

THE DISTRIBUTION OF SOLUTIONS  
OF THE CONGRUENCE  $x_1x_2x_3 \dots x_n \equiv c \pmod{p}$

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ABSTRACT. For a cube  $\mathcal{B}$  of size  $B$ , we obtain a lower bound on  $B$  so that  $\mathcal{B} \cap V$  is nonempty, where  $V$  is the algebraic subset of  $\mathbb{F}_p^n$  defined by

$$x_1x_2x_3 \dots x_n \equiv c \pmod{p},$$

$n$  a positive integer and  $c$  an integer not divisible by  $p$ . For  $n = 3$  we obtain that  $\mathcal{B} \cap V$  is nonempty if  $B \gg p^{\frac{2}{3}}(\log p)^{\frac{2}{3}}$ , for  $n = 4$  we obtain that  $\mathcal{B} \cap V$  is nonempty if  $B \gg \sqrt{p} \log p$ , and for  $n \geq 5$  we obtain that  $\mathcal{B} \cap V$  is nonempty if  $B \gg p^{\frac{1}{4} + \frac{1}{\sqrt{2(n+4)}}}(\log p)^{\frac{3}{2}}$ . Using the assumption of the Grand Riemann Hypothesis we obtain  $\mathcal{B} \cap V$  is nonempty if  $B \gg_{\epsilon} p^{\frac{2}{n} + \epsilon}$ .

1. INTRODUCTION

We use multiplicative characters to study the congruence

$$(1) \quad x_1x_2x_3 \dots x_n \equiv c \pmod{p},$$

where  $c$  is an integer not divisible by  $p$ , and  $n > 2$  is a positive integer. In particular if  $V$  is the algebraic subset of  $\mathbb{F}_p^n$  defined by (1), and  $\mathcal{B}$  the cube of size  $B$  defined by

$$(2) \quad \mathcal{B} = \{\mathbf{x} \in \mathbb{F}_p^n : a_i + 1 \leq x_i \leq a_i + B, 1 \leq i \leq n\},$$

we find how large  $B$  must be to guarantee that  $\mathcal{B} \cap V$  is nonempty. More generally, if  $\mathcal{B}$  is a box having sides of arbitrary lengths,

$$(3) \quad \mathcal{B} = \{\mathbf{x} \in \mathbb{F}_p^n : a_i + 1 \leq x_i \leq a_i + B_i, 1 \leq i \leq n\},$$

then our interest is in finding how large the cardinality  $|\mathcal{B}|$  of  $\mathcal{B}$  must be to guarantee  $\mathcal{B} \cap V$  is nonempty. For  $n = 2$  it is known for a cube of type (2) that  $\mathcal{B} \cap V$  is nonempty if  $B \gg p^{\frac{3}{4}}$ . This follows from Weil's bound on the Kloosterman sum. R. A. Smith [4] conjectured that for a cube centered at the origin,  $\mathcal{B} \cap V$  is nonempty if  $B \gg p^{\frac{2}{3}}$ . He was able to prove this result on the assumption of a conjecture of Hooley.

In this paper we consider larger values of  $n$ , and we have the following main theorems.

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**Theorem 1.** *Let  $\mathcal{B}$  be a box of type (3), and  $V$  the algebraic subset of  $\mathbb{F}_p^n$  defined by (1). Then*

(i) *For  $n = 3$ ,  $\mathcal{B} \cap V$  is nonempty if  $|\mathcal{B}| \gg p^2 \log^2 p$ . In particular if  $\mathcal{B}$  is a cube of size  $B$ , then  $\mathcal{B} \cap V$  is nonempty if  $B \gg p^{\frac{2}{3}} (\log p)^{\frac{2}{3}}$ .*

(ii) *For  $n = 4$ ,  $\mathcal{B} \cap V$  is nonempty if  $|\mathcal{B}| \gg p^2 \log^4 p$ . In particular if  $\mathcal{B}$  is a cube of size  $B$ , then  $\mathcal{B} \cap V$  is nonempty if  $B \gg \sqrt{p} \log p$ .*

With extra work using other methods we can obtain a slight saving in this theorem. When  $n = 3$  we can show that for a box of type (3),  $\mathcal{B} \cap V$  is nonempty if  $|\mathcal{B}| \gg p^2$ . For  $n = 4$  we can save a factor of  $\sqrt{\log p}$  on the size  $B$ , and show that for any cube  $\mathcal{B}$  of type (2),  $\mathcal{B} \cap V$  is nonempty if  $B \gg \sqrt{p \log p}$ . The details will appear in forthcoming work.

