

REMARK ABOUT HEAT DIFFUSION ON PERIODIC SPACES

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ABSTRACT. Let M be a complete Riemannian manifold with a free cocompact \mathbb{Z}^k -action. Let $k(t, m_1, m_2)$ be the heat kernel on M . We compute the asymptotics of $k(t, m_1, m_2)$ in the limit in which $t \rightarrow \infty$ and $d(m_1, m_2) \sim \sqrt{t}$. We show that in this limit, the heat diffusion is governed by an effective Euclidean metric on \mathbb{R}^k coming from the Hodge inner product on $H^1(M/\mathbb{Z}^k; \mathbb{R})$.

1. INTRODUCTION

Let M be a complete connected oriented n -dimensional Riemannian manifold. Let $k(t, m_1, m_2)$ be the time- t heat kernel on M . The usual ansatz to approximate $k(t, m_1, m_2)$ is to say that

$$(1.1) \quad k(t, m_1, m_2) \sim P(t, m_1, m_2) e^{-\frac{d(m_1, m_2)^2}{4t}}$$

where $e^{-\frac{d(m_1, m_2)^2}{4t}}$ is considered to be the leading term and $P(t, m_1, m_2)$ is a correction term which can be computed iteratively. There are results which make this precise. For example [1], if m_1 and m_2 are nonconjugate, then as $t \rightarrow 0$,

$$(1.2) \quad k(t, m_1, m_2) = \sum_{\gamma} \frac{(\det d(\exp_{m_1})_{v_{\gamma}})^{-1/2}}{(4\pi t)^{n/2}} e^{-\frac{d(m_1, m_2)^2}{4t}} (1 + O(t)).$$

Here the sum is over minimal geodesics $\gamma : [0, 1] \rightarrow M$ joining m_1 to m_2 of the form $\gamma(s) = \exp_{m_1}(sv_{\gamma})$. For another example, if M has bounded geometry, then lower and upper heat kernel bounds [4], [5] imply that (1.1) is a good approximation if $d(m_1, m_2) \gg t$, in the sense that $-\ln(k(t, m_1, m_2))$ is well-approximated by $\frac{d(m_1, m_2)^2}{4t}$.

One can ask if the ansatz (1.1) is relevant for other asymptotic regimes. In this paper we look at the case when M has a periodic metric, meaning that \mathbb{Z}^k acts freely by orientation-preserving isometries on M , with $X = M/\mathbb{Z}^k$ compact. We consider the asymptotic regime in which $t \rightarrow \infty$ and $d(m_1, m_2) \sim \sqrt{t}$. As the typical time- t Brownian path will travel a distance comparable to \sqrt{t} , this is the regime which contains the bulk of the diffusing heat. We show that, in this regime, (1.1) is no longer a valid approximation. Instead, the heat diffusion is governed by

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an effective Euclidean metric on \mathbb{R}^k . This metric is constructed using the Hodge inner product on $H^1(X; \mathbb{R})$.

To state the precise result, let \mathcal{F} be a fundamental domain in M for the \mathbb{Z}^k -action. Given $\mathbf{v} \in \mathbb{Z}^k$, put

$$(1.3) \quad k(t, \mathbf{v}) = \int_{\mathcal{F}} k(t, m, \mathbf{v} \cdot m) d \operatorname{vol}(m).$$

This is independent of the choice of fundamental domain \mathcal{F} .

The covering $M \rightarrow X$ is classified by a map $\nu : X \rightarrow B\mathbb{Z}^k$, defined up to homotopy, which is π_1 -surjective. It induces a surjection $\nu_* : H_1(X; \mathbb{R}) \rightarrow \mathbb{R}^k$ and an injection $\nu^* : (\mathbb{R}^k)^* \rightarrow H^1(X; \mathbb{R})$. Let $\langle \cdot, \cdot \rangle_{H^1(X; \mathbb{R})}$ be the Hodge inner product on $H^1(X; \mathbb{R})$.

Definition 1. The inner product $\langle \cdot, \cdot \rangle_{(\mathbb{R}^k)^*}$ on $(\mathbb{R}^k)^*$ is given by

$$(1.4) \quad \langle \cdot, \cdot \rangle_{(\mathbb{R}^k)^*} = \frac{(\nu^*)^* \langle \cdot, \cdot \rangle_{H^1(X; \mathbb{R})}}{\operatorname{vol}(X)}.$$

The inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^k}$ is the dual inner product on \mathbb{R}^k .

