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QUADRATIC BASE CHANGE FOR p-ADIC $\mathrm{SL}(2)$ AS A THETA CORRESPONDENCE I: OCCURRENCE

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ABSTRACT. The local theta correspondence is considered for reductive dual pairs $(\mathrm{SL}_2(F), \mathrm{O}(F))$ where F is a p-adic field of characteristic zero and O is the orthogonal group attached to a quaternary quadratic form with coefficients in F and of Witt rank one over F. It is shown that certain representations of $\mathrm{SL}_2(F)$ occur in the correspondence.

This paper is the first in a series of papers examining in detail Howe's local theta correspondence [H] for a reductive dual pair $(\mathrm{SL}_2(F), \mathrm{O}(F))$ where F is a p-adic field of characteristic zero and O is the orthogonal group attached to a quaternary quadratic form with coefficients in F and of Witt rank one over F. In this paper I show that certain representations of $\mathrm{SL}_2(F)$ occur in the correspondence; see Theorem 2.3 and Corollary 2.4. In future papers, among other results, I will show that these are all the representations of $\mathrm{SL}_2(F)$ that occur in the case p odd; I will also provide strong evidence that this is also true for p even.

The idea behind our argument is relatively simple and can be explained as follows. First, recall the general setting of theta correspondences for symplectic and orthogonal groups; see, e.g., [MVW]. For $i = 1, 2$ let V_i be a finite-dimensional vector space over F equipped with a nondegenerate bilinear form $\langle\ ,\ \rangle_i$; assume that $\langle\ ,\ \rangle_1$ is skew-symmetric while $\langle\ ,\ \rangle_2$ is symmetric. Equip $W = V_1 \otimes V_2$ with the skew-symmetric form $\langle\ ,\ \rangle$ coming from tensoring the $\langle\ ,\ \rangle_i$. Let G_1, G_2 and G be the isometry groups of $\langle\ ,\ \rangle_1$, $\langle\ ,\ \rangle_2$ and $\langle\ ,\ \rangle$ respectively and identify G_1 and G_2 with subgroups of G via their usual actions on W. Then (G_1, G_2) is called a reductive dual pair in G. Let χ be a nontrivial additive character of F and let ω_χ^∞ denote the (smooth) oscillator representation of $\tilde{G}$ attached to χ where $\tilde{G}$ is the (unique) nontrivial two-fold cover of G. For H a closed subgroup of G let $\tilde{H}$ denote the inverse image of H in G and let $\mathcal{R}_\chi(\tilde{H})$ denote the set of irreducible admissible representations of $\tilde{H}$ which occur as quotients of $\omega_\chi^\infty|_{\tilde{H}}$. Then $\tilde{G}_1$ and $\tilde{G}_2$ commute and $\mathcal{R}_\chi(\widetilde{G_1G_2})$ gives rise to a correspondence between $\mathcal{R}_\chi(\tilde{G}_1)$ and $\mathcal{R}_\chi(\tilde{G}_2)$. These correspondences are called theta correspondences. We denote these correspondences by $\theta\colon \mathcal{R}_\chi(\tilde{G}_1) \to \mathcal{R}_\chi(\tilde{G}_2)$ and $\theta\colon \mathcal{R}_\chi(\tilde{G}_2) \to \mathcal{R}_\chi(\tilde{G}_1)$; the direction of θ will be clear from context. Theta correspondences are known in general to be bijections for p odd [W] and for all p in the cases considered in this paper

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[R]. Further in all cases considered here the space V_2 will be even-dimensional and then the $\tilde{G}_1$ and $\tilde{G}_2$ are trivial covers so that we write, in an abuse of notation, $\mathcal{R}_\chi(G_1)$ and $\mathcal{R}_\chi(G_2)$ instead of $\mathcal{R}_\chi(\tilde{G}_1)$ and $\mathcal{R}_\chi(\tilde{G}_2)$ and elements of these sets will be considered as representations of G_1 and G_2 respectively.