For larger values of  $n$  we use the result of Burgess [2] and prove

**Theorem 2.** *Let  $\mathcal{B}$  be a cube of type (2), and  $V$  the algebraic subset of  $\mathbb{F}_p^n$  defined by (1) with  $n \geq 5$ . Then  $\mathcal{B} \cap V$  is nonempty if*

$$B \gg p^{\frac{1}{4} + \frac{1}{\sqrt{2(n+4)}}} (\log p)^{\frac{3}{2}}.$$

On the assumption of the generalized Lindelöf hypothesis we are able to sharpen the result of Theorem 2 and prove

**Theorem 3.** *For any cube  $\mathcal{B}$  of type (2), and algebraic set  $V$  defined by (1) with  $n \geq 5$ ,  $\mathcal{B} \cap V$  is nonempty if  $B \gg_{\epsilon} p^{\frac{2}{n} + \epsilon}$ .*

2. LEMMAS

For any prime  $p$ , we let  $\sum_{\chi \neq \chi_o}$  denote a sum over all multiplicative characters  $\chi \pmod{p}$  with  $\chi \neq \chi_o$ , the principal character.

**Lemma 1.**

$$\frac{1}{p-1} \sum_{\chi \neq \chi_o} \left| \sum_{x=a+1}^{a+B} \chi(x) \right|^4 = O(B^2 \log^2 p).$$

This is just Theorem 2 of Ayyad, Cochrane, and Zheng [1].

**Lemma 2.**

$$\sum_{\chi \neq \chi_o} \left| \sum_{x=a+1}^{a+B} \chi(x) \right|^2 \leq (p-1)B.$$

*Proof.*

$$\begin{aligned} \sum_{\chi \neq \chi_o} \left| \sum_{x=a+1}^{a+B} \chi(x) \right|^2 &= \sum_{\chi \neq \chi_o} \left( \sum_{x=a+1}^{a+B} \chi(x) \sum_{y=a+1}^{a+B} \overline{\chi(y)} \right) \\ &= \sum_{x,y=a+1}^{a+B} \left( \sum_{\chi \neq \chi_o} \chi(xy^{-1}) \right) \\ &\leq \sum_{\substack{x,y=a+1 \\ x=y}}^{a+B} \left( \sum_{\chi \neq \chi_o} \chi(xy^{-1}) \right) \\ &\leq (p-1)B. \end{aligned}$$

To obtain results for values of  $n \geq 5$  we need the following result of Burgess.

**Lemma 3** (Burgess [2]). *For any positive integer  $r \geq 2$ , and nonprincipal character  $\chi$ ,*

$$(4) \quad \left| \sum_{x=a+1}^{a+B} \chi(x) \right| \ll B^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{3}{2r}}.$$

**Lemma 4.** *For every integer  $n \geq 5$  there exists an integer  $r \geq 2$  such that*

$$\frac{2r^2 + n - 4}{r(8r + 4n - 16)} < \frac{1}{\sqrt{2(n+4)}}.$$

*Proof.* For any integer  $n \geq 5$  and positive real number  $x$  we have

$$(5) \quad \begin{aligned} \frac{2x^2 + n - 4}{x(8x + 4n - 16)} &< \frac{1}{\sqrt{2(n+4)}} \\ \iff x^2 + \frac{2(n-4)x}{4 - \sqrt{2(n+4)}} - \frac{(n-4)\sqrt{2(n+4)}}{2(4 - \sqrt{2(n+4)})} &< 0. \end{aligned}$$