Let $\operatorname{vol}(\mathbb{R}^k/\mathbb{Z}^k)$ be the volume of a lattice cell in \mathbb{R}^k , measured with $\langle \cdot, \cdot \rangle_{\mathbb{R}^k}$.

Proposition 1. Fix $C > 0$. Then in the region $\{(t, \mathbf{v}) \in \mathbb{R}^+ \times \mathbb{Z}^k : \langle \mathbf{v}, \mathbf{v} \rangle_{\mathbb{R}^k} \leq Ct\}$, as $t \rightarrow \infty$ we have

$$(1.5) \quad k(t, \mathbf{v}) = \frac{\operatorname{vol}(\mathbb{R}^k/\mathbb{Z}^k)}{(4\pi t)^{k/2}} e^{-\langle \mathbf{v}, \mathbf{v} \rangle_{\mathbb{R}^k}/(4t)} + O(t^{-\frac{k+1}{2}})$$

uniformly in \mathbf{v} .

Example. 1. If $M = \mathbb{R}^k$ with a flat metric $\langle \cdot, \cdot \rangle_{flat}$, then one can check that $\langle \cdot, \cdot \rangle_{\mathbb{R}^k} = \langle \cdot, \cdot \rangle_{flat}$, so one recovers the standard flat-space heat kernel.

2. If $n = 2$, then $\langle \cdot, \cdot \rangle_{H^1(X; \mathbb{R})}$ is conformally-invariant. Hence in this case, the heat kernel asymptotics only depend on $\operatorname{vol}(X)$ and the induced complex structure on X .

One can get similar pointwise estimates on $k(t, m_1, m_2)$ by the same methods. We omit the details.

The result of Proposition 1 is an example of the phenomenon of “homogenization”, which has been much-studied for differential operators on \mathbb{R}^n . Homogenization means that in an appropriate scaling limit, the solution to a problem is governed by the solution to a spatially homogeneous problem; see [2] and references therein. Thus it is not surprising that the answer in Proposition 1 has a homogeneous form. The point of the present paper is to show how one can compute the exact asymptotics in the general geometric setting.

We remark that when $t \rightarrow \infty$ and $d(m_1, m_2) \gg t$, the asymptotic expression (1.1) also shows homogenization. This follows from the result of D. Burago [3] that there are a Banach norm $\| \cdot \|$ on \mathbb{R}^k and a constant $c > 0$ such that if $m \in M$ and $\mathbf{v} \in \mathbb{Z}^k$, then $|d(m, \mathbf{v} \cdot m) - \| \mathbf{v} \| | \leq c$. Thus as $t \rightarrow \infty$, if $d(m_1, m_2) \sim \sqrt{t}$, then the effective geometry is $(\mathbb{R}^k, \langle \cdot, \cdot \rangle_{\mathbb{R}^k})$, while if $d(m_1, m_2) \gg t$, then the effective geometry is $(\mathbb{R}^k, \| \cdot \|)$.

It would be interesting if one could extend the results of this paper to the setting in which Γ is a nonabelian discrete group, such as the fundamental group of a closed hyperbolic surface. In this case, the relevant scaling regime should be $t \rightarrow \infty$ and

$d(m_1, m_2) \sim t$, as the typical time- t Brownian path on the hyperbolic plane travels a distance comparable to t .

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2. PROOF OF PROPOSITION 1

We first recall some basic facts about the eigenvalues of a parametrized family of operators [7, Chapter XII].

Let $M_d(\mathbb{C})$ be the vector space of $d \times d$ complex matrices and let $M_d^{sa}(\mathbb{C})$ be the subspace of self-adjoint matrices. Let $f : \mathbb{R}^k \rightarrow M_d(\mathbb{C})$ be a real-analytic map. The eigenvalues $\{\lambda_i(x)\}_{i=1}^d$ of $f(x)$ are algebraic functions of x , meaning the roots of a polynomial whose coefficients are real-analytic functions of x , as they are given by $\det(f(x) - \lambda) = 0$. If $\lambda_1(0)$ is a nondegenerate eigenvalue of $f(0)$, then it extends near $x = 0$ to a real-analytic function $\lambda_1(x)$.

