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STRONGLY EXPOSED POINTS IN UNIFORM ALGEBRAS

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Abstract. In this paper we show that the unit ball of an infinite dimensional function algebra has no strongly exposed points.

1. Introduction

In Banach spaces X one often looks at the geometry of the unit ball, $\mathrm{Ball}(X) = \{x \in X : \|x\| \leq 1\}$, where on the boundary one distinguishes between *extreme* and *exposed* points. A point f in $\partial\mathrm{Ball}(X)$ is called *exposed* if there exists a continuous linear functional L on X such that $L(f) = \|L\| = 1$ and f is the only point in the unit ball that is mapped to 1. In general it is not easy to see if $\partial\mathrm{Ball}(X)$ contains any extreme or exposed points, but in practical situations there are positive results. The famous Krein-Milman theorem, for example, asserts that in a dual Banach space the unit ball is the weak* closed convex hull of its extreme points. (So L^1 is a dual space only in a few trivial cases.)

There is another refinement of the concept of exposedness: we call $f \in \partial\mathrm{Ball}(X)$ *strongly exposed* if there exists $L \in X^*$ with the properties $L(f) = \|L\| = 1$ and for any sequence $(g_n)_1^\infty$ in X such that $\lim_{n\to\infty} \|g_n\| = \lim_{n\to\infty} L(g_n) = 1$, $\lim_{n\to\infty} g_n = f$ in X. Clearly a strongly exposed point is exposed.

In this paper we show that in any infinite dimensional uniform algebra (with identity) the unit ball does not have any strongly exposed points. Of course we can always interpret a uniform algebra as a closed subalgebra of the continuous functions on some compact (Hausdorff) space K, via the Gelfand transform, where it separates the points of K. This is the setting we wish to work in: a closed subalgebra $\mathcal{A}$ of $C(K)$ separating the points of K, containing the constants; in short, a function algebra (on K). Our result was inspired by looking at H^∞ and the disc algebra first (where the proofs can be even more elementary/explicit, albeit slightly longer; for a description of the exposed points in these spaces; see [3], [1] and [2] respectively); the disc algebra (which plays a crucial role in our approach) is the uniform closure of the polynomials on the (closed) unit disc in $\mathbb{C}$. A trivial but important remark is that for f in the (closed) subalgebra $\mathcal{A}$ of $C(K)$, $\|f\| \leq 1$, and g in the disc algebra, $g \circ f \in \mathcal{A}$.

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We state the results in the next section, together with the proofs. Along the way we also show how these results can be used to rule out duality of Banach spaces. Finally, we put the results into perspective.

2. The results

In our proof of the main result we will make use of the following two lemmas, the proofs of which will be outlined at the end of this section.

Lemma 1. *Let $\Omega \subset \mathbb{C}$ be the interior of the Jordan curve $\{z \in \mathbb{C} : |z| \leq 1, |\arg z| = (1-|z|)^2\} \cup \{0\}$. Then*

$$\lim_{n\to\infty} \sup_{z\in\Omega} |1 - z^n| = 1.$$

In what follows $\phi\colon \Delta \to \Omega$ will be a fixed Riemann map of the unit disc to Ω — an element of the disc algebra — that maps 1 to 1.

Lemma 2. *If F is strongly exposed in $\partial Ball(\mathcal{A})$, then for all $x \in K$, $|F(x)| = 1$.*

We are now ready for the main result:

Main Theorem. *The unit ball of an infinite dimensional function algebra $\mathcal{A} \subset C(K)$ has no strongly exposed points.*

Proof. Since $\mathcal{A}$ separates the points of K, K is infinite and there must be infinitely many *strong boundary points* $t \in K$; i.e., given an arbitrary open neighborhood V of t, we can find a function $f \in \mathcal{A}$ such that $f(t) = \|f\| = 1$, while $|f| < 1$ outside of V. (The supremum of $|f(t)|$ over all strong boundary points t equals $\|f\|$ for any $f \in \mathcal{A}$, so there are infinitely many such t; see [5].)