The graph of the quadratic function

$$f(x) = ax^2 + bx + c =: x^2 + \frac{2(n-4)x}{4 - \sqrt{2(n+4)}} - \frac{(n-4)\sqrt{2(n+4)}}{2(4 - \sqrt{2(n+4)})}$$

is a parabola opening upwards. Now

$$\begin{aligned} b^2 - 4ac &= \frac{4(n-4)^2}{(4 - \sqrt{2(n+4)})^2} - \frac{2(4-n)\sqrt{2(n+4)}}{4 - \sqrt{2(n+4)}} \\ &= \frac{128 - 32n - 8(4-n)\sqrt{2(n+4)}}{(4 - \sqrt{2(n+4)})^2}. \end{aligned}$$

We also have

$$\begin{aligned} 128 - 32n - 8(4-n)\sqrt{2(n+4)} &> (4 - \sqrt{2(n+4)})^2 \\ \iff (8n - 24)\sqrt{2(n+4)} &> 34n - 104. \end{aligned}$$

Since the last inequality holds true for  $n \geq 5$  we see that  $b^2 - 4ac > 1$ . Therefore  $f(x)$  has real roots  $x_1 < x_2$ , with  $x_2 - x_1 = \sqrt{b^2 - 4ac} > 1$ . Moreover,

$$x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2} > \frac{n-4}{\sqrt{2(n+4)} - 4} + \frac{1}{2} > 2,$$

for  $n \geq 5$ . Since  $x_2 - x_1 > 1$  and  $x_2 > 2$ , there exists an integer  $r \geq 2$  with  $x_1 < r < x_2$ . Also, since  $f(x) < 0$  on the interval  $(x_1, x_2)$ , we have  $f(r) < 0$ . Thus  $r$  satisfies (5) and so

$$\frac{2r^2 + n - 4}{r(8r + 4n - 16)} < \frac{1}{\sqrt{2(n+4)}}.$$

3. PROOF OF THEOREM 1

Suppose that  $n = 3$  and that  $\mathcal{B}$  is a box of type (3). Then

$$\begin{aligned}
 |\mathcal{B} \cap V| &= \sum_{\substack{\mathbf{x} \in \mathcal{B} \\ x_1 x_2 x_3 = c}} 1 = \sum_{\substack{\mathbf{x} \in \mathcal{B} \\ x_1 x_2 x_3 c^{-1} = 1}} 1 \\
 (6) \quad &= \frac{1}{p-1} \sum_{\chi} \left( \sum_{\substack{x_i = a_i + 1 \\ i=1,2,3}}^{a_i + B_i} \chi(x_1 x_2 x_3 c^{-1}) \right) \\
 &= \frac{|\mathcal{B}|}{p-1} + \frac{1}{p-1} \sum_{\chi \neq \chi_o} \chi(c^{-1}) \sum_{\substack{x_i = a_i + 1 \\ i=1,2,3}}^{a_i + B_i} \chi(x_1) \chi(x_2) \chi(x_3).
 \end{aligned}$$

Using the Cauchy-Schwarz inequality we bound the error term in (6) as follows:

$$\begin{aligned}
 & \left| \sum_{\chi \neq \chi_o} \chi(c^{-1}) \sum_{\substack{x_i = a_i + 1 \\ i=1,2,3}}^{a_i + B_i} \chi(x_1) \chi(x_2) \chi(x_3) \right| \\
 & \leq \sum_{\chi \neq \chi_o} \left( \left| \sum_{x_1 = a_1 + 1}^{a_1 + B_1} \chi(x_1) \right| \left| \sum_{x_2 = a_2 + 1}^{a_2 + B_2} \chi(x_2) \right| \cdot \left| \sum_{x_3 = a_3 + 1}^{a_3 + B_3} \chi(x_3) \right| \right) \\
 & \leq \left( \sum_{\chi \neq \chi_o} \left| \sum_{x_1 = a_1 + 1}^{a_1 + B_1} \chi(x_1) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{\chi \neq \chi_o} \left| \sum_{x_2 = a_2 + 1}^{a_2 + B_2} \chi(x_2) \right|^2 \cdot \left| \sum_{x_3 = a_3 + 1}^{a_3 + B_3} \chi(x_3) \right|^2 \right)^{\frac{1}{2}} \\
 & \leq \left( \sum_{\chi \neq \chi_o} \left| \sum_{x_1 = a_1 + 1}^{a_1 + B_1} \chi(x_1) \right|^2 \right)^{\frac{1}{2}} \cdot \prod_{i=2}^3 \left( \sum_{\chi \neq \chi_o} \left| \sum_{x_i = a_i + 1}^{a_i + B_i} \chi(x_i) \right|^4 \right)^{\frac{1}{4}}.
 \end{aligned}$$