Now consider the case where V_1 is two-dimensional and V_2 is as follows. Let E/F be a quadratic extension and set $V_2 = \left\{A \in M_2(E) \mid \bar{A}^t = A\right\}$ where ^{t} denotes transpose and $\bar{A}$ denotes the matrix obtained from A by applying σ to each entry of A where $\Gamma(E/F) = \langle\sigma\rangle$ is the Galois group of E/F. Equip V_2 with the quadratic form $q(A) = -\det(A)$ where $\det\colon M_2(E) \to E$ is the determinant map (note that $\det|_{V_2}$ actually maps to F). This form is defined over F and of Witt rank one over F. Now write $V_2 = V_0 \oplus H$ where $V_0 = \{A \in V_2 \mid A \text{ is 0 on diagonal}\}$, $H = \{A \in V_2 \mid A \text{ is 0 off diagonal}\}$ and the summands are orthogonal. Let H_1, H_0 and J be the orthogonal groups attached to V_2, V_0 and H, respectively, and let $\mathcal{R}_\chi(G_1)$, $\mathcal{R}^0_\chi(G_1)$ and $\mathcal{R}'_\chi(G_1)$ denote those representations of G_1 occurring in the theta correspondences attached to χ and the pairs (G_1, H_1), (G_1, H_0) and (G_1, J) also respectively. Then the idea behind our argument, roughly speaking, is that if π occurs in $\mathcal{R}_\chi(G_1)$, then π should be the quotient of $\pi_0 \otimes \pi'$ for some π_0 in $\mathcal{R}^0_\chi(G_1)$ and π' in $\mathcal{R}'_\chi(G_1)$ by a seesaw duality argument [K], the sets $\mathcal{R}^0_\chi(G_1)$ and $\mathcal{R}'_\chi(G_1)$ being known [C], [ST].

This paper is organized as follows. In the first section we set notation and briefly recall the parametrization of the admissible dual of G_1 in [LL]. We also recall the portions of the parametrizations of the theta correspondences attached to χ and the pairs (G_1, H_0) and (G_1, J_1) we will need. In the second section we construct, by the method outlined above, representations of G_1 which occur in $\mathcal{R}_\chi(G_1)$. The only other ingredients necessary are Whittaker models and a result of D. Prasad on trilinear forms for $\mathrm{GL}_2(F)$ [P1].

Finally we note that instead of using Prasad's result one could try to use Cognet's thesis [Co]. The results of Cognet that would be necessary, however, are much more difficult to prove than the necessary result of [P1]. Further they do not as readily yield results as complete as we obtain here and do not apply to the trivial representation of G_1. Cognet's thesis, however, does indicate the intimate connection that exists between the theta correspondences attached to (G_1, H_1) and quadratic base change from $\mathrm{SL}_2(F)$ to $\mathrm{SL}_2(E)$, a subject we will turn to in future papers and thus the title of this paper.

1. Some notation and parameters

In this section we establish notation, recall the parametrization of the admissible dual of $G_1 = \mathrm{SL}_2(F)$, and recall the theta correspondences attached to χ and (G_1, H_0) and (G_1, J_1). Since this material is known or easily derived from the literature we will be quite brief in our discussion. For unexplained terminology or notation see [M] or [MVW].

Let F be a nonarchimedean local field of characteristic zero and let p denote the residual characteristic of F. Let $\mathcal{O} = \mathcal{O}_F$, $P = P_F$, $\omega = \omega_F$, $k = k_F$, $q = q_F$, and $|\ \ | = |\ \ |_F$ denote, respectively, the ring of integers, the prime ideal, a uniformizing parameter, the residue field, the order of the residue field, and the absolute value on F normalized so that $|x| = q^{-\nu(x)}$ where $\nu = \nu_F$ denotes the order function on F. Let $U = U_F = \mathcal{O}_F^\times$ and $U^n = U_F^n = 1 + P_F^n$ for n a positive integer. Further, for K/F Galois, let $\Gamma(K/F)$ denote the associated Galois group and if, in addition,

$[K:F]<\infty$ let $N_{K/F}=N$ denote the norm map and $K^1=K^1_F$ the norm-one elements in $K^\times$. Finally, fix an algebraic closure $\bar{F}$ of F and a Weil group W_F; let the associated Weil group notation be as in [T].

