hausdorff pseudoradial space provided there is a *reflecting self-indexed stationary set* in $[\omega_2]^\omega$ and $\mathfrak{c}$ is ω_2.

$\{S_\alpha : \alpha < \omega_2\}$ is a *reflecting self-indexed stationary set* in $[\omega_2]^\omega$ if for any $\lambda < \omega_2$ with $\mathrm{cf}(\lambda) = \omega_1$, $\{S_\xi : \xi < \lambda \text{ and } \sup S_\xi = \xi\}$ is stationary in $[\lambda]^\omega$.

The reflecting self-indexed stationary set is very similar to the notion of stationary coding set defined in [10].

Lemma 4.1. *Let $V \models CH$ and $\mathbb{P} = \{p \subseteq \omega_2 \times [\omega_2]^\omega : p$ is a function with countable domain such that $p(\alpha) \subseteq \alpha$ for $\alpha \in \mathrm{dom}\,\alpha\}$. Let $\mathbb{P}$ have the inverse inclusion as its partial order. Then, in $V^{\mathbb{P}}$, there is a reflecting self-indexed stationary set in $[\omega_2]^\omega$ and CH is true.*

It is easy to see that $\mathbb{P}$ is countably closed and ω_2-cc. Therefore CH still holds in $V[P]$ and cardinals are preserved. We leave the proof to the reader.

Lemma 4.2. *A reflecting self-indexed stationary set is preserved by ccc forcing.*

Proof. It is well-known that after ccc forcing every cub subset of ω_1 contains a ground model cub. The same proof gives that this also holds for cub subsets of $[\lambda]^\omega$ for each $\lambda < \omega_2$. $\square$

Remark. S. Todorcěvić informed us that the existence of a reflecting self-indexed stationary set follows from the existence of the ρ-function in [3]. Indeed, one can easily produce such a stationary set directly from a $\square$-sequence (see [4]).

Next we show that for the ultrafilter p stated in the following Theorem, it is possible to embed $\omega \cup \{p\}$ into a zero-dimensional pseudoradial space.

Definition 4.3 ([2])**.** Given a family $\mathcal{F}$ of subsets of some set S, a family $\mathcal{I}$ is called a refinement of $\mathcal{F}$ if for each F in $\mathcal{F}$, there is an I in $\mathcal{I}$ such that $I \subseteq F$. If $\mathcal{I}$ is an almost disjoint family we say $\mathcal{I}$ is an almost disjoint refinement of $\mathcal{F}$.

Theorem 4.4 ([5])**.** *There is an ultrafilter p on ω which has an almost disjoint refinement $\mathcal{I}$ and a base $\mathcal{B}$ such that*

(i) *for each subset $\mathcal{I}' \in [\mathcal{I}]^{<\mathfrak{c}}$ there is a member of p which is almost disjoint with each member of $\mathcal{I}'$,*
(ii) *for each $B \in \mathcal{B}$* and *$I \in \mathcal{I}$ either $B \cap I$ is finite or B almost contains I, and*
(iii) *each member of $\mathcal{I}$ is almost disjoint with all but finitely many members of $\mathcal{B}$.*

Theorem 4.5. *Suppose p is the ultrafilter stated in Theorem 4.4. If $\mathfrak{c} = \omega_2$ and there is a reflecting self-indexed stationary set in $[\omega_2]^\omega$, then the space $\omega \cup \{p\}$ can be embedded in a zero-dimensional pseudoradial space.*

Corollary 4.6. *It is consistent that, for every P-point p, $\omega \cup \{p\}$ cannot be embedded into any regular pseudoradial space while there is an ultrafilter p on ω such that $\omega \cup \{p\}$ is a subspace of a zero-dimensional pseudoradial space.*

Proof. Let V be the model obtained through Lemma 4.1. Let $Q = \mathrm{Fn}(\omega_2, 2)$. By Theorem 3.1, in V^Q, for every P-point p, $\omega \cup \{p\}$ cannot be embedded into any regular pseudoradial space.