If $k = 1$ and f takes values in $M_d^{sa}(\mathbb{C})$, then the eigenvalues of f form d real-analytic functions $\{\lambda_i(x)\}_{i=1}^d$ on \mathbb{R} . Of course, these functions may cross. If $k > 1$ and f takes values in $M_d^{sa}(\mathbb{C})$, then it may not be true that the eigenvalues form real-analytic functions on \mathbb{R}^k . This can be seen in the example $f(x_1, x_2) = \begin{pmatrix} 0 & x_1 - ix_2 \\ x_1 + ix_2 & 0 \end{pmatrix}$. Its eigenvalues are $\pm \sqrt{x_1^2 + x_2^2}$, which are not the union of two smooth functions on \mathbb{R}^2 . However, if $\gamma(s)$ is a real-analytic curve in \mathbb{R}^2 , then the eigenvalues of $f(\gamma(s))$ do form real-analytic functions in s .

If f is instead an appropriate real-analytic family of operators on a Hilbert space, then one has similar results. We refer to [7, Chapter XII.2] for the precise requirements.

To prove Proposition 1, we use the method of [6, Section VI]. The Pontryagin dual of \mathbb{Z}^k is $T^k = (\mathbb{R}^k)^*/2\pi(\mathbb{Z}^k)^*$. Given $\theta \in T^k$, let $\rho(\theta) : \mathbb{Z}^k \rightarrow U(1)$ be the corresponding representation and let $E(\theta)$ be the flat line bundle on X associated to the representation $\pi_1(X) \xrightarrow{\nu_*} \mathbb{Z}^k \xrightarrow{\rho(\theta)} U(1)$. Let Δ_θ be the Laplacian on $L^2(X; E(\theta))$. Then Fourier analysis gives

$$(2.1) \quad k(t, \mathbf{v}) = \int_{T^k} e^{i\theta \cdot \mathbf{v}} \operatorname{Tr} \left(e^{-t\Delta(\theta)} \right) \frac{d^k \theta}{(2\pi)^k}.$$

Now $\operatorname{Ker}(\Delta(\theta)) = 0$ if $\theta \neq 0$ and $\operatorname{Ker}(\Delta(0)) \cong \mathbb{C}$ consists of the constant functions on X .

In order to write all of the operators $\Delta(\theta)$ as acting on the same Hilbert space, let $\{\tau^j\}_{j=1}^k$ be a set of harmonic 1-forms on X which gives an integral basis of $(\mathbb{Z}^k)^* \subset (\mathbb{R}^k)^* \subseteq H^1(X; \mathbb{R})$. Let $e(\tau^j)$ denote exterior multiplication by τ^j on $C^\infty(X)$ and let $i(\tau^j)$ denote interior multiplication by τ^j on $\Omega^1(X)$. Putting

$$(2.2) \quad d(\theta) = d + i \sum_{j=1}^k \theta_j e(\tau^j)$$

and

$$(2.3) \quad d^*(\theta) = d^* - i \sum_{j=1}^k \theta_j i(\tau^j),$$

$\Delta(\theta)$ is unitarily equivalent to the self-adjoint operator $d^*(\theta)d(\theta)$ (which we shall also denote by $\Delta(\theta)$) acting on $L^2(X)$. Because $\Delta(\theta)$ is quadratic in θ , it is easy

to see that $\{\Delta(\theta)\}_{\theta \in T^k}$ is an analytic family of type (A) in the sense of [7, Chapter XII.2], so we can apply analytic eigenvalue perturbation theory. In particular, if $\{\lambda_i(\theta)\}_{i \in \mathbb{Z}^+}$ are the eigenvalues of $\Delta(\theta)$, arranged in increasing order and repeated if there is a multiplicity greater than one, then $\lambda_1(\theta) \geq 0$ and $\lambda_1(\theta) = 0$ if and only if $\theta = 0$, in which case it is a nondegenerate eigenvalue. Thus λ_1 extends to a real-analytic function in a neighborhood of $\theta = 0$. So for sufficiently small $\epsilon > 0$, there is a neighborhood $U \subseteq T^k$ of $0 \in T^k$ such that

1. If $\theta \notin U$, then $\lambda_1(\theta) > \epsilon$.
2. Restricted to U , λ_1 is a real-analytic function which represents a nondegenerate eigenvalue and $\lambda_2 > \epsilon$.