For G a group and σ a representation of a subgroup H, let $\operatorname{Ind}(G,H;\sigma)$ denote the representation of G induced by σ (form of induction, e.g., compact, L^2, determined by context) and, for g in G, let σ^g denote the representation of $H^g=gHg^{-1}$ defined by $\sigma^g(h)=\sigma\left(g^{-1}hg\right)$ for h in H^g. If J is a subgroup of H, then let $\sigma|_J$ denote the restriction of σ to J. Further, if $J\triangleleft H$ and $\tilde{\sigma}$ is a representation of H/J, then we also view $\tilde{\sigma}$ as a representation of H via inflation. By a character we mean a one-dimensional representation (not necessarily unitary). If χ is a character of $F^\times$, we also view χ as a character of W_F via local class field theory and as a character of $\mathrm{GL}_2(F)$ by composition with det, the determinant map. We say representations π_1 and π_2 of $\mathrm{GL}_2(F)$ are twist equivalent if there exists a character η of $F^\times$ such that $\pi_1\cong\pi_2\otimes\eta$. Finally, if K/F is a finite-dimensional extension of fields and Galois, we also view χ as a character χ_K of $K^\times$ via composition with $N_{K/F}$.

We now briefly recall the parametrization of the admissible dual of $G_1=G_1(F)$ $=\mathrm{SL}_2(F)$ in [LL]. To do this we first recall the parametrization of the admissible dual of $G_1'=G_1'(F)=\mathrm{GL}_2(F)$ in [JL] in a form suitable for our purposes. If μ and ν are characters of $F^\times$ such that $\mu(x)\nu^{-1}(x)\neq|x|$ or $|x|^{-1}$, let $\pi(\mu,\nu)$ denote the irreducibly induced (normalized induction) principal series representation of G_1' attached to μ and ν; note that $\pi(\mu,\nu)\cong\pi(\nu,\mu)$. If $\mu(x)\nu^{-1}(x)=|x|$, write $\mu=\chi|\quad|^{\frac{1}{2}}$ and $\nu=\chi|\quad|^{-\frac{1}{2}}$ with χ a character of $F^\times$ and let $\sigma(\mu,\nu)$ denote the special representation corresponding to the unique invariant subspace of the space of the associated induced representation from the Borel subgroup of G_1' and let $\pi(\mu,\nu)$ $(\cong\chi)$ denote the corresponding quotient. Similarly, if $\mu(x)\nu^{-1}(x)=|x|^{-1}$, let $\sigma(\mu,\nu)$ denote the corresponding special representation (now the quotient) and $\pi(\mu,\nu)$ the corresponding one-dimensional; note that $\sigma(\mu,\nu)\cong\sigma(\nu,\mu)$ and $\pi(\mu,\nu)\cong\pi(\nu,\mu)$. Further, if K/F is quadratic and θ is a character of $K^\times$, we let $\pi(\theta)=\pi(\rho)$ denote the corresponding irreducible smooth representation of G_1' associated to $\rho=\operatorname{Ind}(W_{K/F},W_{K/K};\theta)$; note that $\pi(\theta)\cong\pi\left(\theta^{-1}\right)$ and note also that π is supercuspidal if and only if θ does not factor through $N_{K/F}$ which in turn happens if and only if ρ is irreducible. We call representations of the form $\pi(\theta)$ Weil representations. The irreducible smooth representations of G_1' not of one of the above forms are called exceptional and occur only if $p=2$. These representations are supercuspidal and are parametrized naturally by the local Langlands correspondence in terms of the primitive (i.e., not induced from a proper subgroup) two-dimensional representations of W_F [Ku]; for σ such a representation of W_F, we write $\pi(\sigma)$ for the corresponding exceptional representation. Finally we note that the representations enjoy no other equivalences with the exception that, if μ and ν are characters of $F^\times$ with $\mu\nu^{-1}$ of order two, then $\pi(\mu,\nu)\cong\pi(\mu_K)$ where K/F is the quadratic extension of F associated to $\mu\nu^{-1}$ by local class field theory.