On the other hand, by Lemma 4.2, in V^Q, there is a reflecting self-indexed stationary set on $[\omega_2]^\omega$ and $\mathfrak{c}$ is ω_2. Therefore, by Theorem 4.5, there is an ultrafilter p on ω such that $\omega \cup \{p\}$ is a subspace of a zero-dimensional pseudoradial space. $\square$

We will need the following Lemma in the proof of Theorem 4.5.

Lemma 4.7. *Suppose R is a subset of space X, $x \in X$, and $\mathcal{U}$ is a local subbase at x, i.e, the intersections of finitely many members of $\mathcal{U}$ constitute a neighbourhood base at x. If κ is any cardinal number such that the following are true,*

(i) *for all $\mathcal{U}' \in [\mathcal{U}]^{<\kappa}$, $R \cap \bigcap \mathcal{U}'$ is not empty ;*
(ii) *for all $y \in R$, $\{U \in \mathcal{U} : y \notin U\}$ has cardinality at most κ,*

then there is a sequence $\{y_\alpha : \alpha < \kappa\}$ in R convergent to x.

Proof. For each $y \in R$, let $\{U^y_\alpha : \alpha < \kappa\}$ enumerate, with possible repetition, the set $\{U \in \mathcal{U} : y \notin U\}$. It is straightforward to define, by the induction, a sequence $\{y_\alpha : \alpha < \kappa\}$ such that for all $\alpha < \kappa$

$$(*) \qquad y_\alpha \in R \cap \bigcap \{U^{y_\xi}_\eta : \eta < \alpha, \xi < \alpha\}.$$

We show that $\{y_\alpha\}_{\alpha<\kappa}$ is convergent to x by showing that each member of $\mathcal{U}$ contains a tail of the sequence. Suppose $U \in \mathcal{U}$. We have to show that there is an $\alpha < \kappa$, such that , for all $\beta \in \kappa$, $\beta > \alpha$ implies $y_\beta \in U$. If $\{y_\alpha : \alpha < \kappa\}$ is contained in U, we are done. Otherwise, choose any ξ such that $y_\xi \notin U$. Take $\eta < \kappa$ such that $U = U^{y_\xi}_\eta$. Let $\alpha = \max\{\eta, \xi\}$; then, for any $\beta > \alpha$, by $(*)$, $y_\beta \in U$. We are done. □

The rest of the section is to prove Theorem 4.5.

Proof of Theorem 4.5. We first fix a reflecting self-indexed stationary set $\{S_\alpha\colon \alpha \in \omega_2\}$ in $[\omega_2]^\omega$. Without loss of generality, we assume that, for $\alpha \in \omega_2$ with countable cofinality, $\alpha = \sup S_\alpha$. Let $p = \{A_\alpha\colon \alpha < \omega_2\}$ be the ultrafilter in Theorem 4.4. Let $\{I_A\colon A \in p\}$ be the almost disjoint refinement of p such that $I_A \subseteq A$ and $\mathcal{B}$ is the base with the properties in Theorem 4.4. By induction we can define an almost disjoint family $\{a_\alpha\colon \alpha < \omega_2\}$ and a base $\{F_\alpha\colon \alpha < \omega_2\}$ of p such that:

(i) $(\forall\alpha)(a_\alpha \subseteq F_\alpha \subseteq A_\alpha)$.
(ii) $(\forall\alpha, \beta)(\alpha < \beta)$ implies $a_\alpha \cap F_\beta$ is finite.
(iii) $(\forall\alpha)$ there are only finitely many β such that $a_\alpha \cap F_\beta$ is infinite and if $a_\alpha \cap F_\beta$ is infinite, then $a_\alpha \subseteq F_\beta$.
(iv) $(\forall\lambda)$ if $cf(\lambda)$ is uncountable, then $\{F_\alpha\colon \alpha < \lambda\}$ is a filter base.

