

ON THE NUMBER OF SOLUTIONS OF AN ALGEBRAIC
EQUATION ON THE CURVE $y = e^x + \sin x$, $x > 0$,
AND A CONSEQUENCE FOR O-MINIMAL STRUCTURES

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ABSTRACT. We prove that every polynomial $P(x, y)$ of degree d has at most $2(d+2)^{12}$ zeros on the curve $y = e^x + \sin(x)$, $x > 0$. As a consequence we deduce that the existence of a uniform bound for the number of zeros of polynomials of a fixed degree on an analytic curve does not imply that this curve belongs to an o-minimal structure.

MAIN RESULT

The aim of this note is to estimate the number of real solutions of the system

$$(1) \quad P(x, y) = 0, \quad y = e^x + \sin(x), \quad x > 0,$$

where $P(x, y)$ is a non-zero polynomial of degree d . As we prove below we have the following bound.

Theorem 1. *The number of solutions of the system (1) is not greater than $A(d) = 2(d+2)^{12}$.*

Clearly the condition $x > 0$ in (1) cannot be omitted. For instance the x -axis intersects the graph $y = e^x + \sin(x)$ infinitely many times for $x < 0$.

Using Theorem 1 it is easy to construct a global analytic function with a similar behavior.

Corollary 2. *The number of solutions of the system*

$$(2) \quad P(x, y) = 0, \quad y = e^{x^2} + \sin(x^2),$$

is less than or equal to $2A(2d)$.

Proof. Take a polynomial $P(x, y)$ of degree d and eliminate the variable x from the equations $P(x, y) = 0$, $x^2 = v$. The resultant $R(v, y) = \text{Res}_x(P(x, y), x^2 - v)$ is a polynomial of degree $\leq 2d$ in variables v, y .

If (x, y) is a solution of the system (2), then (v, y) , where $v = x^2$, is a solution of the system

$$R(v, y) = 0, \quad y = e^v + \sin(v), \quad v \geq 0.$$

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Therefore by Theorem 1 the number of solutions of system (2) is not greater than $2A(2d)$. \square

In the end of this note we give an application of Theorem 1 to the theory of o-minimal structures. This application was our original motivation to study the above question.

PROOF OF THE THEOREM

First we will show that it is enough to prove Theorem 1 for $P(x, y) = 0$ being smooth. We use the following

Lemma 3. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be an analytic function such that $\#\{x : f(x) = \epsilon\} \leq A$ for sufficiently small $|\epsilon| \neq 0$. Then $\#\{x : f(x) = 0\} \leq A$.*

Proof. We adopt here the proof of Lemma 4.2.6 in [BR]. Suppose first that $\#\{x : f(x) = 0\}$ is finite. We divide the set of zeros of f into: n_1 zeros of odd multiplicity, n_2 local minima and n_3 local maxima.

Taking $\epsilon > 0$ if $n_2 \geq n_3$ and $\epsilon < 0$ if $n_2 < n_3$ it is easily seen that

$$\#\{x : f(x) = 0\} = n_1 + n_2 + n_3 \leq \#\{x : f(x) = \epsilon\} \leq A.$$

Note that the same argument shows that $\#\{x : f(x) = 0\}$ is finite provided A is also. \square

Now fix a polynomial $P(x, y)$ of degree d and assume that 0 is a critical value of P . Since the polynomials have only finitely many critical values, for sufficiently small $|\epsilon| \neq 0$, the curve $P(x, y) = \epsilon$ is smooth. Suppose that the estimate of Theorem 1 holds true for $P(x, y) - \epsilon$, that is $\#\{x : f(x) = \epsilon, x > 0\} \leq A(d)$ for $f(x) = P(x, e^x + \sin(x))$. By Lemma 3 $\#\{x : f(x) = 0, x > 0\} \leq A(d)$, and therefore Theorem 1 holds true for $P(x, y)$ as well.

