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FINITE-DIMENSIONAL RIGHT IDEALS IN SOME ALGEBRAS ASSOCIATED WITH A LOCALLY COMPACT GROUP

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ABSTRACT. Let G be a discrete group, a commutative discrete cancellative semigroup or a locally compact abelian group. Let $UC(G)$ be the space of bounded, uniformly continuous, complex-valued functions on G. With an Arens-type product, the conjugate $UC(G)^*$ becomes a Banach algebra. We prove, that unlike left ideals, finite-dimensional right ideals exist in $UC(G)^*$ if and only if G is compact.

INTRODUCTION AND PRELIMINARIES

Let G be a locally compact group, $C(G)$ the space of bounded, continuous, complex-valued functions on G, and $UC(G)$ the subspace of $C(G)$ which consists of those functions which are left uniformly continuous, i.e.,

$$UC(G) = \{f \in C(G) \,:\, s \mapsto f_s \,:\, G \to C(G) \text{ is norm continuous}\},$$

where f_s is the left translate of f by s defined by $f_s(t) = f(s\,t)$ for all $t \in G$. Then $UC(G)^*$ is a Banach algebra under the product

$$(\mu\,\nu)(f) = \mu(f_\nu) \quad \text{for all} \quad f \in UC(G), \quad \text{where}$$
$$f_\nu(s) = \nu(f_s) \quad \text{for all} \quad s \in G.$$

In [7], we have dealt with a number of algebras including $UC(G)^*$, and we have determined all the finite-dimensional left ideals of these algebras. We then deduced that this type of ideals exists in $UC(G)^*$ if and only if G is amenable, i.e., there is $\mu \in UC(G)^*$ such that $\mu \neq 0$ and $\mu(f_s) = \mu(f)$ for all $f \in UC(G)$ and $s \in G$. As already remarked in [1, Section 4] and [7], the finite-dimensional right ideals are determined in the same way when the two Arens products coincide in the algebra; for example in $WAP(G)^*$, where $WAP(G)$ is the space of weakly almost periodic functions (see [2, Section 4.2]), or in the group algebra $L^1(G)$ and the measure algebra $M(G)$ when G is compact. However, in [1, Section 4], we have given a class of locally compact abelian groups for which the non-trivial right ideals in $UC(G)^*$ are all of infinite dimension. In this paper, we let G be either a discrete group, a commutative discrete cancellative semigroup or a locally compact abelian group, and we show that, in fact, finite-dimensional right ideals exist in $UC(G)^*$ if and only if G is compact. This is achieved by using the algebraic structure of

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the uniform compactification UG of G. We recall that UG may be regarded as the spectrum of $UC(G)$ equipped with the relative weak*-topology inherited from $UC(G)^*$. By the spectrum of $UC(G)$, we mean the set of all nonzero multiplicative elements x of $UC(G)^*$, i.e., $x(fg) = x(f)x(g)$ for all $f,\ g \in UC(G)$. It is known that the restriction of the operation of $UC(G)^*$ to UG makes UG into a compact right topological semigroup. This means that the operation is defined for x and y in UG by

$$xy(f) = x(f_y) \quad \text{for all} \quad f \in UC(G).$$

This operation is of course associative and is such that the mappings $x \mapsto xy$ and $x \mapsto sx$ from UG into UG are continuous for each $y \in UG$ and $s \in G$. Note that when G is discrete, the space $UC(G)$ and the space of all bounded complex-valued functions on G (usually denoted by $\ell^\infty(G)$) are identical, and so UG and the Stone-Čech compactification βG of G are identical.

Recall that the Gelfand mapping $f \mapsto \tilde{f}$, where $\tilde{f}(x) = x(f)$ for $x \in UG$ and $f \in UC(G)$, identifies $UC(G)$ with $C(UG)$ (see [2, Theorem 3.1.7]). Hence the Banach spaces $UC(G)^*$ and $C(UG)^*$ may also be identified by the mapping $\mu \mapsto \tilde{\mu}$, where $\tilde{\mu}(\tilde{f}) = \mu(f)$ for $\mu \in UC(G)^*$ and $f \in UC(G)$.

The closure in UG of a subset A of UG is denoted by $\overline{A}$. If A is a subset of G, then A^* will denote $\overline{A} \setminus A$. For more information on UG, see [2] and [4].

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Right ideals in $UC(G)^*$

We begin with some results concerning the algebraic structure of UG. The correspondence between UG and $UC(G)^*$ enables us then to prove our main theorem.