Then viewing G_1 as a subgroup of G_1' we have

Theorem 1.1 ([LL])**.** *Let π_1 be an irreducible smooth representation of G_1. Then there exists an irreducible smooth representation π of G_1' unique up to twist equivalence which contains π_1 upon restriction to G_1. The L-packet of π_1 is of the form $\{\pi_1,\ldots,\pi_s\}$ where the π_i are distinct irreducible smooth representations of G_1 and the restriction of π to G decomposes as $\bigoplus_{i=1}^s\pi_i$. Further, given $1\le i,j\le s$, there exists g in G' such that $\pi_i^g\cong\pi_j$. Moreover:*

(i) *If π is not a Weil representation, then $s = 1$.*
(ii) *If $\pi = \pi(\theta)$ with θ a character of $K^\times$ not of order two, then $s = 2$ and $\pi_i^g \cong \pi_i$ if and only if $\det g$ is a norm from $K^\times$. If π is supercuspidal in this setting then ρ is singly imprimitive (i.e., can only be induced nontrivially from $W_{K/K}$).*
(iii) *If $\pi = \pi(\theta)$ with θ a character of $K^\times$ of order two, then $s = 4$. In this case, ρ is triply imprimitive and if K_i, $i = 1, 2, 3$, are the fields such that ρ may be induced from W_{K_i/K_i} and L is their composite, then $\Gamma(L/F) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\pi_i^g \cong \pi_i$ if and only if $\det g$ is a norm from $L^\times$.*
(iv) *The collection of distinct L-packets partitions $G_1^\wedge$. Further another representation π' in $(G_1')^\wedge$ gives rise to the same L-packet as π if and only if π and π' can be realized as follows:*
 a) *$\pi = \pi(\mu, \nu)$ and $\pi' = \pi(\mu', \nu')$ with $\mu\nu^{-1} = (\mu')(\nu')^{-1}$,*
 b) *$\pi = \sigma(\mu, \nu)$ and $\pi' = \sigma(\mu, \nu')$ with $\mu\nu^{-1} = (\nu')(\nu')^{-1}$,*
 c) *$\pi = \pi(\theta)$ and $\pi' = \pi(\theta')$ with θ and θ' on $K^\times$ with $\theta(\theta')^{-1}|_{K^1} = 1$,*
 d) *$\pi = \pi(\sigma)$ and $\pi' = \pi(\sigma')$ with σ and σ' primitive projectively equivalent representations.*

In what follows we will distinguish among the π_i by their Whittaker models. In particular recall that if π is an infinite-dimensional irreducible smooth representation of G' and η is a nontrivial character of F, then π has, up to scaling, a unique Whittaker model with respect to η and, of course, if π is a finite-dimensional smooth representation, then it has no Whittaker models. Thus for infinite-dimensional π we let $\pi(\mu, \nu, \eta)$ denote the component of $\pi(\mu, \nu)$ upon restriction to G_1 with η-Whittaker model and similarly for $\sigma(\mu, \nu)$, $\pi(\theta)$ and $\pi(\sigma)$. The only remaining irreducible admissible representation of G_1 is the trivial representation which we denote by 1. Finally for a in $F^\times$ and π in $G_1^\wedge$ set $\pi^a = \pi^g$ where g is an element of G_1' such that $\det g = a$. It is easy to check that π^a is well defined. We note that $\pi(\mu, \nu, \eta)^a \cong \pi(\mu, \nu, \eta_a)$ for all a in $F^\times$ and similarly for $\sigma(\mu, \nu)$, $\pi(\theta)$ and $\pi(\sigma)$.

We close this section with two lemmas which briefly recall the theta correspondences attached to χ and the pairs (G_1, H_0) and (G_1, J_1).

Lemma 1.2 ([C]). *If θ is a character of $E^\times$ and a is an element of $N_{E/F}(E^\times)$, then $\pi(\theta, \chi_a)$ occurs in $\mathcal{R}_\chi^0(G_1)$ and these are the only representations of G_1 that occur. In particular for any character θ of $E^\times$ precisely half of the representations in the L-packet of G_1 attached to $\pi(\theta)$ occur.*

Remark 1.3. Although the statements concerning Whittaker models in Lemma 1.2 are not explicitly made in [C], they are easy consequences of the results of [C] and Theorem 1.1.