We construct $\{a_\alpha, F_\alpha\colon \alpha < \omega_1\eta\}$ by induction on η. At stage η, the only non trivial case is when η is a successor ordinal. Suppose $\eta = \xi + 1$. We define $a_{\omega_1\xi+\alpha}$ and $F_{\omega_1\xi+\alpha}$ for $\alpha \in \omega_1$ by induction on ω_1. Let $f\colon \omega_1 \to [\omega_1\xi + \omega_1]^{<\omega}$ be an onto mapping such that $f(\alpha) \subseteq \omega_1\xi + \alpha$ for each $\alpha \in \omega_1$. At stage α, take a B in $\mathcal{B}$ such that B is almost disjoint with a_ζ for each $\zeta < \omega_1\xi + \alpha$. Now simply define $F_{\omega_1\xi+\alpha}$ to be $A_{\omega_1\xi+\alpha} \cap B \cap \bigcap\{F_\zeta\colon \zeta \in f(\alpha)\}$ and let $a_{\omega_1\xi+\alpha}$ be $I_{F_{\omega_1\xi+\alpha}}$.

Let $\mathcal{H} = \{b \subseteq \omega\colon (\forall\alpha \in \omega_2)(a_\alpha \subseteq^* b)\}$. Let $\mathcal{A}$ be the subalgebra of $P(\omega)$ generated by $[\omega]^{<\omega} \cup \{a_\alpha, F_\alpha\colon \alpha \in \omega_2\} \cup \mathcal{H}$.

We will define a subspace X of the Stone space $\mathrm{St}(\mathcal{A})$ such that X contains a copy of $\omega \cup \{p\}$ and X is pseudoradial. X will be of the form $\omega \cup \{x_\alpha : \alpha \in \Gamma\}$, where Γ is a subset of ω_2 and each x_α is an ultrafilter of $\mathcal{A}$. We specify the set Γ and x_α for $\alpha \in \Gamma$ as follows.

For convenience, we define a subset x^+_α of $\mathcal{F}$ for each $\alpha \in \omega_2$. For successor ordinal $\alpha + 1$, $x^+_{\alpha+1}$ is the finite of F_β's which contains a_α. For limit α with $\mathrm{cf}(\alpha) = \omega$, x^+_α is $\{F_\alpha\colon \alpha \in S_\alpha\}$, where S_α is from the reflecting self-indexed stationary set. For limit λ with $\mathrm{cf}(\lambda) = \omega_1$, x^+_λ is $\{F_\alpha\colon \alpha < \lambda\}$.

First of all, Γ contains all the successor ordinals and the ordinals with uncountable cofinality.

For each α, since no member of $\mathcal{A}$ splits a_α, the cofinite subsets of a_α generate an ultrafilter on $\mathcal{A}$. Let $x_{\alpha+1}$ be this ultrafilter. The point x_{ω_2} will simply be the ultrafilter $p \cap \mathcal{A}$. Before we handle other ordinals, we prove the following.

Fact 4.8. *If $\lambda < \omega_2$ has cofinality ω_1, then $x_\lambda^+ \cup \{\omega - F_\beta : F_\beta \notin x_\lambda^+\} \cup \mathcal{H}$ is actually $\{F_\alpha, \omega \setminus F_\beta \colon \alpha < \lambda \text{ and } \beta \geq \lambda\} \cup \mathcal{H}$ and generates an ultrafilter x_λ on $\mathcal{A}$. If $\alpha < \omega_2$ has cofinality ω and $x_\alpha^+ \cup \{\omega - F_\beta : F_\beta \notin x_\alpha^+\} \cup \mathcal{H}$ has the finite intersection property, then it generates an ultrafilter x_α on $\mathcal{A}$.*

Proof of Fact 4.8. If $\lambda < \omega_2$ has cofinality ω_1, then $\{F_\alpha \colon \alpha < \lambda\}$ is a filter base and it is easy to see that $\{F_\alpha, \omega \setminus F_\beta \colon \alpha < \lambda \text{ and } \beta \geq \lambda\} \cup \mathcal{H}$ has the finite intersection property. To prove that it generates an ultrafilter, it is sufficient to show that for each $\beta \in \omega_2$, $\omega \setminus a_\beta$ contains some member of the family. Indeed, if $\beta < \lambda$, then $F_{\beta+1}$ is almost disjoint with a_β; hence $F_{\beta+1} \subseteq^* \omega \setminus a_\beta$. If $\beta \geq \lambda$, then $\omega \setminus F_\beta$ is almost contained in $\omega \setminus a_\beta$. The proof for the case when α has countable cofinality is virtually the same.