Suppose $F \in \partial\text{Ball}(\mathcal{A})$ is exposed w.r.t. the functional L. Using the Hahn-Banach theorem it is easy to see that L must be of the form

$$L : g \in \mathcal{A} \mapsto \int_K g\overline{F}\, d\mu,$$

for a certain probability measure μ (that must be supported on the set where $|F| = 1$). We can find a sequence of open sets $(V_n)_1^\infty$, each containing a strong boundary point k_n with corresponding $f_n \in \mathcal{A}$, such that

$$\sum_{n=1}^{\infty} \mu(V_n) < \infty.$$

But then the functions $1 - (\phi \circ f_n)^n \in \mathcal{A}$ tend to 1 boundedly and pointwise outside of $\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty V_n$, a set of μ-measure 0, as $n \to \infty$. An easy application of the dominated convergence theorem yields that $L\big((1-(\phi\circ f_n)^n)F\big) \to 1$, whereas $\|(1-(\phi\circ f_n)^n)F\| \to 1$, yet

$$\|(1-(\phi\circ f_n)^n)F - F\| = \|(\phi\circ f_n)^n F\| \geq |F(k_n)|,$$

so that by Lemma 2, F cannot be strongly exposed. □

Proof of Lemma 1. By the maximum modulus theorem we need to show that

$$\lim_{n\to\infty} \sup_{z\in\partial\Omega} |1 - z^n| = 1.$$

For $z \in \partial\Omega$:

$$|1 - z^n|^2 = 1 + |z|^{2n} - 2|z|^n \cos\big(n(1-|z|)^2\big).$$

If $n(1-|z|)^2 \leq \frac{1}{\sqrt{n}}$, then the cosine is at least $\frac{1}{2}$ and $|1-z^n|^2 \leq 1$. On the other hand, if $n(1-|z|)^2 \geq \frac{1}{\sqrt{n}}$, then $|z|^n$ is bounded above by $(1-\frac{1}{n^{3/4}})^n$ which behaves like $e^{-n^{1/4}}$. □

Proof of Lemma 2. Take $e^{i\theta} \in \mathbb{T}$. $E_\theta := \{x \in K\colon F(x) = e^{i\theta}\}$. The function $G = G_\theta = \phi \circ (e^{-i\theta} F)$ in $\mathcal{A}$ peaks on E_θ, and is such that $\limsup_{n\to\infty} \|1-G^n\| \leq 1$, and $1-G^n \to 1$ boundedly and pointwise outside of E_θ. As in the proof of the theorem it follows that $\mu(E_\theta) \neq 0$ if $E_\theta \neq \emptyset$. Hence $E = F(F^{-1}(\mathbb{T}))$ is a countable compact subset of $\mathbb{T}$, thus a peak set for a function ψ in the disc algebra. But then $(\psi \circ F)F$ and F coincide on the support of μ showing that $(\psi \circ F)F = F$, and thus $F(x) = 0$ or $F(x) \in E$.

Now if E were infinite, it would contain infinitely many isolated points $e^{i\theta_n}$, along which $\mu(E_{\theta_n}) \to 0$. Take a function ψ_n in the disc algebra that peaks on $E \setminus E_{\theta_n}$, such that $|\psi_n(e^{i\theta_n})| \leq \frac{1}{2}$. It is easy to check that $L\big((\psi_n \circ F)F\big) \to 1$, while $\|(\psi_n \circ F)F - F\| \geq \frac{1}{2}$ contradicting the assumption that F is strongly exposed. Hence E is a finite set, say $E = \{e^{i\theta_1}, \ldots, e^{i\theta_N}\}$. Using standard arguments it follows that $1_{E_{\theta_1}}, \ldots, 1_{E_{\theta_N}} \in \mathcal{A}$. Also, $1 \in \mathcal{A}$ so that $1_{F^{-1}(0)} \in \mathcal{A}$. Now the exposing functional L cannot distinguish between the elements F and $F + 1_{F^{-1}(0)}$ of ∂Ball, so that in fact F is nowhere zero. We conclude that F only assumes the values $e^{i\theta_1}, \ldots, e^{i\theta_N}$ on K; in particular, for all $x \in K, |F(x)| = 1$. □

Corollary. *No (infinite dimensional) separable function algebra is a dual space.*

One way of proving this result is by using the fact that in a separable dual space the unit ball is the closed convex hull of its strongly exposed points; see [6]. For example, the disc algebra is not a dual space. We thank the referee for pointing out that the corollary is also a consequence of the fact that the space c_0 of sequences on $\mathbb{N}$ that tend to zero, can be isometrically embedded into each separable function algebra (use the generalized Rudin-Carleson theorem, pg. 191 in [4] with a suitably chosen compact set F), combined with the absence of extreme points in the unit ball of c_0.

Looking back, we see that for function algebras, the condition $L(g_n) \to 1$ in the definition of a strongly exposed point f translates to convergence of the functions g_n to f with respect to a single measure, which is of course for $C(K)$ not as strong as the desired uniform convergence (unless K is pathological). A simple proof using Urysohn's lemma uses $C(K)$'s algebraic structure only in a minimal way. Couple that with the fact that any Banach space is linearly and isometrically embeddable as a (closed) *subspace* of some $C(K)$, and it is no longer obvious why strongly exposed points should not exist for general *subalgebras* of $C(K)$.

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