From (2.1), we have

$$(2.4) \quad k(t, \mathbf{v}) = \int_{T^k} e^{i\theta \cdot \mathbf{v}} \sum_{i=1}^{\infty} e^{-t\lambda_i(\theta)} \frac{d^k \theta}{(2\pi)^k}.$$

Then it is easy to show that

$$(2.5) \quad k(t, \mathbf{v}) = \int_U e^{i\theta \cdot \mathbf{v}} e^{-t\lambda_1(\theta)} \frac{d^k \theta}{(2\pi)^k} + O(e^{-\epsilon t/2}),$$

uniformly in \mathbf{v} .

Lemma 1. *The Taylor’s series of $\lambda_1(\theta)$ near $\theta = 0$ starts off as*

$$(2.6) \quad \lambda_1(\theta) = \langle \theta, \theta \rangle_{(\mathbb{R}^k)^*} + O(|\theta|^3).$$

Proof. It suffices to compute $\frac{d\lambda_1(s\vec{w})}{ds} \Big|_{s=0}$ and $\frac{d^2\lambda_1(s\vec{w})}{ds^2} \Big|_{s=0}$ for all $\vec{w} \in (\mathbb{R}^k)^*$. For simplicity, denote $\Delta(s\vec{w})$ by $\Delta(s)$ and $\lambda_1(s\vec{w})$ by $\lambda(s)$. As $\lambda(s)$ is nonnegative and $\lambda(0) = 0$, we must have $\lambda'(0) = 0$. Let $\psi(s)$ denote a nonzero eigenfunction with eigenvalue $\lambda(s)$; we can assume that it is real-analytic in s with $\psi(0) = 1$. Differentiation of $\Delta(s)\psi(s) = \lambda(s)\psi(s)$ gives

$$(2.7) \quad \Delta'(0)\psi(0) + \Delta(0)\psi'(0) = 0$$

and

$$(2.8) \quad \Delta''(0)\psi(0) + 2\Delta'(0)\psi'(0) + \Delta(0)\psi''(0) = \lambda''(0)\psi(0).$$

Taking the inner product of (2.8) with $\psi(0)$ gives

$$(2.9) \quad \langle \psi(0), \Delta''(0)\psi(0) \rangle + 2\langle \psi(0), \Delta'(0)\psi'(0) \rangle = \lambda''(0)\langle \psi(0), \psi(0) \rangle.$$

Let G be the Green’s operator for $\Delta(0)$. From (2.7),

$$(2.10) \quad \psi'(0) = c\psi(0) - G\Delta'(0)\psi(0)$$

for some constant c . Changing $\psi(s)$ to $e^{-cs}\psi(s)$, we may assume that $c = 0$. Substituting (2.10) into (2.9) gives

$$(2.11) \quad \langle \psi(0), \Delta''(0)\psi(0) \rangle - 2\langle \psi(0), \Delta'(0)G\Delta'(0)\psi(0) \rangle = \lambda''(0)\langle \psi(0), \psi(0) \rangle.$$

It remains to compute $\langle \psi(0), \Delta''(0)\psi(0) \rangle$ and $\langle \psi(0), \Delta'(0)G\Delta'(0)\psi(0) \rangle$. Put $D(s) = d_{s\vec{w}}$ and $D^*(s) = d_{s\vec{w}}^*$. Then $\Delta(s) = D^*(s)D(s)$. From (2.2) and (2.3), $D(s)$ and $D^*(s)$ are linear in s , with

$$(2.12) \quad D'(0) = i \sum_{j=1}^k w_j e(\tau^j)$$

and

$$(2.13) \quad (D^*)'(0) = -i \sum_{j=1}^k w_j i(\tau^j).$$

Then

$$(2.14) \quad \begin{aligned} \langle \psi(0), \Delta''(0)\psi(0) \rangle &= 2\langle \psi(0), (D^*)'(0)D'(0)\psi(0) \rangle \\ &= 2|D'(0)\psi(0)|_{H^1(X;\mathbb{C})}^2 \\ &= 2\left| \sum_{j=1}^k w_j \tau^j \right|_{H^1(X;\mathbb{C})}^2. \end{aligned}$$

Now

$$(2.15) \quad \begin{aligned} \Delta'(0)\psi(0) &= [(D^*)'(0)D(0) + D^*(0)D'(0)]\psi(0) \\ &= d^* \left(-i \sum_{j=1}^k w_j \tau^j \right) = 0. \end{aligned}$$

Substituting (2.14) and (2.15) into (2.11) and using the fact that $\langle \psi(0), \psi(0) \rangle = \text{vol}(X)$, the lemma follows. \square