We resume the proof of the theorem. For λ with cofinality ω_1, let λ be in Γ and x_λ be the ultrafilter defined in Fact 4.8. For an ordinal α with $\mathrm{cf}(\alpha) = \omega$, $\alpha \in \Gamma$ if $x_\alpha^+ \cup \{\omega - F_\beta : F_\beta \notin x_\alpha^+\} \cup \mathcal{H}$ generate a filter; we let x_α be the ultrafilter defined in Fact 4.8

By above discussion Γ is well defined and, for each $\alpha \in \Gamma$, x_α is a point in the Stone space $\mathrm{St}(\mathcal{A})$. The following claim is trivial.

Claim 4.1. $\omega \cup \{x_{\omega_2}\}$ is homomorphic to $\omega \cup \{p\}$.

For simplicity we will abuse the notation of Stone duality. If $F \subset \omega$ is a member of $\mathcal{A}$, we also let F denote the set of points of X which as an ultrafilter on $\mathcal{A}$ contain F as a member. Of course, F is clopen; hence, to say $x_\alpha \in F$ is equivalent to saying $F \in x_\alpha$.

Let X_0 be all x_α with α a successor and X_1 be all x_α with $\mathrm{cf}(\alpha) = \omega$. Recall that $x_{\alpha+1}$ corresponds to a_α. Let X_2 be all x_α such that α has uncountable cofinality.

Fact 4.9. *The sets $X_1 \cup X_2$ and X_2 are closed in X. X_2 is homomorphic to a subspace of ω_2, namely $\{\alpha \in \omega_2 \colon \mathrm{cf}(\alpha)$ is uncountable $\}$ and so X_2 is radial.*

Proof of Fact 4.9. For the first statement of the claim, the only case that requires proof is that a point x_α of X_1 is not in the closure of X_2. If α is countable, then $\omega - F_{\alpha+1}$ is disjoint from X_2. On the other hand, if α is uncountable, then we can find $\gamma < \beta < \alpha$, so that $F_\beta \in x_\alpha^+$ and $F_\gamma \notin x_\alpha^+$. Note that $F_\beta \setminus F_\gamma$ is not a member of x_λ for any λ with uncountable cofinality. For the second statement, it is sufficient to see that, for any $\lambda < \omega_2$ with uncountably cofinality, the typical neighhourhood of x_λ in the subspace X_2 is $(F_\alpha \setminus F_\lambda) \cap X_2 = \{x_\xi \in X_2 \colon \alpha < \xi \leq \lambda\}$ for some $\alpha < \lambda$. Similarly, a typical neighbourhood of x_{ω_2} is $\{x_\xi \in X_2 \colon \alpha < \xi \leq \omega_2\}$.

Claim 4.2 will finish the proof of Theorem 4.5.

Claim 4.2. X is a zero-dimensional pseudoradial space.

Proof of Claim 4.2. Since X is a subspace of a $\mathrm{St}(\mathcal{A})$, X is zero-dimensional. Suppose X is not pseudoradial. Let R be a radially closed subset of X which is not closed. Let $x_\alpha \in \operatorname{cl} R \setminus R$ and α is the minimal such index. Obviously $x_\alpha \in X_1 \cup X_2$ because other points have countable neighbourhood bases. We will prove $x_\alpha \in R$ to produce a contradiction. We will need the following fact.

Fact 4.10. $x_\alpha \in \operatorname{cl}(R \setminus \omega)$.

Proof of Fact 4.10. Suppose $x_\alpha \notin \operatorname{cl}(R \setminus \omega)$. Then $x_\alpha \in \operatorname{cl}(R \cap \omega) \setminus \operatorname{cl}(R \setminus \omega)$. Let U be a clopen neighbourhood of x_α such that $U \cap (R \setminus \omega) = \emptyset$. Hence $b = R \cap U$ is a subset of ω. Since R is radially closed and U is closed , b is also radially closed. For any $\beta \in \omega_2$, if $a_\beta \cap b$ is infinite, then $x_{\beta+1}$ would be in b contradicting that $b \subseteq \omega$. Hence, b is almost disjoint with each a_β, i.e, $\omega \setminus b$ is in $\mathcal{H}$. Thus $\omega \setminus b$ are in the ultrafilter x_α, which contradict that x_α is in the closure of b.

