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LOCAL SPECTRAL THEORY AND ORBITS OF OPERATORS

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Abstract. For $T \in \mathcal{L}(X)$, we give a condition that suffices for $\varphi(T)$ to be hypercyclic where φ is a nonconstant function that is analytic on the spectrum of T. In the other direction, it is shown that a property introduced by E. Bishop restricts supercyclic phenomena: if $T \in \mathcal{L}(X)$ is finitely supercyclic and has Bishop's property (β), then the spectrum of T is contained in a circle.

Introduction

An operator T is cyclic provided there is a vector x whose orbit under T has dense linear span. If the orbit $\{T^n x\}_{n\geq 0}$ is itself dense, then T is hypercyclic. An example of such an operator was given by Rolewicz [33] in terms of the backward shift on the Hardy space H^2: If B is given by $\sum_0^\infty a_n z^n \xrightarrow{B} \sum_0^\infty a_{n+1} z^n$, then λB is hypercyclic for each λ, $|\lambda| > 1$. Here the factor λ is crucial; B is itself not hypercyclic, and this leads to an intermediate notion of cyclicity. If the homogeneous orbit $\{\lambda T^n x : n \geq 0, \lambda \in \mathbb{C}\}$ is dense for some x, the operator T is said to be supercyclic. This definition is due to Hilden and Wallen, who show that the adjoint of every injective unilateral weighted shift is supercyclic [20]. In contrast to the case of cyclic vectors, if an operator has a hypercyclic (supercyclic) vector, then there is a T-invariant, dense G_δ set of hypercyclic (supercyclic) vectors [22].

C. Kitai [22], and independently Gethner and J. Shapiro [15], established a sufficient condition for hypercyclicity that has been applied in a variety of settings and is a principal tool in Theorem 1 below.

Theorem (Kitai, Gethner and Shapiro)**.** *Suppose that X is a separable Fréchet space and that T is a continuous linear operator on X. If there exist dense T-invariant subsets $\mathcal{M}$ and $\mathcal{N}$ of X such that $T^n x \to 0$ for each $x \in \mathcal{M}$ and a sequence of (not necessarily continuous) functions $S_n : \mathcal{N} \to \mathcal{N}$ satisfying $S_n(x) \to 0$ and $T^n S_n(x) = x$ for each $x \in \mathcal{N}$, then the operator T is hypercyclic.*

Gethner and Shapiro apply this criterion to backward shifts on weighted sequence spaces. Godefroy and Shapiro [16] later use this condition to obtain the classical results of Birkoff [6] and MacLane [27], and also to obtain hypercyclic and supercyclic vectors for Banach space operators in the double commutant of generalized backward shifts and multipliers on very general Hilbert spaces of analytic functions.

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In the negative direction, it was observed by Hilden and Wallen [20] that neither the unilateral forward shift on H^2 nor any normal operator is supercyclic. Ansari and Bourdon [5] recently showed that Banach space isometries are not supercyclic, and Bourdon proved that hyponormal operators have no supercyclic vectors [8]. Hyponormal operators and isometries are examples of operators that satisfy a condition introduced by E. Bishop [7] in the late 1950's. In Theorem 2 below, we show how this condition restricts these cyclic phenomena.

Local spectral theory

Let X be a complex Banach space. For T a bounded linear operator on X, that is, for $T \in \mathcal{L}(X)$, we denote as usual the spectrum and the approximate point spectrum of T by $\sigma(T)$ and $\sigma_{ap}(T)$. The surjectivity spectrum of T is $\sigma_{su}(T) = \{\lambda \in \mathbb{C} : (\lambda - T)X \neq X\}$. Notice that if $T^* \in \mathcal{L}(X^*)$ is the adjoint of T, then $\sigma_{su}(T) = \sigma_{ap}(T^*)$ and $\sigma_{ap}(T) = \sigma_{su}(T^*)$. The complement of the surjectivity spectrum is $\rho_{su}(T)$, and so forth. If $T \in \mathcal{L}(X)$ and if Y is a closed, T-invariant subspace of X, let $T|_Y \in \mathcal{L}(Y)$ be the restriction of T to Y. Below, we give the basics of local spectral theory that we will employ.