Now consider (G_1, J_1). Parametrize $J_1^\wedge$ as follows. First identify $F^\times$ with the connected component of the identity, J_1^0, of J_1 by letting a in $F^\times$ act as follows:

$$a \cdot \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} ax & 0 \\ 0 & a^{-1}y \end{pmatrix}$$

where $x, y \in F$. Let σ' denote the element of J_1 defined by

$$\sigma' \cdot \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}.$$

Then $J_1 = J_1^0 \rtimes \langle\sigma'\rangle$. Then one easily checks that if ρ is a character of $F^\times$, then $\mathrm{Ind}\left(J_1, J_1^0; \rho\right)$ is irreducible if and only if $\rho^2 \neq 1$. If $\rho^2 \neq 1$, then, in a slight abuse of notation, we write $\rho = \mathrm{Ind}\left(J_1, J_1^0; \rho\right)$ and note that $\rho = \rho^{-1}$ as representations of J_1. If $\rho^2 = 1$, then we let ρ^+ and ρ^- denote the two components of $\mathrm{Ind}\left(J_1, J_1^0; \rho\right)$. These representations are the extensions of ρ specified by requiring that $\rho^+(\sigma') = 1$ while $\rho^-(\sigma') = -1$. All these representations enjoy the obvious equivalences and exhaust $J_1^\wedge$.

Lemma 1.4 ([ST]). *The theta correspondence attached to χ and (G_1, J_1) may be parametrized as follows.*

(i) *$\mathcal{R}_\chi(J_1)$ contains all of $J_1^\wedge$ with the exception of 1^-.*
(ii) *If ρ is a character of $F^\times$ such that $\rho^2 \neq 1$, then ρ as a representation of $J_1^\wedge$ pairs with the representation $\pi(\rho, 1)$ of $G_1^\wedge$.*
(iii) *If ρ is a character of $F^\times$ of order two, then ρ^+ and ρ^- pair with the two components of $\pi(\rho, 1)$ upon restriction to G_1.*
(iv) *The trivial representation of J_1 pairs with the trivial representation of G_1.*

Remark 1.5. The relevant statements in [ST] are 1.1.4 and 1.1.5 on p. 1054. It is stated, however, in 1.1.5 that $F^\times$ (N_1 in their notation) has a unique character of order two. Although this is incorrect, the proofs given are correct and yield 1.4 as above.

2. Some representations which occur

We need two preliminary results. The first is a statement of seesaw duality which can be stated in general as follows. Suspending earlier notation for the moment, a pair of reductive dual pairs (G_1, G_2) and (G_1', G_2') in a symplectic group G is called a seesaw pair if $G_1 \supset G_1'$ and $G_2' \supset G_2$. This terminology is suggested by the following diagram:

$$\begin{tikzcd} G_1 \arrow[d, dash] \arrow[dr, dash] & G_2' \arrow[d, dash] \arrow[dl, dash] \\ G_1' & G_2 \end{tikzcd}$$

Now if (π, V) is an irreducible smooth representation of $\tilde{G}_1$ and $\left(\omega_\chi^\infty, \mathcal{S}\right)$ is the oscillator representation of $\tilde{G}$, we set $S(\pi) = \bigcap \ker f$ where the intersection is over all f in $\mathrm{Hom}_{\tilde{G}_1}(\mathcal{S}, V)$. Now set $S[\pi] = \mathcal{S} \diagup S(\pi)$ and view it as a $\tilde{G}_1 \times \tilde{G}_2$-space. Then either $S[\pi] = 0$ in which case π does not occur in $\mathcal{R}_\chi(\tilde{G}_1)$ and we set $\theta_0(\pi) = 0$ or $S[\pi]$ is nonzero and by Lemma III.4 of Chapter II of [MVW] there is a smooth representation $\theta_0(\pi)$ of $\tilde{G}_2$ unique up to isomorphism such that $S[\pi] \cong \pi \otimes \theta_0(\pi)$. Then the strong form of Howe's duality conjecture is that either $\theta_0(\pi)$ is zero or is admissible of finite length with unique irreducible quotient $\theta(\pi)$ (in particular the theta correspondence is a bijection); however, we only need the following concerning $\theta_0(\pi)$.

Lemma 2.1 (see, e.g., [P2]). *Let π an irreducible smooth representation of $\tilde{G}_2$ and π' an irreducible smooth representation of $\tilde{G}_1'$, then*

$$\mathrm{Hom}_{\tilde{G}_1'}\left(\theta_0(\pi), \pi'\right) \cong \mathrm{Hom}_{\tilde{G}_2}\left(\theta_0(\pi'), \pi\right).$$

Finally, we need

Lemma 2.2 ([P1]). *Let π_i, $i = 1, 2, 3$, be infinite-dimensional irreducible smooth representations of* $\mathrm{GL}_2(F)$ *such that the product of their central characters is trivial and at least one of the representations is not discrete series. Then there exists a unique* (*up to scalar*) $\mathrm{GL}_2(F)$*-invariant trilinear form on* $\pi_1 \otimes \pi_2 \otimes \pi_3$.

