PERIODIC SOLUTIONS OF A PERIODIC DELAY PREDATOR-PREY SYSTEM

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Abstract. The existence of a positive periodic solution for
\[\frac{dH(t)}{dt} = r(t)H(t) \left[1 - \frac{H(t - \tau(t))}{K(t)}\right] - \alpha(t)H(t)P(t),\]
\[\frac{dP(t)}{dt} = -b(t)P(t) + \beta(t)P(t)H(t - \sigma(t))\]
is established, where \(r, K, \alpha, b, \beta\) are positive periodic continuous functions with period \(\omega > 0\), and \(\tau, \sigma\) are periodic continuous functions with period \(\omega\).

1. Introduction

As pointed out by Freedman and Wu [1] and Kuang [2], it would be of interest to study the global existence of periodic solutions for systems with periodic delays, representing predator-prey or competition systems. The purpose of this article is to consider the following periodic delay predator-prey model:

\[
\begin{align*}
\frac{dH(t)}{dt} &= r(t)H(t) \left[1 - \frac{H(t - \tau(t))}{K(t)}\right] - \alpha(t)H(t)P(t), \\
\frac{dP(t)}{dt} &= -b(t)P(t) + \beta(t)P(t)H(t - \sigma(t)),
\end{align*}
\]
where \(r, K, \alpha, b, \beta\) are positive periodic continuous functions with period \(\omega > 0\), and \(\tau, \sigma\) are periodic continuous functions with period \(\omega > 0\). The system (1.1) was introduced by May in [3, p. 103].

In Section 2, we will use the continuation theorem of coincidence degree theory, which was proposed in [4] by Gaines and Mawhin, to establish the existence of at least one positive \(\omega\)-periodic solution of system (1.1).
For convenience we introduce a continuation theorem [4, p. 40] as follows.

**Lemma 1.1.** Let $X$ be a Banach space and $L$ a Fredholm mapping of index zero. Assume that $N : \overline{\Omega} \rightarrow X$ is $L$-compact on $\overline{\Omega}$ with $\Omega$ open bounded in $X$. Furthermore assume:

(a) for each $\lambda \in (0, 1)$, $x \in \partial \Omega \cap \text{Dom } L$, 
$$Lx \neq Nx;$$
(b) for each $x \in \partial \Omega \cap \text{Ker } L$, 
$$QNx \neq 0$$

and

$$\text{deg}\{QNx, \Omega \cap \text{Ker } L, 0\} \neq 0.$$

Then $Lx = Nx$ has at least one solution in $\Omega$.

2. Main result

In what follows, we use the following notation:

$$\bar{u} = \frac{1}{\omega} \int_{0}^{\omega} u(t) dt, \quad (u)_M = \max_{t \in [0, \omega]} |u(t)|, \quad (u)_m = \min_{t \in [0, \omega]} |u(t)|,$$

where $u$ is a periodic continuous function with period $\omega$.

Now we state our fundamental theorem about the existence of a positive $\omega$-periodic solution of system (1.1).

**Theorem 2.1.** Assume the following:

(i) $((b/\beta)M^2 \omega)^{2/\omega} < (K)_m$;
(ii) $\bar{r} > (r/K)b/\beta$.

Then system (1.1) has at least one positive $\omega$-periodic solution.

**Proof.** Consider the system

$$\begin{cases}
\frac{dx(t)}{dt} = r(t) \left[ 1 - \frac{e^{x(t-\tau(t))}}{K(t)} \right] - \alpha(t)e^{y(t)}, \\
\frac{dy(t)}{dt} = -b(t) + \beta(t)e^{x(t-\sigma(t))},
\end{cases}$$