An operator T on a complex Banach space X is decomposable in the sense of Foiaş [1] provided that whenever $\{U_1, U_2\}$ is an open cover of $\mathbb{C}$, there exist closed, T-invariant subspaces Y_k such that $X = Y_1 + Y_2$ and $\sigma(T|_{Y_k}) \subset U_k$, $k = 1, 2$. This class of operators is quite large; for example, all normal operators on a Hilbert space, compact operators and generalized scalar operators on Banach spaces are decomposable. Although decomposable operators generally have no functional calculus beyond the basic analytic functional calculus of Riesz, these operators possess many of the spectral properties of normal operators. For the basic theory of decomposable operators, refer to [10] and [38].

Let U be open in $\mathbb{C}$ and let $\mathcal{O}(U, X)$ be the space of analytic X-valued functions on U. Endowed with the topology of uniform convergence on compact subsets of U, the space $\mathcal{O}(U, X)$ is a Fréchet space. An operator $T \in \mathcal{L}(X)$ induces, for each open $U \subset \mathbb{C}$, a continuous linear mapping T_U on $\mathcal{O}(U, X)$ defined by $(T_U f)(\lambda) = (\lambda - T)f(\lambda)$ for every $f \in \mathcal{O}(U, X)$ and $\lambda \in U$.

Corresponding to each closed $F \subset \mathbb{C}$ there is also an associated analytic subspace, $X_T(F)$, of X consisting of all vectors x for which there is an analytic function $f : \mathbb{C} \setminus F \to X$ with $(\lambda - T)f(\lambda) = x$ for each $\lambda \in \mathbb{C} \setminus F$. Equivalently, viewing $x \in X$ as a constant function, $X_T(F) = X \cap \operatorname{ran}(T_{\mathbb{C} \setminus F})$. For an arbitrary $T \in \mathcal{L}(X)$, the spaces $X_T(F)$ are T-invariant, generally nonclosed linear manifolds in X. Also, if F and K are disjoint closed subsets of the plane, then $X_T(F) \perp X^*_{T^*}(K)$; that is, $\langle x, x^* \rangle = 0$ whenever $x \in X_T(F)$ and $x^* \in X^*_{T^*}(K)$ [14].

If $x \in X$, the local spectral radius of x with respect to T is the quantity $r_T(x) = \limsup_{n\to\infty} \|T^n x\|^{1/n}$. We will use the fact that for any $T \in \mathcal{L}(X)$ and $x \in X$, the vector x is contained in $X_T(\overline{B(0, r)})$ if and only if $r_T(x) \leq r$ [29]. It follows, in particular, that $\bigcup_{n\geq 0} \ker(\lambda - T)^n \subset X_T(\{\lambda\})$ for every $\lambda \in \mathbb{C}$. A consequence of a deep result of Leiterer is that $X_T(\sigma_{su}(T)) = X$ for any $T \in \mathcal{L}(X)$ [26, Theorem 5.1].

For an arbitrary $T \in \mathcal{L}(X)$, consider the set of all $\lambda \in \mathbb{C}$ such that $\lambda - T$ has closed range and $\ker(\lambda - T) \subset \bigcap_{n\geq 0} (\lambda - T)^n X$. This set has been referred to by various names, but let us call it $\rho_K(T)$, the Kato resolvent of T; the Kato spectrum is $\sigma_K(T) = \mathbb{C} \setminus \rho_K(T)$. Clearly, $\mathbb{C} \setminus (\sigma_{su}(T) \cap \sigma_{ap}(T)) \subset \rho_K(T)$, and it is well-known that $\rho_K(T)$ is open and preserved by adjoints [21]. Also, if $\lambda \in \rho_K(T)$, then $(\lambda - T)^n X$ is closed for each n, and $\bigcup_n \ker(\lambda - T)^n \subset \bigcap_n (\lambda - T)^n X$ [30]. There is a spectral mapping theorem for the Kato spectrum [35], and if G is a component of $\rho_K(T)$, then $\bigcap_{n\geq 0}(\lambda - T)^n X = X_T(\mathbb{C} \setminus G)$ for each $\lambda \in G$. Indeed, the closed spaces $\bigcap_{n\geq 0}(\lambda - T)^n X$ are invariant for $\lambda \in G$ [17]; let $Y = \bigcap_{n\geq 0}(\lambda - T)^n X$ for $\lambda \in G$. If $x \in X$ and if $\lambda \in G$, then $(\lambda - T)x \in Y$ if and only if $x \in Y$ ([30] or [24]), and therefore $G \subset \rho_{su}(T|_Y)$. The assertion now follows from Leiterer's theorem.