Since $R \setminus \omega$ is also radially closed in X, we can now assume $R \cap \omega = \emptyset$. By Fact 4.9, X_2 is radial and closed so we can assume that $R \cap X_2 = \emptyset$ by again restricting to a neighbourhood of x_α. For the rest of the proof we assume that $R \subseteq X_0 \cup X_1$. Recall that $x_\alpha \in X_1 \cup X_2$.

We claim that $\alpha \neq \omega_2$. Indeed fix any $M \prec H(\omega_3)$ such that $|M| = \omega_1$ and M contains all relevant sets such as R and X, and so that the cofinality of $\delta = M \cap \omega_2$ is uncountable. We show that x_δ is in the closure of R which will contradict that R is assumed to be disjoint from X_2 and that α is the minimal index of a limit point which is not in R. A typical neighbourhood of x_δ has the form $H \cap \bigcap\{F_\xi : \xi \in f\} \setminus \bigcup\{F_\gamma : \gamma \in g\}$ where $f \in [\delta]^{<\omega}$, $g \in [\omega_2 \setminus \delta]^{<\omega}$ and $H \in \mathcal{H}$. However, $\bigcap\{F_\xi : \xi \in f\}$ is a neighbourhood of x_{ω_2} and is a member of M. Choose any $\beta \in M$ such that $x_\beta \in R \cap \bigcap\{F_\xi : \xi \in f\}$. Clearly $x_\beta \notin F_\gamma$ for all $\gamma > \delta$; hence x_β is in the above neighbourhood of x_δ.

Next we apply Lemma 4.7 to prove that there exists a sequence in R convergent to x_α and thus finish the proof.

Case 1. $\operatorname{cf}(\alpha) = \omega$.

To apply Lemma 4.7, let κ be ω, $\mathcal{U}$ be $x_\alpha^+ \cup \{\omega \setminus F_\beta \colon x_\beta \notin x_\alpha^+\} \cup \mathcal{H}$, and x be x_α. $\mathcal{U}$ is a local subbase of x. We show x, R and $\mathcal{U}$ satisfy the conditions in Lemma 4.7. Since $x \in \operatorname{cl}(R)$, the condition (i) is trivial. To verify the condition (ii), let $x_\xi \in R$. Then $\operatorname{cf}(\xi) \leq \omega$. It is easy to see that

$$\{U \in \mathcal{U} \colon x_\xi \notin U\} \subseteq x_\alpha^+ \cup \{\omega \setminus F \colon F \in x_\xi^+\}.$$

But both x_α^+ and x_ξ^+ are countable. Therefore $\{U \in \mathcal{U} \colon x_\xi \notin U\}$ is countable.

Case 2. $cf(\alpha) = \omega_1$.

We let κ be ω_1, and $\mathcal{U}$ be $x_\alpha^+ \cup \{\omega \setminus F_\beta \colon x_\beta \notin x_\alpha^+\} \cup \mathcal{H}$, and x be x_α. Again $\mathcal{U}$ is a local subbase at x. The second condition of Lemma 4.7 is verified as in the Case 1. We are left to show that for any $\mathcal{U}' \in [\mathcal{U}]^\omega$, $R \cap \bigcap \mathcal{U}' \neq \emptyset$. Since for each H in $\mathcal{H}$, $R \subseteq X \setminus \omega \subseteq H$, we can assume $\mathcal{U}' = \{F_{\alpha_i}, \omega \setminus F_{\beta_i} : i \in \omega\}$. We take a countable elementary submodel M of $H(\omega_3)$, such that $\{X, R, \mathcal{U}', \alpha\} \subseteq M$ and $M \cap \alpha = S_\gamma$ and $\gamma = \sup(M \cap \alpha)$, where S_γ is from the reflecting self-indexed stationary set. Since $S_\gamma = M \cap \alpha$ and x_γ^+ is a filter base, $x_\gamma^+ \cup \{\omega \setminus F_\xi : F_\xi \notin x_\gamma^+\} \cup \mathcal{H}$ has the finite intersection property. Therefore it generates an ultrafilter on $\mathcal{A}$. Therefore $\gamma \in \Gamma$, i.e., x_γ is a point of our X. Next we show that $x_\gamma \in R \cap \bigcap \mathcal{U}'$.