From now on we assume that $P(x, y) = 0$ is a smooth curve.

Step 1. A bound on the number of connected components of the set

$$C = \{(x, y) : P(x, y) = 0, e^x - 1 \leq y \leq e^x + 1\}.$$

First we estimate the number of intersections of the curve $P(x, y) = 0$ with the graphs $y = e^x + 1$ and $y = e^x - 1$. This is precisely the number of solutions of equations $P(x, e^x + 1) = 0$ and $P(x, e^x - 1) = 0$. By Khovansky's theorem, see for instance [BR, 4.1.1],

$$\#\{x : P(x, e^x + 1) = 0\} + \#\{x : P(x, e^x - 1) = 0\} \leq 2d(d + 1).$$

Since the curve $P(x, y) = 0$ is smooth, it is a finite union of topological ovals and intervals. By Harnack's theorem [BR, 4.4.4, 5.3.2] their number is not greater than $(d^2 - d + 2)/2$. Hence $\#(\text{connected components of } C) \leq (d^2 - d + 2)/2 + 2d(d + 1) = (5d^2 + 3d + 2)/2$.

Step 2. A bound on the "size" of a connected component of C .

Denote

$$L = d(d + 1)^2(d + 2)^2 \ln 2/4 + (d + 1)^4(d + 2)^3/8.$$

This part of the proof is based on

Lemma 4. *Assume that points $(0, y_1)$, (x_2, y_2) belong to the same component of C . Then $x_2 \leq L$.*

The proof of Lemma 4 is given in the next section. Lemma 4 has the following consequence.

Corollary 5. *Assume that points (a, c_1) , (b, c_2) , where $0 \leq a \leq b$, belong to the same component of C . Then $b - a \leq L$.*

Proof. Assume that points (a, c_1) , (b, c_2) , $0 \leq a \leq b$, belong to the same component of the set C . It is easy to see that there exist (possibly different than (a, c_1) , (b, c_2)) points (a, d_1) , $(b, d_2) \in C$ and a connected set $D \subset C \cap \{(x, y) : a \leq x \leq b\}$ joining them. We shall write D and C in a new system of coordinates \bar{x} , \bar{y} such that $x = \bar{x} + a$, $y = e^a \bar{y}$.

For every point $(x, y) \in D$ we have

$$P(x, y) = 0, \quad e^x - 1 \leq y \leq e^x + 1, \quad a \leq x \leq b.$$

Therefore in the new coordinates

$$P(\bar{x} + a, e^a \bar{y}) = 0, \quad e^{\bar{x}+a} - 1 \leq e^a \bar{y} \leq e^{\bar{x}+a} + 1, \quad a \leq \bar{x} + a \leq b,$$

and consequently

$$\bar{P}(\bar{x}, \bar{y}) = 0, \quad e^{\bar{x}} - 1 \leq \bar{y} \leq e^{\bar{x}} + 1, \quad 0 \leq \bar{x} \leq b - a,$$

where $\bar{P}(\bar{x}, \bar{y}) = P(\bar{x} + a, e^a \bar{y})$.

Set $\bar{C} = \{(\bar{x}, \bar{y}) : \bar{P}(\bar{x}, \bar{y}) = 0, e^{\bar{x}} - 1 \leq \bar{y} \leq e^{\bar{x}} + 1\}$ and $\bar{D} = \{(\bar{x}, \bar{y}) : (\bar{x} + a, e^a \bar{y}) \in D\}$. As we have already checked, \bar{D} is a subset of \bar{C} . It is also clear that \bar{D} is connected and that points $(0, e^{-a}d_1)$, $(b - a, e^{-a}d_2)$ belong to \bar{D} . Consequently, $b - a \leq L$ follows easily from Lemma 4. \square

Step 3. A bound on the number of intersections of a connected component $D \subset C$ with the graph $y = e^x + \sin(x)$, $x > 0$.