(2.1)

where $r$, $K$, $\alpha$, $b$, $\beta$, $\tau$, $\sigma$ are the same as those in system (1.1). It is easy to see that if the system (2.1) has an $\omega$-periodic solution $(x^*(t), y^*(t))$, then $(e^{x^*(t)}, e^{y^*(t)})$ is a positive $\omega$-periodic solution of system (1.1). Therefore, for (1.1) to have at least one positive $\omega$-periodic solution it is sufficient that (2.1) has at least one $\omega$-periodic solution. In order to apply Lemma 1.1 to system (2.1), we take

$$X = \{(x(t), y(t))^T \in C(R, R^2) : x(t + \omega) = x(t), \ y(t + \omega) = y(t)\}$$

and

$$\| (x, y)^T \| = \max_{t \in [0, \omega]} |x(t)| + \max_{t \in [0, \omega]} |y(t)|.$$
With this norm, \( X \) is a Banach space. Let
\[
N \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r(t) \left( 1 - \frac{e^{x(t-\tau(t))}}{K(t)} \right) - \alpha(t)e^{y(t)} \\ -b(t) + \beta(t)e^{x(t-\sigma(t))} \end{bmatrix},
\]
\[
L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega x(t)dt \\ \frac{1}{\omega} \int_0^\omega y(t)dt \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} \in X.
\]

Since \( \text{Ker} \ L = \mathbb{R}^2 \) and \( \text{Im} \ L \) is closed in \( X \), \( L \) is a Fredholm mapping of index zero. Furthermore, we have that \( N \) is \( L \)-compact on \( \mathbb{R} \) [4]; here \( \Omega \) is any open bounded set in \( X \).

Corresponding to equation (1.2), we have
\[
\begin{align*}
\frac{dx(t)}{dt} &= \lambda \left( r(t) \left[ 1 - \frac{e^{x(t-\tau(t))}}{K(t)} \right] - \alpha(t)e^{y(t)} \right), \\
\frac{dy(t)}{dt} &= \lambda [-b(t) + \beta(t)e^{x(t-\sigma(t))}].
\end{align*}
\]

Suppose that \( (x(t), y(t))^T \in X \) is a solution of system (2.2) for a certain \( \lambda \in (0, 1) \).

By integrating (2.2) over the interval \([0, \omega]\), we obtain
\[
\int_0^\omega \left\{ r(t) \left[ 1 - \frac{e^{x(t-\tau(t))}}{K(t)} \right] - \alpha(t)e^{y(t)} \right\} dt = 0
\]
and
\[
\int_0^\omega [-b(t) + \beta(t)e^{x(t-\sigma(t))}] dx = 0.
\]

Thus
\[
\int_0^\omega \left[ \frac{r(t)e^{x(t-\sigma(t))}}{K(t)} + \alpha(t)e^{y(t)} \right] dt = \int_0^\omega r(t)dt
\]
and
\[
\int_0^\omega \beta(t)e^{x(t-\sigma(t))} dt = \int_0^\omega b(t)dt.
\]

From (2.2)–(2.4), it follows that
\[
\int_0^\omega |\dot{x}(t)| dt \leq \lambda \int_0^\omega \left| r(t) \left[ 1 - \frac{e^{x(t-\tau(t))}}{K(t)} \right] - \alpha(t)e^{y(t)} \right| dt
\]
\[
< \int_0^\omega r(t)dt + \int_0^\omega \left[ \frac{r(t)e^{x(t-\tau(t))}}{K(t)} + \alpha(t)e^{y(t)} \right] dt
\]
\[
= 2 \int_0^\omega r(t)dt = 2\omega
\]
and
\[
\int_0^\omega |\dot{y}(t)| dt \leq \lambda \int_0^\omega \left| -b(t) + \beta(t)e^{x(t-\sigma(t))} \right| dt < 2\omega.
\]

That is,
\[
\int_0^\omega |\dot{x}(t)| dt < 2\omega
\]
and

\[ (2.6) \quad \int_0^\omega |\dot{y}(t)| \, dt < 2b\omega. \]

Moreover, (2.4) implies that there exists a point \( \xi_1 \in [0, \omega] \) such that

\[ x(\xi_1 - \sigma(\xi_1)) = \log \frac{b(\xi_1)}{\beta(\xi_1)} \leq \log \left( \frac{b}{\beta} \right)_M \]

hence

\[ |x(\xi_1 - \sigma(\xi_1))| \leq \max_{t \in [0, \omega]} \left| \log \frac{b(t)}{\beta(t)} \right| \equiv M_1. \]