Now by Lemma 1 and Lemma 2 we obtain the following bound on the error term in (6):

$$\begin{aligned}
 |\text{error}| &\ll \frac{1}{p-1} \sqrt{(p-1)B_1} \cdot \prod_{i=2}^3 ((p-1)B_i^2 \log^2 p)^{\frac{1}{4}} \\
 &\ll |\mathcal{B}|^{\frac{1}{2}} \log p.
 \end{aligned}$$

Thus

$$|\mathcal{B} \cap V| = \frac{|\mathcal{B}|^3}{p-1} + O\left(|\mathcal{B}|^{\frac{1}{2}} \log p\right).$$

For  $\mathcal{B} \cap V$  not to be empty it suffices that

$$\frac{|\mathcal{B}|^3}{p-1} \gg |\mathcal{B}|^{\frac{1}{2}} \log p,$$

that is,

$$|\mathcal{B}| \gg p^2 \log^2 p.$$

When  $n = 4$ , we proceed in a similar manner to obtain

$$(7) \quad |\mathcal{B} \cap V| = \frac{|\mathcal{B}|}{p-1} + \frac{1}{p-1} \sum_{\chi \neq \chi_o} \chi(c^{-1}) \sum_{\substack{x_i = a_i + 1 \\ i=1,2,3,4}}^{a_i + B_i} \chi(x_1)\chi(x_2)\chi(x_3)\chi(x_4).$$

Using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} & \left| \sum_{\chi \neq \chi_o} \chi(c^{-1}) \sum_{\substack{x_i = a_i + 1 \\ i=1,2,3,4}}^{a_i + B_i} \chi(x_1)\chi(x_2)\chi(x_3)\chi(x_4) \right| \\ & \leq \sum_{\chi \neq \chi_o} \left( \left| \sum_{x_1 = a_1 + 1}^{a_1 + B_1} \chi(x_1) \sum_{x_2 = a_2 + 1}^{a_2 + B_2} \chi(x_2) \right| \cdot \left| \sum_{x_3 = a_3 + 1}^{a_3 + B_3} \chi(x_3) \sum_{x_4 = a_4 + 1}^{a_4 + B_4} \chi(x_4) \right| \right) \\ & \leq \left( \sum_{\chi \neq \chi_o} \left| \prod_{i=1}^2 \sum_{x_i = a_i + 1}^{a_i + B_i} \chi(x_i) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{\chi \neq \chi_o} \left| \prod_{i=3}^4 \sum_{x_i = a_i + 1}^{a_i + B_i} \chi(x_i) \right|^2 \right)^{\frac{1}{2}} \\ & \leq \prod_{i=1}^4 \left( \sum_{\chi \neq \chi_o} \left| \sum_{x_i = a_i + 1}^{a_i + B_i} \chi(x_i) \right|^4 \right)^{\frac{1}{4}}. \end{aligned}$$

Now by Lemma 2 we obtain the following bound on the error term in (7):

$$(8) \quad \begin{aligned} |\text{error}| & \ll \frac{1}{p-1} \prod_{i=1}^4 (pB_i^2 \log^2 p)^{\frac{1}{4}} \\ & \ll \sqrt{B_1 B_2 B_3 B_4} \log^2 p = |\mathcal{B}|^{\frac{1}{2}} \log^2 p. \end{aligned}$$

Therefore we obtain

$$|\mathcal{B} \cap V| = \frac{|\mathcal{B}|}{p-1} + O\left(|\mathcal{B}|^{\frac{1}{2}} \log^2 p\right).$$

Thus for  $\mathcal{B} \cap V$  not to be empty it suffices that

$$\frac{|\mathcal{B}|}{p-1} \gg |\mathcal{B}|^{\frac{1}{2}} \log^2 p,$$

that is,

$$|\mathcal{B}| \gg p^2 \log^4 p.$$

#### 4. PROOF OF THEOREM 2

For any cube  $\mathcal{B}$  of size  $B$  we have

$$(9) \quad |\mathcal{B} \cap V| = \frac{B^n}{p-1} + \frac{1}{p-1} \sum_{\chi \neq \chi_o} \chi(c^{-1}) \sum_{\substack{x_i = a_i + 1 \\ i=1,2,\dots,n}}^{a_i + B} \chi(x_1)\chi(x_2)\dots\chi(x_n).$$