Let D be a fixed connected component of C . From Corollary 5 it follows that there are constants $0 \leq a \leq b$, $b - a \leq L$ such that for all $(x, y) \in D$, $x > 0$, we have $a \leq x \leq b$.

In particular, the number of intersections of D with the graph $y = e^x + \sin(x)$, $x > 0$, is not greater than the number of solutions of the system

$$P(x, e^x + \sin(x)) = 0, \quad a \leq x \leq b.$$

By a theorem of Khovansky [K2, 1.4] this number is $\leq 4d(d + 2)^2(b - a)/\pi \leq 4d(d + 2)^2L/\pi$.

Now we are ready to finish the proof. It is clear that all solutions of (1) belong to C . In Step 3 we have estimated the number of solutions of (1) which belong to a given connected component of C . In Step 1 we have bounded a number of connected components of C . Summing up, the number of solutions of (1) is less than or equal to

$$(4d(d + 2)^2L/\pi)((5d^2 + 3d + 2)/2) =$$

$$d(d + 1)^2(d + 2)^4(5d^2 + 3d + 2)(d \ln 4 + (d + 1)^2(d + 2))/\pi \leq 2(d + 2)^{12}.$$

PROOF OF LEMMA 4

Suppose, contrary to our claim, that there exists a connected component D of the set $C = \{(x, y) : P(x, y) = 0, e^x - 1 \leq y \leq e^x + 1\}$ joining two points $(0, y_1)$ and (x_2, y_2) such that $x_2 > L$ where $L = d(d+1)^2(d+2)^2 \ln 2/4 + (d+1)^4(d+2)^3/8$.

Put $n = (d+1)(d+2)/2, t = nd \ln 2 + n^2(d+1)$. We have $L = nt$. Let v_i be the vertical segment $\{(x, y) : x = it, e^x - 1 \leq y \leq e^x + 1\}$. For each $i = 1, \dots, n$ we have $0 < it < x_2$. Since D is connected, it must intersect each segment v_i ($i = 1, \dots, n$). Thus for each $i = 1, \dots, n$ there exists ϵ_i such that $P(it, e^{it} + \epsilon_i) = 0$ and $|\epsilon_i| \leq 1$.

Writing the polynomial P as a sum of monomials $P(x, y) = \sum_{k+l \leq d} a_{kl} x^k y^l$ we get a square system of n linear equations

$$(3) \quad \sum_{k+l \leq d} a_{kl} (it)^k (e^{it} + \epsilon_i)^l = 0, \quad i = 1, \dots, n,$$

with respect to coefficients a_{kl} .

To get a contradiction it is enough to check that the determinant of this system does not vanish. Indeed, in this case the system (3) has only the zero solution $P \equiv 0$.

To compute this determinant we arrange the set of indices $\{(k, l) \in \mathbb{N}_0^2 : k+l \leq d\}$ in a sequence $\{(\alpha_i, \beta_i)\}_{1 \leq i \leq n}$ ordered as follows: if $i < j$, then $\beta_i < \beta_j$ or $\beta_i = \beta_j$ and $\alpha_i < \alpha_j$.

This sequence splits in a natural way into $d+1$ subsequences. In each of them the numbers β_i are constant. More precisely, there exists a partition $N_0 \cup \dots \cup N_d = \{1, \dots, n\}$ such that

$$\beta_j = i, \quad 0 \leq \alpha_j \leq d - i \quad \text{for } j \in N_i, \quad i = 0, \dots, d.$$