The operator T has Bishop's property (β) provided that for each open $U \subset \mathbb{C}$, the mapping T_U is injective and has closed range in $\mathcal{O}(U, X)$. That decomposable operators have property (β) is due to Albrecht [1]. It is a theorem of Bishop [7] that an operator $T \in \mathcal{L}(X)$ with (β) has the property that the dual space decomposes as $X^* = X^*_{T^*}(\overline{U}) + X^*_{T^*}(\overline{V})$ whenever U and V are open with $\mathbb{C} \subset U \cup V$. This is the decomposition property (δ); specifically, $T \in \mathcal{L}(X)$ has property (δ) provided that for any open cover $\{U, V\}$ of $\sigma(T)$, the space X can be written as the sum of the analytic subspaces: $X = X_T(\overline{U}) + X_T(\overline{V})$.

If T has property (β), then $\sigma(T) = \sigma_{su}(T)$, and, for every closed $F \subset \mathbb{C}$, the analytic subspace $X_T(F)$ is closed [25]. Thus an operator T is decomposable if and only if T has both properties (β) and (δ) [3]. Albrecht and Eschmeier [2] have shown that the properties (β) and (δ) are completely dual; an operator T has one of these exactly when its adjoint has the other. Moreover, they characterize operators with Bishop's property (β) as those similar to the restriction of a decomposable operator, and operators with the decomposition property (δ) as those similar to a quotient of a decomposable operator.

That hyponormal operators are subscalar and thus subdecomposable is due to Putinar [32]. Surjective Banach space isometries are generalized scalar [10, 5.1.4], and it is a result of Douglas that every isometry has a surjective extension [11]. If decomposability is the appropriate generalization of normal operators to Banach spaces, the relation between subnormal and subdecomposable operators is even stronger. For example, Eschmeier and Prunaru applied the Scott Brown technique, originated to show the existence of invariant subspaces for subnormal operators, to prove that Banach space operators with property (β) and thick spectra have invariant subspaces [12].

For a systematic treatment of local spectral theory, we refer the reader to [13] and [23].

Main results

Hilbert spaces of analytic functions with bounded point evaluations have a rich supply of dense subspaces spanned by the reproducing kernels. Godefroy and Shapiro use these with the Kitai-Gethner-Shapiro condition to obtain the optimal result regarding multipliers on such spaces: Let U be a domain in $\mathbb{C}^n$ and H a Hilbert space of analytic functions on U with bounded point evaluations at each λ in U. If φ is a nonconstant analytic function such that the multiplier $M_\varphi(f) = \varphi f$ is bounded, then M^*_φ is supercyclic; in fact, M^*_φ is hypercyclic if and only if $\varphi(U)$ meets the unit circle [16, Theorem 4.5]. Under the conditions of Theorem 1 below, the local spectral theory provides similar dense spaces of eigenvectors, and we obtain a comparable conclusion.

Theorem 1. *Suppose that X is a Banach space and $T \in \mathcal{L}(X)$. Let G be a component of the Kato resolvent of T. The analytic subspace $X_T(F)$ corresponding to a closed subset F of G is dense in X if and only if for every $\lambda \in G$, the space $\bigcup_{n\geq 0} \ker(\lambda - T)^n$ is dense in X. In this case, $\rho_K(T) = \rho_{su}(T)$; if X is separable and if φ is a nonconstant analytic function defined on a neighborhood of $\sigma(T)$, then $\varphi(T)$ is supercyclic. If $\varphi(G)$ intersects the unit circle, then $\varphi(T)$ is hypercyclic.*

