THE MAXIMAL IDEAL SPACE OF $H^\infty(D)$
WITH RESPECT TO THE HADAMARD PRODUCT

HERMANN RENDER

(Communicated by Albert Baernstein II)

Abstract. It is shown that the space of all regular maximal ideals in the Banach algebra $H^\infty(D)$ with respect to the Hadamard product is isomorphic to $\mathbb{N}_0$. The multiplicative functionals are exactly the evaluations at the $n$-th Taylor coefficient. It is a consequence that for a given function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H^\infty(D)$ and for a function $F(z)$ holomorphic in a neighborhood $U$ of 0 with $F(0) = 0$ and $a_n \in U$ for all $n \in \mathbb{N}_0$ the function $g(z) = \sum_{n=0}^{\infty} F(a_n) z^n$ is in $H^\infty(D)$.

Introduction

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be power series on $D$. Then the Hadamard product of $f$ and $g$ is defined by $f \ast g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. The Hadamard product on the space $H(D)$ of all holomorphic functions on $D$ is continuous with respect to the topology of compact convergence. In [1] R. Brooks has shown that the space of all maximal ideals in the space $H(D)$ is isomorphic to the Stone-Čech-compactification $\beta N_0$ of $N_0 := N \cup \{0\}$ and the multiplicative functionals on $H(D)$ are given by the coefficient functionals $\delta_n : H(D) \to \mathbb{C}$ defined by $\delta_n(f) := a_n$ (where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $|z| < 1$ and $n \in N_0$). In this note we discuss the subalgebra $H^\infty(D)$ of all bounded holomorphic functions which has been considered for example in [3]. Our main result states that the non-trivial multiplicative functionals on $H^\infty(D)$ are of the form $\delta_n, n \in N_0$ (as in the case of $H(D)$). In contrast to the algebra $H(D)$ the space $H^\infty(D)$ is even a Banach algebra with respect to the supremum norm which is denoted by $\|f\|_\infty$ for $f \in H^\infty(D)$. It follows that the maximal modular ideals of $H^\infty(D)$ are the kernels of the multiplicative functionals and therefore the space of all maximal modular ideals of $H^\infty(D)$ is isomorphic to $N_0$. Note that $H(D)$ possesses a unit element $\gamma(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ which is not in the subalgebra $H^\infty(D)$.

The results

Let $B$ be the space of all $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{n=0}^{\infty} |a_n| < \infty$; clearly, $\|f\|_\infty \leq \sum_{n=0}^{\infty} |a_n|$, so $B \subset H^\infty(D)$. We note that if $f = \sum_{n=0}^{\infty} a_n z^n \in H^\infty(D)$, then $\sum_{n=0}^{\infty} |a_n|^2 = \|f\|_2^2 \leq \|f\|_\infty^2 < \infty$, where $\|f\|_2 := \sqrt{\sum_{n=0}^{\infty} |a_n|^2}$. Hence for any $f,g \in H^\infty(D)$, we have $f \ast g \in B$, since $\sum_{n=0}^{\infty} |a_n b_n| \leq \|f\|_2 \|g\|_2$ by the

Received by the editors March 27, 1997 and, in revised form, August 19, 1997.
1991 Mathematics Subject Classification. Primary 46J15; Secondary 30B10.
Key words and phrases. Hadamard product, bounded analytic functions.
Cauchy-Schwarz inequality; this also shows that $H^\infty(\mathbb{D})$ is a Banach algebra under Hadamard multiplication.

**Proposition 1.** Let $A$ be the Banach algebra obtained by adjoining a unit to $H^\infty(\mathbb{D})$. If $f = \sum_{n=0}^\infty a_n z^n \in B$, then $\sigma_A(f) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

**Proof.** We must show that if $\lambda \notin \{a_n : n \in \mathbb{N}_0\}$ and $\lambda \neq 0$, then $\lambda - f$ is invertible in $A$ (the other inclusion is easy). Let $g(z) = \sum_{n=0}^\infty \frac{a_n}{\lambda - a_n} z^n$; since $|a_n| < |\lambda|/2$ for sufficiently large $n$, we have $|a_n/(\lambda - a_n)| \leq (2/|\lambda|)|a_n|$ for sufficiently large $n$, so $g \in B \subset H^\infty(\mathbb{D})$. Since

$$\lambda - f = \sum_{n=0}^\infty \frac{\lambda a_n - \lambda a_n + a_n^2}{\lambda - a_n} z^n = f * g,$$

we see that $(\lambda - f) * (1 + g) = \lambda$, so $\lambda - f$ is invertible in $A$. $\square$

The next result is the main step of our proof. Although we need it only for Banach algebras, it is valid for the larger class of all Fréchet algebras, cf. [4] for definition. We denote by $\Delta_A$ the set of all continuous multiplicative non-trivial functionals.