Definition 1. A subset V of G is said to be *sparse* if it is countably infinite and $s\,V \cap t\,V$ is finite whenever s and t are distinct elements of G.

These sets exist and were used in [3], [5] and [6] to show the following results.

Theorem 1. *Let G be either a discrete group or a commutative cancellative discrete semigroup. Let V be a sparse subset of G. Then*

(1) *each $x \in V^*$ satisfies $y\,x \neq z\,x$ whenever $y \neq z$ in βG, i.e., x is right cancellative in βG; and*
(2) *$(\beta G)\,x_1 \cap (\beta G)x_2 = \emptyset$ whenever x_1 and x_2 are distinct elements in V^*.*

As the theorem below shows, these results are also valid in UG when G is a non-compact, locally compact abelian group. For the proof, we need to recall the following facts used in [4] to transfer properties from βG when G is discrete to UG when G is not discrete. Write, by [9, Theorem 24.30], $G = \mathbb{R}^n \times H$, where $n \in \mathbb{N}$ and H is a locally compact abelian group containing a compact open subgroup K. Let $\phi : H \to H/K$ be the quotient mapping, and $\psi : \mathbb{Z}^n \times H \to \mathbb{Z}^n \times H/K$ be the mapping defined by $\psi(m,h) = (m, \phi(h))$. By [2, Theorem 4.4.4], let $\widetilde{\psi} : U(\mathbb{Z}^n \times H) \to \beta(\mathbb{Z}^n \times H/K)$ be the continuous homomorphism which extends ψ to $U(\mathbb{Z}^n \times H)$. We recall also from [4] that $U(\mathbb{Z}^n \times H) = \overline{\mathbb{Z}^n \times H}$ (the closure is taken in UG), and that each x in UG can be written as $x = (s,e)\bar{x}$ where $s \in [0,1]^n$, e is the identity in H and $\bar{x} \in U(\mathbb{Z}^n \times H)$.

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Lemma 1. *Let y and z be elements of $U(\mathbb{Z}^n \times H)$, and suppose that $(k,e)\,y \neq z$ for all $k \in \{0,\,1\}^n$. Then $(s,e)\,y \neq z$ in $U(\mathbb{R}^n \times H)$ for all s in $[0,\,1]^n$.*

Proof. The case of $n = 0$ is trivial, so we start with $n = 1$, and suppose that $y \neq z$ and $(1,e)\,y \neq z$. We pick a function $f \in UC(\mathbb{Z} \times H)$ such that

$$\tilde{f}(y) = \tilde{f}((1,e)\,y) = 0 \quad \text{and} \quad \tilde{f}(z) = 1.$$

We extend f to a function g which is defined on $\mathbb{R} \times H$ in the following way. We write each $u \in \mathbb{R}$ as $u = m + s$, where $m \in \mathbb{Z}$ and $s \in [0,1]$, and let

$$\begin{aligned} g(u,h) &= g(m+s,h) \\ &= \big(f(m+1,h) - f(m,h)\big)s + f(m,h) \quad \text{for all} \quad (u,\,h) \in \mathbb{R} \times H. \end{aligned}$$

This means that, for each fixed $h \in H$, the function g_h defined on $\mathbb{R}$ by $g_h(u) = g(u,h)$ is linear in the interval $[m, m+1]$, $m \in \mathbb{Z}$. Then it is not difficult to verify that the function g is uniformly continuous on $\mathbb{R} \times H$. Let $\tilde{g}$ be the continuous extension of g to $U(\mathbb{R} \times H)$, and let (y_α, h_α) be a net in $(\mathbb{Z} \times H)$ which converges to y in $U(\mathbb{R} \times H)$. Then we have, for each $s \in [0,\,1]$,

$$\begin{aligned} \tilde{g}((s,e)\,y) &= \lim_\alpha g(y_\alpha + s,\, h_\alpha) \\ &= \lim_\alpha \big(f(y_\alpha + 1,\, h_\alpha) - f(y_\alpha,\, h_\alpha)\big)s + \lim_\alpha f(y_\alpha,\, h_\alpha) = 0, \end{aligned}$$

whereas it is clear that $\tilde{g}(z) = \tilde{f}(z) = 1$. Thus $(s,e)\,y \neq z$.