Continuing with the proof of Proposition 1, by Morse theory and Lemma 1, we can find a change of coordinates near $0 \in T^k$ with respect to which λ_1 becomes quadratic. That is, if $B_r(0)$ denotes the ball of radius r in $(\mathbb{R}^k)^*$, we can find an $r > 0$, a neighborhood U of $0 \in T^k$ and a diffeomorphism $\phi : B_r(0) \rightarrow U$ such that $\phi(0) = 0$, $d\phi_0 = \text{Id}$ and $\lambda_1(\phi(x)) = \langle x, x \rangle_{(\mathbb{R}^k)^*}$. Then there is some $\alpha > 0$ such that as $t \rightarrow \infty$,

$$(2.16) \quad k(t, \mathbf{v}) = \int_{B_r(0)} e^{i\phi(x) \cdot \mathbf{v}} e^{-t\langle x, x \rangle_{(\mathbb{R}^k)^*}} \det(d\phi_x) \frac{d^k x}{(2\pi)^k} + O(e^{-\alpha t}),$$

uniformly in \mathbf{v} . Multiplying by a cutoff function on $(\mathbb{R}^k)^*$, we can write

$$(2.17) \quad \begin{aligned} k(t, \mathbf{v}) &= \int_{(\mathbb{R}^k)^*} e^{i\phi(x) \cdot \mathbf{v}} e^{-t\langle x, x \rangle_{(\mathbb{R}^k)^*}} g(x) \frac{d^k x}{(2\pi)^k} + O(e^{-\alpha' t}) \\ &= t^{-\frac{k}{2}} \int_{(\mathbb{R}^k)^*} e^{i\phi\left(\frac{x}{\sqrt{t}}\right) \cdot \mathbf{v}} e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} g\left(\frac{x}{\sqrt{t}}\right) \frac{d^k x}{(2\pi)^k} + O(e^{-\alpha' t}) \end{aligned}$$

for some $g \in C_0^\infty((\mathbb{R}^k)^*)$ with $g(0) = 1$ and some $\alpha' > 0$. (Here ϕ has been extended to become a map $\phi : (\mathbb{R}^k)^* \rightarrow (\mathbb{R}^k)^*$ which is the identity outside of a compact set.)

We have now reduced to a stationary-phase-type integral. Let

$$(2.18) \quad g(x) = 1 + (\nabla g)(0) \cdot x + E(x)$$

be the beginning of the Taylor’s expansion of g . We can write

$$\begin{aligned}
 (2.19) \quad & t^{-\frac{k}{2}} \int_{(\mathbb{R}^k)^*} e^{i\phi(\frac{x}{\sqrt{t}})\cdot\mathbf{v}} e^{-\langle x,x \rangle_{(\mathbb{R}^k)^*}} g\left(\frac{x}{\sqrt{t}}\right) \frac{d^k x}{(2\pi)^k} \\
 &= t^{-\frac{k}{2}} \int_{(\mathbb{R}^k)^*} e^{i\frac{x}{\sqrt{t}}\cdot\mathbf{v}} e^{-\langle x,x \rangle_{(\mathbb{R}^k)^*}} \left[1 + (\nabla g)(0) \cdot \frac{x}{\sqrt{t}} + E\left(\frac{x}{\sqrt{t}}\right)\right] \frac{d^k x}{(2\pi)^k} \\
 &+ t^{-\frac{k}{2}} \int_{(\mathbb{R}^k)^*} e^{i\frac{x}{\sqrt{t}}\cdot\mathbf{v}} \left[e^{i\left(\phi(\frac{x}{\sqrt{t}}) - \frac{x}{\sqrt{t}}\right)\cdot\mathbf{v}} - 1\right] e^{-\langle x,x \rangle_{(\mathbb{R}^k)^*}} g\left(\frac{x}{\sqrt{t}}\right) \frac{d^k x}{(2\pi)^k}.
 \end{aligned}$$