For $i \in \omega$, since, $\mathcal{U}' \in M$, $\alpha_i \in M \cap \alpha = S_\gamma$. Thus, $F_{\alpha_i} \in x_\gamma^+$ while $F_{\beta_i} \notin x_\gamma^+$ because $\beta_i \geq \alpha > \gamma$. This proves that $x_\gamma \in \bigcap \mathcal{U}'$.

To prove $x_\gamma \in R$, it is sufficient to prove that $x_\gamma \in \mathrm{cl}(R)$ because α is the minimal index such that $x_\alpha \in \mathrm{cl}(R) \setminus R$. Take a neighbourhood W of x_γ; we have to show $R \cap W \neq \emptyset$. We can assume $W = \bigcap \mathcal{F} \setminus \bigcup \mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are finite subsets of x_γ^+ and $\mathcal{B} \setminus x_\gamma^+$ respectively. Obviously $\mathcal{F}$ and $\mathcal{G} \cap M$ are members of M. If $F_\xi \in \mathcal{G} \cap M$, then by the definition of x_γ^+ and the fact $S_\gamma = M \cap \alpha$, $\xi \geq \alpha$. Therefore, $(R \cap \bigcap \mathcal{F}) \setminus \bigcup(\mathcal{G} \cap M)$ is a neighbourhood of x_α and it is in M. Take a $x_\xi \in M \cap R \cap \bigcap \mathcal{F} \setminus \bigcup(\mathcal{G} \cap M)$. Since x_ξ^+ is a countable member of M, $x_\xi^+ \subseteq M$; hence $\mathcal{G} \setminus M$ is disjoint with x_ξ^+. Therefore $x_\xi \notin \bigcup(\mathcal{G} \setminus M)$ and $x_\xi \in R \cap \bigcap \mathcal{F} \setminus \bigcup \mathcal{G} = R \cap W$. We are done. □

We finish with the following question.

Question 4.11. *Does MA or* $\mathfrak{p} = \mathfrak{c}$ *imply that, for each ultrafilter* p *on* ω, $\omega \cup \{p\}$ *is a subspace of a regular pseudo-radial space?*

By Lemma 2.1, the answer is yes for the non P-point p.

References

[1] A.V. Arhangel'skiĭ, R. Isler, and G.Tironi. On pseudo-radial spaces. *Comment. Math. Univ. Carolinae*, 27(1):137–154, 1986.

[2] Bohuslav Balcar and Petr Simon. Disjoint refinement. In J.K. Monk and R. Bonnet, editors, *Handbook of Boolean Algebra*, chapter 9. Elsevier Science Publishers, B.V., 1989. CMP 21:10

[3] Mohamed Bekkali. *Topics in Set Theory*. Springer-Verlag, 1991. MR **92m:**03070

[4] Alan Dow. The completion of $\mathcal{P}(\omega)/\mathrm{Fin}$ is not equal to the regular open algebra of $\beta\mathbb{R} \setminus \mathbb{R}$. *to appear.*

[5] Alan Dow and Jinyuan Zhou. Two real ultrafilters on ω. *on Topology and its Applications*. to appear

[6] Kenneth Kunen. *Set Theory*. Elsevier Science Publishers, B.V., 1980. MR **82f:**03001

[7] Peter J. Nyikos. Convergence in topology. In M. Hušek and J. van Mill, editors, *Recent Progress in General Topology*, chapter 17. Elsevier Science Publishers B.V., 1992. CMP 93:15

[8] Saharon Shelah. *Proper Forcing*. Springer-Verlag, 1982. MR **84h:**03002

[9] Jinyuan Zhou. On subspaces of pseudo-radial spaces. *Comment. Math. Univ. Carolinae*, 34(3):583–586, 1994. MR **94h:**54003

[10] William S. Zwicker. $P_\kappa\lambda$ combinatorics I: Stationary coding sets rationalize the club filter. In Donald A. Martin James E. Baumgartner and Saharon Shelah, editors, *Axiomatic Set Theory*, pages 243–259. American Mathematical Society, 1984. MR **86e:**03046

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