Proof. For any operator T and for any complex λ, we have that $\bigcup_{n\geq 0} \ker(\lambda - T)^n \subset X_T(\{\lambda\})$. Therefore, if $\bigcup_{n\geq 0} \ker(\lambda - T)^n$ is dense in X for some $\lambda \in G$, there is a closed $F \subset G$ such that $X_T(F)$ is dense. Conversely, suppose that $F \subset G$ is closed and that $X_T(F)$ is dense. Since $\mathbb{C} \setminus G$ and F are disjoint, it follows that $X_T^*(\mathbb{C} \setminus G)$ annihilates $X_T(F)$, and thus $X_{T^*}^*(\mathbb{C} \setminus G) = 0$. Since $\rho_K(T^*) = \rho_K(T)$, we have that $0 = X_{T^*}^*(\mathbb{C} \setminus G) = \bigcap_{n\geq 0}(\lambda - T^*)^n X^*$ for each $\lambda \in G$. Because $(\lambda - T^*)^n X^*$ is norm-closed for each n, it is weak-* closed, and so $0 = (\bigcup_{n\geq 0} \ker(\lambda - T)^n)^\perp$ as required.

Suppose now that $\bigcup_{n\geq 0} \ker(\lambda - T)^n$ is dense for every $\lambda \in G$. We show $\rho_K(T) \subset \rho_{su}(T)$, the other inclusion being apparent. If $\omega \in \sigma(T) \setminus G$ and if $\lambda \in G$, then again $X_{T^*}^*(\{\omega\}) \perp X_T(\{\lambda\})$ implies that $0 = X_{T^*}^*(\{\omega\})$; in particular, $0 = \ker(\omega - T^*)$. It follows that $\rho_K(T) \setminus G$ is contained in $\rho_{su}(T) = \rho_{ap}(T^*)$. If $\lambda \in G$, then $\bigcap_{n\geq 0}(\lambda - T)^n X$ is closed and contains the dense set $\bigcup_{n\geq 0} \ker(\lambda - T)^n$. Thus $X = \bigcap_{n\geq 0}(\lambda - T)^n X = X_T(\mathbb{C} \setminus G)$, and $G \subset \rho_{su}(T)$.

Finally, assume that X is separable, that G is a component of $\rho_{su}(T)$ as above, and φ is a nonconstant analytic function defined in a neighborhood of $\sigma(T)$ with $\varnothing \neq \varphi(G) \cap \{z : |z| = 1\}$. Choose a_1 and a_2 in G so that $|\varphi(a_1)| < 1$ and $|\varphi(a_2)| > 1$. Let $\delta > 0$ be such that, for $j = 1, 2$, the closed disks $\overline{B(a_j, \delta)}$ are contained in G, and $|\varphi(z) - \varphi(a_j)| \leq |1 - |\varphi(a_j)||/2$ whenever $|z - a_j| \leq \delta$. Let $B_j = \overline{B(a_j, \delta/2)}$ and, for each $x \in X_T(B_j)$, let $f_{j,x} : \mathbb{C} \setminus B_j \to X$ be such that $x = T_{\mathbb{C}\setminus B_j} f_{j,x}$ in $\mathcal{O}(\mathbb{C} \setminus B_j, X)$. By Leiterer's theorem, we may choose $g_x \in \mathcal{O}(G, X)$ such that $x = T_G g_x$. Let $h_{j,x} = f_{j,x} - g_x$ on $G \setminus B_j$ and define $\mathcal{M}_j = \operatorname{span}\{h_{j,x}(z) : x \in X_T(B_j), |z - a_j| = \delta\}$. Notice that each $h_{j,x}(z)$ is an eigenvector for T and therefore for $\varphi(T)$. In particular, if $|z - a_1| = \delta$, then $\|\varphi(T)^n h_{1,x}(z)\| = |\varphi(z)|^n \|h_{1,x}(z)\| \to 0$ as $n \to \infty$.

We claim that each $\mathcal{M}_j$ is dense in X. Indeed, fix j, and let us suppress the subscript j in the definitions above. If $x^* \in X^*$ annihilates $\mathcal{M}$, then for every $x \in X_T(B)$ and $\lambda \in G$ such that $|\lambda - a| > \delta$, we have

$$\begin{aligned}
0 &= \frac{1}{2\pi} \int_{|z-a|=\delta} \langle h_x(z), x^* \rangle (z - \lambda)^{-1}\, dz \\
&= \frac{1}{2\pi} \int_{|z-a|=\delta} \langle f_x(z), x^* \rangle (z - \lambda)^{-1}\, dz - \frac{1}{2\pi} \int_{|z-a|=\delta} \langle g_x(z), x^* \rangle (z - \lambda)^{-1}\, dz \\
&= \frac{1}{2\pi} \int_{|z-a|=\delta} \langle f_x(z), x^* \rangle (z - \lambda)^{-1}\, dz \\
&= \langle f_x(\lambda), x^* \rangle.
\end{aligned}$$