The error term in (9) is bounded above by

$$\frac{1}{p-1} \sum_{\chi \neq \chi_o} \left( \prod_{i=1}^n \left| \sum_{x_i = a_i + 1}^{a_i + B} \chi(x_i) \right| \right).$$

Thus

$$(10) \quad |\mathcal{B} \cap V| \geq \frac{B^n}{p-1} - \frac{1}{p-1} \sum_{\chi \neq \chi_o} \left( \prod_{i=1}^n \left| \sum_{x_i=a_i+1}^{a_i+B} \chi(x_i) \right| \right).$$

The term

$$\frac{1}{p-1} \sum_{\chi \neq \chi_o} \left( \prod_{i=1}^n \left| \sum_{x_i=a_i+1}^{a_i+B} \chi(x_i) \right| \right)$$

in (10) may be bounded as follows:

$$\begin{aligned} & \frac{1}{p-1} \sum_{\chi \neq \chi_o} \left( \prod_{i=1}^n \left| \sum_{x_i=a_i+1}^{a_i+B} \chi(x_i) \right|^n \right) \\ & \leq \prod_{i=5}^n \left( \max_{\chi \neq \chi_o} \left| \sum_{x_i=a_i+1}^{a_i+B} \chi(x_i) \right| \right) \cdot \frac{1}{p-1} \sum_{\chi \neq \chi_o} \left( \prod_{i=1}^4 \left| \sum_{x_i=a_i+1}^{a_i+B} \chi(x_i) \right| \right). \end{aligned}$$

Inserting the upper bound of Burgess, Lemma 3, and the upper bound in (8) we obtain

$$\begin{aligned} \frac{1}{p-1} \sum_{\chi \neq \chi_o} \left( \prod_{i=1}^n \left| \sum_{x_i=a_i+1}^{a_i+B} \chi(x) \right| \right) & \ll \left( B^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{3}{2r}} \right)^{n-4} B^2 \log^2 p \\ & = B^{2+\frac{nr-4r-n+4}{r}} p^{\frac{nr+n-4r-4}{4r^2}} (\log p)^{\frac{4r+3n-12}{2r}}. \end{aligned}$$

Therefore

$$|\mathcal{B} \cap V| = \frac{B^n}{p-1} + O \left( B^{2+\frac{nr-4r-n+4}{r}} p^{\frac{nr+n-4r-4}{4r^2}} (\log p)^{\frac{4r+3n-12}{2r}} \right).$$

Thus  $\mathcal{B} \cap V$  is nonempty if

$$\frac{B^n}{p-1} \gg B^{2+\frac{nr-4r-n+4}{r}} p^{\frac{nr+n-4r-4}{4r^2}} (\log p)^{\frac{4r+3n-12}{2r}},$$

that is,

$$(11) \quad B \gg p^{\frac{4r^2+nr+n-4r-4}{8r^2+4rn-16r}} (\log p)^{\frac{4r+3n-12}{4r+2n-8}}.$$

Now the power of  $p$  in (11) is

$$\frac{4r^2 + rn + n - 4r - 4}{8r^2 + 4rn - 16r} = \frac{1}{4} + \frac{2r^2 + n - 4}{r(8r + 4n - 16)}.$$

By Lemma 4 for any integer  $n \geq 5$  there exists an integer  $r \geq 2$  such that

$$\frac{2r^2 + n - 4}{r(8r + 4n - 16)} < \frac{1}{\sqrt{2(n+4)}}.$$

For such choice of  $r$  the power of  $p$  in (11) satisfies

$$\frac{4r^2 + rn + n - 4r - 4}{8r^2 + 4rn - 16r} < \frac{1}{4} + \frac{1}{\sqrt{2(n+4)}}.$$