We now revert to our previous notation to state and prove the main result of this paper.

Theorem 2.3. *If π is an irreducible smooth representation of G_1 whose L-packet is not that of a Weil representation coming from E, then π is in $\mathcal{R}_\chi(G_1)$.*

Proof. First assume that π is infinite-dimensional. Let π_3' be an irreducible admissible representation of G_1' whose restriction to G_1 decomposes as the L-packet of π. Let π_2' be a Weil representation of G_1' attached to E and singly imprimitive. Finally let π_1' denote an infinite-dimensional nondiscrete series representation of G_1' with associated singleton L-packet for G_1 and also chosen so that the product of the central characters of the π_i' is trivial. Clearly such a choice is possible. Then, by Lemma 2.2, $\pi_1' \otimes \pi_2' \otimes \pi_3'$ has a G_1'-invariant trilinear form. Thus there exist π_i, $i = 1, 2, 3$, in the L-packet of G_1 associated to π_i' such that $\pi_1 \otimes \pi_2 \otimes \pi_3$ has a nontrivial G_1-invariant trilinear form.

Now since $\pi_1 \otimes \pi_2 \otimes \pi_3$ has a G_1-invariant trilinear form so does $\pi_1^a \otimes \pi_2^a \otimes \pi_3^a$ for all a in $F^\times$. Further, by Theorem 1.1, $\pi_1^a \cong \pi$ for all a in $F^\times$ and thus $\pi_1 \otimes \pi_2^a \otimes \pi_3^a$ has a G_1-invariant trilinear form for all a in $F^\times$. Now considering the effect of conjugation on Whittaker models it follows from Theorem 1.1 and Lemma 1.2 that there exists an a in $F^\times$ such that π_2^a is in $\mathcal{R}_\chi^0(G_1)$; fix such an a.

Now suppose $\pi_3^a \cong \pi^\vee$, the contragredient of π. Then $\pi_1 \otimes \pi_2^a \otimes \pi^\vee$ has a G_1-invariant trilinear form. Moreover π_2^a and π_1 are in $\mathcal{R}_\chi^0(G_1)$ and $\mathcal{R}_\chi'(G_1)$ respectively, in the latter case by Lemma 1.4. Let σ_0 and σ_1 be representations in $H_0^\wedge$ and $J_1^\wedge$ respectively such that $\theta(\sigma_0) = \pi_2^a$ and $\theta(\sigma_1) = \pi_1$. Now consider the seesaw pair

$$\begin{tikzcd} G_1 \times G_1 \arrow[d, dash] \arrow[dr, dash] & H_1 \arrow[d, dash] \arrow[dl, dash] \\ G_1 & H_0 \times J_1. \end{tikzcd}$$

Now let $W_1 = V_1 \otimes H$ and $W_2 = V_1 \otimes V_0$ with symplectic forms as in the introduction. Then, as usual (see, e.g., [MVW, p. 37]), the restriction of ω_χ on $\tilde{G}$ to $\widetilde{Sp(W_1)} \times \widetilde{Sp(W_2)}$ is given by the (outer) tensor product of $\omega_{\chi,1}$ and $\omega_{\chi,2}$ the oscillator representations of $\widetilde{Sp(W_1)}$ and $\widetilde{Sp(W_2)}$, respectively, and thus $\theta(\sigma_0 \times \sigma_1) = \pi_1 \otimes \pi_2^a$. Then the existence of the G_1-invariant trilinear form implies that $\mathrm{Hom}_{G_1}(\theta_0(\sigma_0 \otimes \sigma_1), \pi)$ is nonzero whence by Lemma 2.1 $\theta_0(\pi)$ is nonzero so that π is in $\mathcal{R}_\chi(G_1)$ as desired.