It follows that $0 = \langle f_x(\lambda), x^* \rangle$ for every $x \in X_T(B)$ and $\lambda \in \mathbb{C} \setminus B$; in particular, if $|\lambda| > \|T\|$, then for every $x \in X_T(B), 0 = \langle f_x(\lambda), x^* \rangle = \langle (\lambda - T)^{-1} x, x^* \rangle = \langle x, (\lambda - T^*)^{-1} x^* \rangle$. Since $X_T(B)$ is dense, $x^* = 0$, and the claim is established.

Define $S : \mathcal{M}_2 \to \mathcal{M}_2$ by $S(\sum_{k=1}^n \zeta_k h_{x_k}(z_k)) = \sum_{k=1}^n \frac{\zeta_k}{\varphi(z_k)} h_{x_k}(z_k)$ whenever $n \geq 1$, $\{\zeta_k\}_{k=1}^n \subset \mathbb{C}$, $\{x_k\}_{k=1}^n \subset X_T(B_2)$, and $|z_k - a_2| = \delta$, $k = 1, \ldots, n$. That

S is well-defined follows from the fact that for any polynomial p, each $h_{x_k}(z_k)$ is an eigenvector for $p(\varphi(T))$ with eigenvalue $p(\varphi(z_k))$. Now, $S^n x \to 0$ as $n \to \infty$, and $\varphi(T)Sx = x$ for every $x \in \mathcal{M}_2$. Since $\varphi(T)^n x \to 0$ on $\mathcal{M}_1$, the Kitai-Gethner-Shapiro condition implies that $\varphi(T)$ is hypercyclic.

Of course, in the setting of Hilbert spaces of analytic functions, the Godefroy-Shapiro result is stronger, but an immediate corollary of Theorem 1 generalizes [16, Theorem 4.11]. Specifically, we do not require that $\ker(T)$ be finite dimensional. Theorem 1 of [19] is also subsumed by the following.

Corollary 1. *Suppose that X is separable, and that $T \in \mathcal{L}(X)$. If $\lambda - T$ is surjective and $\bigcup_{n\geq 0} \ker(\lambda - T)^n$ is dense for some λ, then $\varphi(T)$ is supercyclic whenever φ is a nonconstant analytic function on a neighborhood of $\sigma(T)$. If G is the component of $\rho_{su}(T)$ containing λ, and if $\varphi(G) \cap \{z : |z| = 1\} \neq \varnothing$, then $\varphi(T)$ is hypercyclic.*

Example. A semi-shift is an isometry $S \in \mathcal{L}(X)$ such that $\bigcap_{n\geq 0} S^n X = 0$. If $T \in \mathcal{L}(X)$ is such that T^* is a semi-shift, then T is in some sense a backward shift. Since the spectrum of T^* is the closed unit disk, and its appropriate point spectrum the unit circle, an application of the corollary yields in particular that λT is hypercyclic for every λ, $|\lambda| > 1$.

It is natural to consider orbits of finite sets. We will call an operator T finitely supercyclic if there is a finite set of vectors F whose homogeneous orbit, $\{\lambda T^n x : n \geq 0, \lambda \in \mathbb{C}, x \in F\}$, is dense in X [18]. Finitely hypercyclic operators are defined similarly. If F is a finite subset of X with dense homogeneous orbit under T, we say F is a minimal supercyclic set for T provided that no proper subset of F has dense homogeneous orbit. It follows immediately that a supercyclic set F is minimal if and only if for each x in F, the homogeneous orbit of x under T is not nowhere dense, and in this case, each $x \in F$ is a cyclic vector for T. Moreover, the minimality of F implies that for each $x \in F$, every vector in $\{\lambda T^n x : n \geq 0, \lambda \in \mathbb{C}\}^-$ is a limit of a sequence of the form $(\lambda_k T^{n_k} x)_k$, where $(\lambda_k)_k \subset \mathbb{C}$ and $(n_k)_k$ is a sequence of natural numbers satisfying $n_k \to \infty$ as $k \to \infty$ [28]. In particular, it follows that $\mathrm{ran}(T)$ is dense in X.