Since the power of  $\log p$  in (11) satisfies

$$\frac{4r + 3n - 12}{4r + 2n - 8} < \frac{3}{2},$$

we have that  $\mathcal{B} \cap V$  is nonempty if

$$B \gg p^{\frac{1}{4} + \frac{1}{\sqrt{2(n+4)}}} (\log p)^{\frac{3}{2}}.$$

**The optimal choice of  $r$  in (11).** The best choice of  $r$  is that integer which minimizes the power of  $p$  in (11). Using calculus it is easy to see that the power of  $p$  in (11) is minimal when

$$(8r^2 + 4rn - 16)(8r + n - 4) - (4r^2 + nr - 4r + n - 4)(16r + 4n - 16) = 0,$$

that is,

$$r^2(2n - 8) + r(16 - 4n) + n(8 - n) - 16 = 0.$$

Therefore for  $n \geq 5$  we take  $r$  to be

$$r = \left\lceil 1 + \frac{\sqrt{2n^3 - 20n^2 + 64n - 64}}{2n - 8} \right\rceil \text{ or } \left\lceil 1 + \frac{\sqrt{2n^3 - 20n^2 + 64n - 64}}{2n - 8} \right\rceil + 1.$$

The following table gives the optimal choice of  $r$  for various values of  $n$ . We also include the corresponding power of  $p$  in (11).

$n$	$r$	power of $p$
5	2	0.4749
10	3	0.4166
20	4	0.375
100	8	0.3125
1000	23	0.2714
1000000	708	0.2507

### 5. PROOF OF THEOREM 3

It is conjectured that

$$(12) \quad \left| \sum_{n \leq x} \chi(n) \right| \ll_{\epsilon} x^{\frac{1}{2}} p^{\epsilon},$$

for any nonprincipal character  $\chi \pmod p$ . As Montgomery and Vaughan [3] have pointed out, the conjecture is known to be true under the assumption of the Grand Riemann Hypothesis. It is actually a consequence of the generalized Lindelöf hypothesis. Under the assumption of (12) we can substantially sharpen the result of Theorem 2, and prove Theorem 3 as follows.

In (10) we have shown

$$|\mathcal{B} \cap V| \geq \frac{B^n}{p-1} - \frac{1}{p-1} \sum_{\chi \neq \chi_o} \left( \prod_{i=1}^n \left| \sum_{x_i=a_i+1}^{a_i+B} \chi(x_i) \right| \right).$$

Also

$$\begin{aligned} & \frac{1}{p-1} \sum_{\chi \neq \chi_o} \left( \prod_{i=1}^n \left| \sum_{x_i=a_i+1}^{a_i+B} \chi(x_i) \right| \right) \\ & \leq \prod_{i=5}^n \left( \max_{\chi \neq \chi_o} \left| \sum_{x_i=a_i+1}^{a_i+B} \chi(x_i) \right| \right) \cdot \frac{1}{p-1} \sum_{\chi \neq \chi_o} \left( \prod_{i=1}^4 \left| \sum_{x_i=a_i+1}^{a_i+B} \chi(x_i) \right| \right). \end{aligned}$$

Inserting the upper bounds of (12) and (8) we obtain

$$\begin{aligned} \frac{1}{p-1} \sum_{\chi \neq \chi_o} \left( \prod_{i=1}^n \left| \sum_{x_i=a_i+1}^{a_i+B} \chi(x_i) \right| \right) &\ll_{\epsilon} \left( B^{\frac{1}{2}} p^{\epsilon} \right)^{n-4} B^2 (\log p)^2 \\ &= B^{\frac{n}{2}} p^{(n-4)\epsilon} (\log p)^2. \end{aligned}$$

Thus by (10) we have

$$|\mathcal{B} \cap V| \geq \frac{B^n}{p-1} - c(\epsilon) B^{\frac{n}{2}} p^{(n-4)\epsilon} (\log p)^2,$$

where  $c(\epsilon)$  is a constant depending on  $\epsilon$ . Therefore  $\mathcal{B} \cap V$  is nonempty if

$$B \gg_{\epsilon} p^{\frac{2}{n} + \frac{2\epsilon(n-4)}{n}} (\log p)^{\frac{4}{n}}.$$

It suffices to take

$$B \gg_{\epsilon} p^{\frac{2}{n} + \epsilon}.$$

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