Now suppose $\pi_3^a \cong \pi^\vee$ does not hold. The existence of a G_1-invariant trilinear form on $\pi_1 \otimes \pi_2^a \otimes \pi_3^a$ implies the existence of such a form on $\pi_1^b \otimes \pi_2^{ba} \otimes \pi_3^{ba}$ for all b in $F^\times$. Now, if b is in $N_{E/F}(E^\times)$, then Theorem 1.1 implies $\pi_2^{ba} \cong \pi_2^a$ and $\pi_1^b \cong \pi_1$. Further $\pi^\vee = \pi^{-1}$ where -1 is viewed as an element of $F^\times$ [MVW, p.

91]. It thus suffices to show that there exists b in $N_{E/F}(E^\times)$ such that $\pi_3^{-ba} \cong \pi$. This, however, follows from Theorem 1.1 and our hypothesis that π is not in the L-packet of a Weil representation attached to E.

The only remaining case is π trivial. In this case let π_2 be the component of $\pi(\omega_{E/F}, 1)$ restricted to G_1 that occurs in $\mathcal{R}^0_\chi(G_1)$ where $\omega_{E/F}$ is the character of $F^\times$ associated to E/F by class field theory. Further, let $\pi_1 = \pi_2^\vee$ which in turn is π_2^{-1} and thus is in $\mathcal{R}'_\chi(G_1)$ by Lemma 1.4. Now another seesaw duality argument, as above, gives the result. □

Corollary 2.4. *Let π be an irreducible smooth representation of G_1. Then π occurs in $\mathcal{R}_\chi(G_1)$ if π is trivial or if for some b in $N_{E/F}(E^\times)$, π^b has a Whittaker model with respect to χ.*

Proof. For representations not coming from Weil representations attached to E this follows from the previous theorem. For representations coming from Weil representations attached to E this comes from the usual inclusion $\mathcal{R}^0_\chi(G_1) \subseteq \mathcal{R}_\chi(G_1)$ (i.e., persistence) in the theory of towers (see, e.g., [MVW]). □

Corollary 2.5. *Let $\{\pi_i\}_{i=1}^s$ be an L-packet for G_1. Then*

$$|\mathcal{R}_\chi(G_1) \cap \{\pi_i\}| \geq \frac{s}{2}$$

if the packet comes from a Weil representation attached to E and

$$|\mathcal{R}_\chi(G_1) \cap \{\pi_i\}_{i=1}^s| = s$$

otherwise.

Proof. Immediate. □

Remark 2.6. The quadratic spaces (V_2, q) considered here are not the only Witt-rank-one quaternary quadratic spaces over F. To consider all such possible spaces take V_2 as before but set $q_{E,a}(A) = aq(A)$ where a is an element of $F^\times$ and the subscript E denotes the dependence of q on E. Then as E and a vary the $(V_2, q_{E,a})$ do exhaust (up to isomorphism) the Witt-rank-one quaternary quadratic spaces over F. Moreover $q_{E,a}$ is equivalent to $q_{E',a'}$ if and only if $E = E'$ (recall that we fixed an algebraic closure) and $\left(a^{-1}a', a^{-1}a'\right) = \omega_{E/F}\left(a^{-1}a'\right)$ where $(\ ,\ )$ is the Hilbert symbol and $\omega_{E/F}$ is the character of $F^\times$ associated to E/F by local class field theory. This can all be checked by comparing discriminants and Hasse invariants. Thus to get all possible theta correspondences it suffices to consider the effect of changing the parameter a. This is a straightforward exercise (see, e.g., [M, Remark 2.4(iii)]). The answer is that in Corollary 2.4 π occurs in $\mathcal{R}_\chi(G_1)$ if π is trivial or, if for some b in $N_{E/F}(E^\times)$, π^b has a Whittaker model with respect to χ_a.

Remark 2.7. (i) As we noted in the first paragraph of this paper, we can show that the representations shown here to be in $\mathcal{R}_\chi(G_1)$ do in fact exhaust $\mathcal{R}_\chi(G_1)$ at least for p odd. This strengthens Corollary 2.4 to an “if and only if” statement and thereby the inequality in Corollary 2.5 becomes an equality. These stronger results are not amenable to the methods of this paper, however.

(ii) The results of this paper and the indicated strengthening above are consistent with the general conjectures of Prasad [P3]. They are also consistent with Kudla's conservation conjecture.

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