If T has property (β) and if x is a cyclic vector for T, clearly $x \in X_T(K)$ for some closed K implies that $X = X_T(K)$. Therefore, since $T|_{X_T(K)}$ also has property (β), we conclude that $\sigma(T) = \sigma_{su}(T|_{X_T(K)}) \subset K$. Because operators with Bishop's property (β) are power regular [4], that is, $\lim_{n\to\infty} \|T^n x\|^{1/n}$ exists for every $x \in X$, it follows that for a cyclic vector x, the sequence $(\|T^n x\|^{1/n})_{n\geq 1}$ converges to the spectral radius of T.

Theorem 2. *Suppose that $T \in \mathcal{L}(X)$ has Bishop's property (β). If T is finitely supercyclic, then the spectrum of T is contained in a circle.*

Proof. Since both property (β) and finite supercyclicity are preserved under similarity, we may assume that there is a decomposable operator $S \in \mathcal{L}(Y)$ with X a closed, S-invariant subspace of Y and $T = S|_X$. We may also assume that T has spectral radius 1. Let $q : Y^* \to X^*$ be the restriction map: $\langle x, qy^* \rangle = \langle x, y^* \rangle$ for every $y^* \in Y^*$ and $x \in X$. We claim that $Y^*_{S^*}(\overline{B(0,r)}) \subset \ker(q)$ for every $r < 1$. Indeed, suppose that for some $r < 1$ there exists $y^* \in Y^*_{S^*}(\overline{B(0,r)})$ such that $qy^* \neq 0$. If $F \subset X$ is a minimal supercyclic set for T, there is a vector $x \in F$ and a sequence

$(n_k)_{k=1}^{\infty}$ so that $n_k \to \infty$ as $k \to \infty$ and

$$0 < \delta := \inf_k |\langle \|T^{n_k}x\|^{-1}T^{n_k}x, y^*\rangle|.$$

Let $S_r^* = S^*|_{Y_{S^*}^*(\overline{B(0,r)})}$. Since S^* is decomposable, $\sigma(S_r^*) \subset \overline{B(0,r)}$; choose ρ, $r < \rho < 1$, and apply the Riesz functional calculus to S_r^*. We obtain for each k,

$$\begin{aligned} |\langle T^{n_k}x, y^*\rangle| = |\langle x, S_r^{*n_k}y^*\rangle| &= \left| \frac{1}{2\pi} \int_{|z|=\rho} \langle x, z^{n_k}(z - S_r^*)^{-1}y^*\rangle \, dz \right| \\ &\leq \|x\| \max_{|z|=\rho} \|(z - S_r^*)^{-1}y^*\|\rho^{1+n_k}. \end{aligned}$$

Thus $0 < \delta \leq \liminf_k C_\rho \rho^{n_k}\|T^{n_k}x\|^{-1}$, where C_ρ is independent of k, and therefore

$$1 = \liminf_k \delta^{1/n_k} \leq \liminf_k (C_\rho \rho^{n_k}\|T^{n_k}x\|^{-1})^{1/n_k} = \rho \liminf_k \|T^{n_k}x\|^{-1/n_k}.$$

Consequently, $1 > \rho \geq \limsup_k \|T^{n_k}x\|^{1/n_k}$. But x is a cyclic vector for T, and, since T is power regular [4], this contradicts our assumption that T has spectral radius 1.

Now, if $0 < \eta < 1$, write $Y^* = Y_{S^*}^*(\overline{B(0, \frac{1+\eta}{2})}) + Y_{S^*}^*(\mathbb{C} \setminus B(0,\eta))$. Because $qS^* = T^*q$, it follows that

$$X^* = qY^* = q(Y_{S^*}^*(\mathbb{C} \setminus B(0,\eta))) \subset X_{T^*}^*(\mathbb{C} \setminus B(0,\eta))$$

since $Y_{S^*}^*(\overline{B(0, \frac{1+\eta}{2})})$ is contained in $\ker(q)$. Thus

$$\sigma_{ap}(T) = \sigma_{su}(T^*) \subset \bigcap_{0<\eta<1} (\mathbb{C} \setminus B(0,\eta)) = \mathbb{C} \setminus B(0,1).$$

Since T has dense range, it follows that T is invertible, and since $\partial\sigma(T) \subset \sigma_{ap}(T)$, we have that $B(0,1) \subset \rho(T)$. Thus $\sigma(T)$ is a subset of the unit circle.