Recall that the measure $\frac{d^k x}{(2\pi)^k}$ on $(\mathbb{R}^k)^*$ derives from the product measure on $T^k = (\mathbb{R}^*/2\pi\mathbb{Z}^*)^k$. Let $\langle \cdot, \cdot \rangle_{prod}$ be the standard product Euclidean metric on $(\mathbb{R}^*)^k$. Let Q be the self-adjoint operator on $(\mathbb{R}^k)^*$ such that $\langle x, x \rangle_{(\mathbb{R}^k)^*} = \langle x, Qx \rangle_{prod}$. Then a standard calculation gives

$$(2.20) \quad t^{-\frac{k}{2}} \int_{(\mathbb{R}^k)^*} e^{i\frac{x}{\sqrt{t}}\cdot\mathbf{v}} e^{-\langle x,x \rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k} = \frac{(\det Q)^{-1/2}}{(4\pi t)^{k/2}} e^{-\langle \mathbf{v}, \mathbf{v} \rangle_{\mathbb{R}^k}/(4t)}.$$

On the other hand,

$$(2.21) \quad (\det Q)^{-1/2} = \text{vol}(\mathbb{R}^k/\mathbb{Z}^k).$$

By symmetry,

$$(2.22) \quad t^{-\frac{k}{2}} \int_{(\mathbb{R}^k)^*} e^{i\frac{x}{\sqrt{t}}\cdot\mathbf{v}} e^{-\langle x,x \rangle_{(\mathbb{R}^k)^*}} (\nabla g)(0) \cdot \frac{x}{\sqrt{t}} \frac{d^k x}{(2\pi)^k} = 0.$$

Let $c > 0$ be such that $|E(x)| \leq c\langle x, x \rangle_{(\mathbb{R}^k)^*}$ for all $x \in (\mathbb{R}^k)^*$. Then

$$\begin{aligned}
 (2.23) \quad & \left| \int_{(\mathbb{R}^k)^*} e^{i\frac{x}{\sqrt{t}}\cdot\mathbf{v}} e^{-\langle x,x \rangle_{(\mathbb{R}^k)^*}} E\left(\frac{x}{\sqrt{t}}\right) \frac{d^k x}{(2\pi)^k} \right| \\
 & \leq \frac{c}{t} \int_{(\mathbb{R}^k)^*} \langle x, x \rangle_{(\mathbb{R}^k)^*} e^{-\langle x,x \rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k}.
 \end{aligned}$$

Finally,

$$\begin{aligned}
 (2.24) \quad & \left| \int_{(\mathbb{R}^k)^*} e^{i\frac{x}{\sqrt{t}}\cdot\mathbf{v}} \left[e^{i\left(\phi(\frac{x}{\sqrt{t}}) - \frac{x}{\sqrt{t}}\right)\cdot\mathbf{v}} - 1\right] e^{-\langle x,x \rangle_{(\mathbb{R}^k)^*}} g\left(\frac{x}{\sqrt{t}}\right) \frac{d^k x}{(2\pi)^k} \right| \\
 & \leq \|g\|_\infty \int_{(\mathbb{R}^k)^*} 2 \left| \sin\left(\frac{1}{2} \left[\phi\left(\frac{x}{\sqrt{t}}\right) - \frac{x}{\sqrt{t}}\right] \cdot \mathbf{v}\right) \right| e^{-\langle x,x \rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k}.
 \end{aligned}$$

We can find a constant $c' > 0$ such that

$$(2.25) \quad 2 \left| \sin\left(\frac{1}{2} [\phi(x) - x] \cdot \mathbf{v}\right) \right| \leq c' \langle x, x \rangle_{(\mathbb{R}^k)^*} \|\mathbf{v}\|_{\mathbb{R}^k}$$

for all $x \in (\mathbb{R}^k)^*$ and $\mathbf{v} \in \mathbb{Z}^k$. Then

$$(2.26) \quad \begin{aligned} \|g\|_\infty &\int_{(\mathbb{R}^k)^*} 2 \left| \sin \left(\frac{1}{2} \left[\phi \left(\frac{x}{\sqrt{t}} \right) - \frac{x}{\sqrt{t}} \right] \cdot \mathbf{v} \right) \right| e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k} \\ &\leq \frac{c'}{\sqrt{t}} \frac{\|\mathbf{v}\|_{\mathbb{R}^k}}{\sqrt{t}} \|g\|_\infty \int_{(\mathbb{R}^k)^*} \langle x, x \rangle_{(\mathbb{R}^k)^*} e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k}. \end{aligned}$$

By assumption,

$$(2.27) \quad \frac{\|\mathbf{v}\|_{\mathbb{R}^k}}{\sqrt{t}} \leq \sqrt{C}.$$

The proposition follows from combining equations (2.17)–(2.27).

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