**Theorem 2.** If $A$ is a unital Fréchet algebra, and $S$ is a countable subset of $\Delta_A$ with the property that $\sigma_A(f^2) = \{\varphi(f^2) : \varphi \in S\}$ for all $f \in A$, then $S = \Delta_A$.

**Proof.** Let $S = \{\varphi_n : n \in \mathbb{N}\}$. Suppose that there exists $\varphi \in \Delta_A \setminus S$. As $\varphi \neq \varphi_n$ for all $n \in \mathbb{N}$, the sets $A_n := \ker(\varphi_n - \varphi)$ and $B_n := \ker(\varphi_n + \varphi)$ are closed hyperplanes, in particular they are nowhere dense. By the Baire category theorem there exists $f \in A$ such that $f \notin A_n$ and $f \notin B_n$ for all $n \in \mathbb{N}$. This means that $\varphi_n(f) \neq \varphi(f)$ and $\varphi_n(f) \neq -\varphi(f)$ for all $n \in \mathbb{N}$. On the other hand we know that $\lambda := \varphi(f) \in \sigma_A(f)$ since $\varphi$ is multiplicative. Hence $\lambda^2 \in \sigma_A(f^2)$. By assumption there exists $n \in \mathbb{N}$ with $\lambda^2 = \varphi_n(f^2) = (\varphi_n(f))^2$. Hence $\lambda = \varphi_n(f)$ or $\lambda = -\varphi_n(f)$, a contradiction. $\square$

**Theorem 3.** The non-trivial multiplicative functionals on $H^\infty(\mathbb{D})$ are of the form $\delta_n, n \in \mathbb{N}_0$.

**Proof.** By the above, $f^2 \in B$ for all $f \in H^\infty(\mathbb{D})$. Now apply Proposition 1 and Theorem 2. $\square$

**Theorem 4.** Let $U$ be an open neighborhood of zero and $F: U \to \mathbb{C}$ holomorphic with $F(0) = 0$. If $f(z) = \sum_{n=0}^\infty a_n z^n$ is in $H^\infty(\mathbb{D})$ and $a_n \in U$ for all $n \in \mathbb{N}_0$, then $F(f)(z) := \sum_{n=0}^\infty F(a_n) z^n$ is in $H^\infty(\mathbb{D})$.

**Proof.** This is just the functional calculus for Banach algebras (without unit element) using the fact that $\sigma_A(f) \subset U$ by Theorem 3. $\square$

**Remark 5.** There is no continuous functional calculus on $H^\infty(\mathbb{D})$. Consider for example $F(x) = |x|$. Let $g(z) = (1 - z)^{-1} = \sum_{n=0}^\infty b_n z^n$. Then $F(g) = \sum_{n=0}^\infty |b_n| z^n$ is not bounded since $|b_n| \geq \frac{1}{n}$ and $\sum_{n=0}^\infty |b_n|$ is divergent; cf. [5, p. 68].

One should observe that analyticity plays no role, other than in the proof that $H^\infty(\mathbb{D})$ is a Banach algebra under Hadamard multiplication; since $H^\infty(\mathbb{D})$ is isometrically imbedded in $L^\infty(\mathbb{T})$, and the Hadamard product is just convolution, one can just as easily state and prove the corresponding theorem for $L^\infty(\mathbb{T})$, or any of
its subspaces having the form $E = \{ f \in L^\infty(\mathbb{T}) : \hat{f}_n = 0 \text{ for all } n \notin S \}$, where $\hat{f}_n$ is the $n$th Fourier coefficient of $f$, and $S$ is any subset of $\mathbb{Z}$. Of course, $H^\infty(\mathbb{D})$ is the special case of $S = \mathbb{N}_0$. Each such $E$ is a Banach algebra under convolution, and every nontrivial homomorphism of $E$ to $\mathbb{C}$ has the form $f \mapsto \hat{f}_n$ for some $n \in S$.

**References**

nentheorie*. Springer Berlin 1986. MR 88d:01046

Universität Duisburg, Fachbereich Mathematik, Lotharstr. 65, D-47057 Duisburg, Federal Republic of Germany

E-mail address: render@math.uni-duisburg.de