Denote \( \xi_1 + \sigma(\xi_1) = t_1 + n_1 \omega, t_1 \in [0, \omega], \) and \( n_1 \) is an integer; then

\[ x(t_1) \leq \log \left( \frac{b}{\beta} \right)_M \quad \text{and} \quad |x(t_1)| \leq M_1. \]

In view of this and (2.5), we have

\[ (2.7) \quad x(t) \leq x(t_1) + \int_0^\omega |\dot{x}(t)| \, dt \leq \log \left( \frac{b}{\beta} \right)_M + 2\bar{r}\omega \]

and

\[ |x(t)| \leq |x(t_1)| + \int_0^\omega |\dot{x}(t)| \, dt \leq M_1 + 2\bar{r}\omega \equiv M_2. \]

By (2.3), (2.7) and assumption (i), we find that there exists a point \( \xi_2 \in [0, \omega] \) such that

\[ \frac{r(\xi_2)e^{\alpha(\xi_2)} - \alpha(\xi_2)}{K(\xi_2)} = r(\xi_2), \]

which implies that

\[ e^{\gamma(\xi_2)} < \frac{r(\xi_2)}{\alpha(\xi_2)} \leq \left( \frac{r}{\alpha} \right)_M \]

and

\[ e^{\gamma(\xi_2)} = \frac{r(\xi_2)}{\alpha(\xi_2)} \left[ 1 - e^{\alpha(\xi_2)} \right] \geq \frac{r(\xi_2)}{\alpha(\xi_2)} \left[ 1 - \frac{(b/\beta)_M e^{2\bar{r}\omega}}{K(\xi_2)} \right] \geq \left( \frac{r}{\alpha} \right)_M \left[ 1 - \frac{(b/\beta)_M e^{2\bar{r}\omega}}{(K)_M} \right] \equiv M_3 > 0. \]

Thus,

\[ |y(\xi_2)| < \max \left\{ \left| \log \left( \frac{r}{\alpha} \right)_M \right|, \left| \log M_3 \right| \right\} \equiv M_4. \]

In view of this and (2.6), we obtain that

\[ |y(t)| \leq y(\xi_2) + \int_0^\omega |\dot{y}(t)| \, dt < M_4 + 2b\omega \equiv M_5. \]
Clearly, $M_i$ ($i = 1, 2, 3, 4, 5$) are independent of $\lambda$, and under the assumption (ii) of the theorem, the system of algebraic equations

\[
\begin{cases}
\bar{r} - \left(\frac{r}{K}\right) u - \bar{\alpha} v = 0, \\
-\bar{b} + \bar{\beta} u = 0
\end{cases}
\]

(2.8)

has a unique solution $(u^*, v^*)$ which satisfies $u^* > 0$ and $v^* > 0$. Denote $M = M_2 + M_5 + C$, where $C > 0$ is taken sufficiently large so that the unique solution of system (2.8) satisfies $\| (u^*, v^*)^T \| = |u^*| + |v^*| < M$. Now we take $\Omega = \{ (x(t), y(t))^T \in X: \| (x, y)^T \| < M \}$. This satisfies condition (a) of Lemma 1.1. When $(x, y)^T \in \partial \Omega \cap \ker L = \partial \Omega \cap R^2$, $(x, y)^T$ is a constant vector in $R^2$ with $|x| + |y| = M$. Then

\[
QN \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{r} - \left(\frac{r}{K}\right)e^x - \bar{\alpha} e^y \\ -\bar{b} + \bar{\beta} e^x \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Furthermore, it can easily be seen that

\[
\text{deg}\{QN(x, y)^T, \Omega \cap \ker L, (0, 0)^T \} = \text{sign}[\bar{\alpha} \bar{\beta} u^* v^*] \neq 0.
\]

By now we know that $\Omega$ verifies all the requirements of Lemma 1.1 and then (2.1) has at least one $\omega$-periodic solution. This completes the proof.

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REFERENCES


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