Examples. (a) If T is a unilateral weighted shift with the decomposition property (δ), then T^* has property (β) and is supercyclic [20]. Because a unilateral weighted shift has spectrum equal to a disk [36], it follows that $\sigma(T) = \sigma(T^*) = \{0\}$ and therefore T is decomposable. We do not know, generally, which unilateral weighted shifts have property (β).

(b) A weaker condition than Bishop's property (β) is the single-valued extension property, which an operator $T \in \mathcal{L}(X)$ possesses provided that, for every open subset U of the plane, T_U is injective on $\mathcal{O}(U,X)$. There is a supercyclic operator with the single-valued extension property whose spectrum is not contained in a circle. Consider the weighted shift T given by $f(z) \mapsto zf(z)$ on $H^2(\beta)$ where the sequence $(\beta_n)_{n\geq 0}$ satisfies $0 < r := \lim_n \sup_{k\geq 0}(\beta_{n+k}/\beta_k)^{1/n}$ and $0 = \liminf_n \beta_n^{1/n}$; see [36]. Then $\sigma(T) = \{\lambda : |\lambda| \leq r\}$, and the second condition gives that T^* has point spectrum $\{0\}$; in particular, T^* has the single-valued extension property. That T^* is supercyclic is again the result of Hilden and Wallen [20].

(c) An automorphism φ of the unit disk is said to be parabolic provided that φ has a unique fixed point $a \in \mathbb{C}_\infty$, necessarily on the unit circle, and $\varphi'(a) = 1$. A typical example is $\varphi(z) = ((1+i)z - i)(1 - i + iz)^{-1}$. Suppose that φ is a parabolic automorphism of the disk, and let C_φ be defined on the Hardy space H^2 by $C_\varphi f = f \circ \varphi$. Cyclic phenomena of composition operators have been studied extensively by Bourdon and Shapiro. By [9, Theorem 2.2], the operator C_φ is hypercyclic. By [37, Theorem 1.1], the spectrum of C_φ is the unit circle, and C_φ is generalized scalar, in particular, decomposable.

The conclusion of the following is known, but we include it because its proof is quite different from that in [28].

Corollary 2. *If T is an isometry on an infinite dimensional Banach space, then T is not finitely supercyclic.*

Proof. We argue by contradiction. Suppose that T is a finitely supercyclic isometry on X. Then T has property (β) as mentioned above, and so Theorem 2 implies that T is a surjective isometry. Proposition 5.1.4 of [10] implies that T is a generalized scalar operator. If T has spectrum a singleton $\{\lambda\}$, then $T - \lambda I$ is nilpotent [10, 4.3.5], contradicting the assumption that T is finitely supercyclic on an infinite dimensional space. Thus there is an open set U so that $U \cap \sigma(T) \neq \varnothing$ and $\sigma(T) \not\subset \overline{U}$; in particular, $0 \neq X_T(\overline{U}) \neq X$. If F is a minimal supercyclic set for T, then there is an $x \in F$ and a sequence of natural numbers $(n_k)_{k\geq 1}$ so that $(T^{n_k}x)_{k\geq 0}$ converges to a vector $y \in X_T(\overline{U})$. Our contradiction is at hand; $X_T(\overline{U})$ is closed, invariant under T^{-1}, and $T^{-n_k}y \to x$. Since x is a cyclic vector for T, it follows that $X_T(\overline{U}) = X$.

Corollary 3. *If T is hyponormal on an infinite dimensional Hilbert space, then T is not finitely supercyclic.*

Proof. Again we argue by contradiction. If T is hyponormal on H, then T is subscalar [32], and therefore has property (β). If T is finitely supercyclic, then, by Theorem 2, $\sigma(T)$ is contained in a circle. In this case, Putnam's theorem [31] implies that T is normal; in fact, $T = rU$ where U is unitary and $0 \leq r = \|T\|$. The assumption that T is finitely supercyclic implies that $r > 0$, but this leads to a contradiction of Corollary 2.

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