The determinant $D = \det((it)^{\alpha_j} (e^{it} + \epsilon_i)^{\beta_j})_{1 \leq i \leq n, 1 \leq j \leq n}$ of the system (3) is by definition equal to

$$\begin{aligned} D &= \sum_{\sigma \in \text{Perm}\{1, \dots, n\}} \text{sgn}(\sigma) (\sigma(1)t)^{\alpha_1} (e^{\sigma(1)t} + \epsilon_{\sigma(1)})^{\beta_1} \dots (\sigma(n)t)^{\alpha_n} (e^{\sigma(n)t} + \epsilon_{\sigma(n)})^{\beta_n} \\ &= t^s \sum_{\sigma \in \text{Perm}\{1, \dots, n\}} \text{sgn}(\sigma) \sigma(1)^{\alpha_1} \dots \sigma(n)^{\alpha_n} (e^{\sigma(1)t} + \epsilon_{\sigma(1)})^{\beta_1} \dots (e^{\sigma(n)t} + \epsilon_{\sigma(n)})^{\beta_n}. \end{aligned}$$

Here $s = \sum_{i=1}^n \alpha_i$. Write $D = Wt^s$ i.e.

$$W = \sum_{\sigma \in \text{Perm}\{1, \dots, n\}} \text{sgn}(\sigma) \sigma(1)^{\alpha_1} \dots \sigma(n)^{\alpha_n} (e^{\sigma(1)t} + \epsilon_{\sigma(1)})^{\beta_1} \dots (e^{\sigma(n)t} + \epsilon_{\sigma(n)})^{\beta_n}.$$

Put $K = \max_{\sigma \in \text{Perm}\{1, \dots, n\}} \sum_{i=1}^n \sigma(i) \beta_i$ and let $S_K \subset \text{Perm}\{1, \dots, n\}$ be the set of permutations satisfying the condition: $\sigma \in S_K$ iff $\sum_{i=1}^n \sigma(i) \beta_i = K$.

Denote

$$W_1 = \sum_{\sigma \in S_K} \text{sgn}(\sigma) \sigma(1)^{\alpha_1} \dots \sigma(n)^{\alpha_n} (e^{\sigma(1)t})^{\beta_1} \dots (e^{\sigma(n)t})^{\beta_n}.$$

For every $\sigma \in S_K$ we have $(e^{\sigma(1)t})^{\beta_1} \dots (e^{\sigma(n)t})^{\beta_n} = e^{Kt}$ and consequently $W_1 = e^{Kt} W_2$, where

$$W_2 = \sum_{\sigma \in S_K} \text{sgn}(\sigma) \sigma(1)^{\alpha_1} \dots \sigma(n)^{\alpha_n}.$$

Let us introduce a notation. Consider a non-empty subset A of $\{1, \dots, n\}$. By $\text{Perm}(A)$ we denote the set of all $\sigma \in \text{Perm}\{1, \dots, n\}$ such that $\sigma(i) = i$ for $i \in \{1, \dots, n\} \setminus A$.

In further computations the following description of S_K will be useful.

Lemma 5. *Every permutation $\sigma \in S_K$ admits a decomposition $\sigma = \sigma_0 \cdots \sigma_d$ where $\sigma_i \in \text{Perm}(N_i)$ for $i = 0, \dots, d$. Moreover, such a decomposition is unique.*

We omit a purely combinatorial proof of this lemma. By Lemma 5

$$W_2 = \prod_{k=0}^d \sum_{\sigma_k \in \text{Perm}(N_k)} \text{sgn}(\sigma_k) \prod_{i \in N_k} \sigma_k(i)^{\alpha_i} = \prod_{k=0}^d \det(i^j)_{i \in N_k, 0 \leq j \leq d-k}.$$

Each determinant in this product is the classical Vandermonde determinant of pairwise distinct integers and hence is a non-zero integer. Therefore W_2 being their product is a non-zero integer. As a consequence

$$(4) \quad |W_1| \geq e^{Kt}.$$

Now we estimate the difference $W - W_1$. From definitions of W and W_1 follows that this number is a sum of at most $2^s n!$ terms of the form

$$\pm \sigma(1)^{\alpha_1} \cdots \sigma(n)^{\alpha_n} e^{K't} \epsilon_1^{\gamma_1} \cdots \epsilon_n^{\gamma_n}.$$

Here $s = \sum_{i=1}^n \alpha_i$ and $\gamma_1, \dots, \gamma_n$ are non-negative integers, $K' < K$. The absolute value of each term of the sum is not greater than $(n!)^d e^{(K-1)t}$. Hence

$$(5) \quad |W - W_1| \leq 2^s (n!)^{(d+1)} e^{(K-1)t}.$$

We have two obvious inequalities: $s = \sum_{i=1}^n \alpha_i \leq nd$ and $n! < e^{n^2}$. Hence $2^s (n!)^{(d+1)} < e^{s \ln 2} (e^{n^2})^{d+1} \leq e^{nd \ln 2 + n^2(d+1)} = e^t$.

By (4) and (5) we have

$$\begin{aligned} |W| &\geq |W_1| - |W - W_1| \geq e^{Kt} - 2^s (n!)^{(d+1)} e^{(K-1)t} \\ &= e^{(K-1)t} (e^t - 2^s (n!)^{(d+1)}) > 0. \end{aligned}$$

The last inequality shows that the determinant D of the system (3) is non-zero and gives us a contradiction, as desired.

MOTIVATIONS

Theorem 1 should be understood in the context of Khovansky's theory [K1], [K2]. Our motivation and inspiration for this problem comes from the theory of o-minimal structures. By an o-minimal structure on $(\mathbb{R}, +, \cdot)$ we mean a collection $\mathcal{M} = \bigcup_{n \in \mathbb{N}} \mathcal{M}_n$, where each \mathcal{M}_n is a family of subsets of \mathbb{R}^n such that:

- (1) each \mathcal{M}_n is closed under finite set-theoretical operations;
- (2) if $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_m$, then $A \times B \in \mathcal{M}_{n+m}$;
- (3) let $A \in \mathcal{M}_{n+m}$ and $\pi : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ be a projection on the first n coordinates; then $\pi(A) \in \mathcal{M}_n$;
- (4) every semialgebraic subset of \mathbb{R}^n belongs to \mathcal{M}_n ;
- (5) \mathcal{M}_1 consists of all finite unions of open intervals and points.

O-minimal structures, invented by model theorists, are natural and important extensions of semialgebraic (or more general subanalytic) geometry. We mention here only two important examples of o-minimal structures; more details and examples can be found in [DM]. Wilkie [W] proved (using results of Khovansky [K1])

that by adding to semialgebraic sets the graph of an exponential function one gets an o-minimal structure (called \mathbb{R}_{exp}). A similar extension of global subanalytic sets was done by L. van den Dries, A. Macintyre, D. Marker in [DMM].

Let \mathcal{M} be an o-minimal structure on $(\mathbb{R}, +, \cdot)$. The following important finiteness property (see [DM], [vD]) can be obtained from a result of Pillay, Steinhorn, and Knight [PS], [KPS]:

Theorem KPS. *Let \mathcal{M} be an o-minimal structure. Suppose that $A \in \mathcal{M}_{n+m}$ and denote by $\pi : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ the projection on the first n coordinates. Then there exists $N \in \mathbb{N}$ such that for each $x \in \mathbb{R}^n$ the fiber $\pi^{-1}(x) \cap A$ has at most N connected components.*

Let us consider the following problem:

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ (or more generally $f : (a, \infty) \rightarrow \mathbb{R}$) be an analytic function. What conditions on f would guarantee that the graph of f belongs to an o-minimal structure?

Using the fact that the space of polynomials in 2 variables of degree $\leq d$ is of finite dimension, we get easily from theorem KPS the following necessary condition: (*) For each $d \in \mathbb{N}$ there is $A(d) \in \mathbb{N}$ such that if $P(x, y)$ is a non-zero polynomial of degree d , then the number of isolated solutions of the system

$$P(x, y) = 0, \quad y = f(x), \quad x > a,$$

is not greater than $A(d)$.

One may conjecture that (*) is also a sufficient condition, but this is not the case. Actually $f(x) = e^x + \sin x$, $x > 0$, is a counter-example. Indeed, by Theorem 1, f satisfies (*) with $A(d) \sim 2d^{12}$. Suppose, contrary to our claim, that the graph of f belongs to some o-minimal structure \mathcal{M} . This would imply (see [DM]) that the derivative f' belongs to \mathcal{M} . Hence the graph of $\sin x - \cos x = f(x) - f'(x)$, $x > 0$, is in \mathcal{M} . But this is impossible since $\{x \in \mathbb{R} : \sin x - \cos x = 0, x > 0\}$ cannot belong to \mathcal{M}_1 .

By a similar argument $g(x) = e^{x^2} + \sin(x^2)$, $x \in \mathbb{R}$, does not belong to any o-minimal structure even though it clearly satisfies condition (*).

Note that, by the Bezout theorem, if f is algebraic, then the function $d \rightarrow A(d)$ can be bounded by a linear one. Actually the converse is also true. To show this suppose that $A(d) \leq \text{const}(d + 1)$. Then, for d sufficiently large, $A(d) < B(d) - 1$, where $B(d) = \frac{1}{2}(d + 1)(d + 2)$ is the dimension of the space of polynomials of degree $\leq d$. Take $B(d) - 1$ points on the graph of f and a nonzero polynomial $P(x, y)$, $\deg P \leq d$, which vanishes at these points. Then, by the definition of $A(d)$, P has to vanish on the graph of f , that is f is algebraic. On the other hand, by Khovansky [K2], if f is pffafian (e.g. $f = e^x$), then $A(d)$ can be bounded by a quadratic function.

In general, from the fact that f belongs to some o-minimal structure we cannot deduce anything about $A(d)$. More precisely, if we are given a sequence $\mathbb{N} \ni d \rightarrow a(d) \in \mathbb{N}$, then there exist an analytic function $f : (a, \infty) \rightarrow \mathbb{R}$, subanalytic at the infinity, and an increasing sequence $k \rightarrow d_k$ of integers such that

$$a(d_k) \leq A(d_k)$$

for all $k \in \mathbb{N}$. We sketch only the idea of construction. One can easily construct by induction: a sequence $b_k \in \mathbb{N}$, two sequences $\varepsilon_k > 0$, $\eta_k > 0$, and a sequence of

polynomials $P_k = c_{1+b_k}t^{1+b_k} + \dots + c_{b_{k+1}}t^{b_{k+1}}$ such that:

- (1) $\|P_k\| \leq \varepsilon_k$,
 (2) if $r : (0, 1) \rightarrow \mathbb{R}$ is continuous, $\sup_{t \in (0,1)} |r(t)| \leq \eta_k$, then

$$\#\{t \in (0, 1) : P_k(t) + r(t) = 0\} \geq a(4b_k),$$

- (3) $\sum_{k>n} \varepsilon_k < \eta_n$ for all $n \in \mathbb{N}$,

where $\|\cdot\|$ is the sum of absolute values of coefficients. Now, put

$$g(t) = \sum_{k=1}^{\infty} P_k(t).$$

We can take P_k so small that the radius of convergence of the series is > 1 . Finally put $f(x) = g\left(\frac{x}{\sqrt{x^2+1}}\right)$, $x > 0$. Let

$$q_k(t, y) = y - \sum_{n=1}^{k-1} P_n(t), \quad k > 2.$$

Clearly q_k is of degree $\leq b_k$ and it has at least $a(4b_k)$ zeros on the graph of $g(t)$, for $t \in [0, 1)$. It is easy to find a polynomial $Q_k(x, y)$ of degree $\leq d_k = 4b_k$ which vanishes on the zeros of $q_k\left(\frac{x}{\sqrt{x^2+1}}, y\right)$. Since Q_k has at least $a(d_k)$ zeros on the graph of f , it follows that $a(d_k) \leq A(d_k)$, as desired.

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