We come now to the general case. Suppose that $(k,e)\,y \neq z$ for all $k \in \{0,\,1\}^n$, and let $s = (s_1, s_2, ..., s_n) \in [0,\,1]^n$. Then the proof given for the case $n = 1$ implies that $(s_1, k, e)y \neq z$ for all $k \in \{0,1\}^{n-1}$ and $s_1 \in [0,1]$. Then we consider $\mathbb{R} \times H$ instead of H. The same argument shows again that $(s_1, s_2, k, e)y \neq z$ for all $k \in \{0,1\}^{n-2}$. Inductively, this leads to the desired result. ☐

Theorem 2. *Let G be a non-compact, locally compact abelian group. Then there are points x_1 and x_2 in $UG \setminus G$ such that*

(1) *x_1 and x_2 are right cancellative in UG, and*
(2) *$(UG)\,x_1 \cap (UG)\,x_2 = \emptyset$.*

Proof. Recall that $G = \mathbb{R}^n \times H$. Let V be a sparse subset of $\mathbb{Z}^n \times H/K$. Then, by Theorem 1, each point of V^* is right cancellative in $\beta(\mathbb{Z}^n \times H/K)$. From [4, Theorem 5.4], it follows that every point of $(\widetilde{\psi})^{-1}(V^*)$ belongs to $UG \setminus G$ and is right cancellative in UG. So Statement (1) follows.

For Statement (2), let a_1 and a_2 be two distinct elements in V^*, and let x_1 and x_2 be in $U(\mathbb{Z}^n \times H)$ such that $\widetilde{\psi}(x_1) = a_1$ and $\widetilde{\psi}(x_2) = a_2$. Let y and z be arbitrary elements in UG. We claim that $yx_1 \neq zx_2$. We write $y = (s,e)\bar{y}$ and $z = (t,e)\bar{z}$, where $s,\, t \in [0,1]^n$ and $\bar{y},\, \bar{z} \in U(\mathbb{Z}^n \times H)$. Then, by Theorem 1,

$$\beta(\mathbb{Z}^n \times H/K)\,a_1 \cap \beta(\mathbb{Z}^n \times H/K)\,a_2 = \emptyset.$$

It follows that, for all $k \in \mathbb{Z}^n$,

$$\widetilde{\psi}((k,e)\bar{y})\,\widetilde{\psi}(x_1) = \widetilde{\psi}((k,e)\bar{y})\,a_1 \neq \widetilde{\psi}(\bar{z})\,a_2 = \widetilde{\psi}(\bar{z})\,\widetilde{\psi}(x_2).$$

Since $\widetilde{\psi}$ is a homomorphism, this implies that

$$\widetilde{\psi}((k,e)\,\bar{y}\,x_1) \neq \widetilde{\psi}(\bar{z}\,x_2)$$

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in $\beta(\mathbb{Z}^n \times H/K)$, and so $(k,e)\,\bar{y}\,x_1 \neq \bar{z}\,x_2$ in $U(\mathbb{Z}^n \times H)$. In particular, $(k,e)\bar{y}\,x_1 \neq (k',e)\bar{z}\,x_2$ for all k and $k' \in \{0,1\}^n$. The lemma above leads to the desired conclusion. □

We come now to the correspondence between UG and $UC(G)^*$. We need to recall the following definitions.

Definition 2. The *total variation* of an element μ of $UC(G)^*$ is denoted by $|\mu|$ and defined first for $f \in UC(G)$, $f \geq 0$ by

$$|\mu|(f) = \sup\{\, |\mu(h)| \,:\, h \in UC(G) \text{ and } |h| \leq f\},$$

then extended in the usual way to an element of $UC(G)^*$.

The *support* of an element μ of $UC(G)^*$ $(= C(UG)^*)$ is denoted by *supp*(μ) and defined by

$$supp(\mu) = \{x \in UG \,:\, |\mu|(f) \neq 0 \quad \text{whenever } f \in UC(G),\ f \geq 0 \text{ and } \tilde{f}(x) \neq 0\}.$$

Remark. If one takes the corresponding $\tilde{\mu}$ in $C(UG)^*$, regards by the Riesz representation theorem (see, e.g., [9, Theorem 14.10]), $\tilde{\mu}$ as a bounded, regular, Borel measure on UG, and defines (as usual) the support of $\tilde{\mu}$ by

$$Supp(\tilde{\mu}) = UG \setminus \bigcup\{U \,:\, U \text{ open in } UG \text{ and } |\tilde{\mu}|U) = 0\},$$

then it is easy to check that $supp(\mu) = Supp(\tilde{\mu})$.

Lemma 2. *Let x be a right cancellative element in UG, and let μ be a nonzero element of $UC(G)^*$. Then*

(1) *$C = \{f_x \,:\, f \in UC(G)\}$ is norm-dense in $UC(G)$ (and so $\mu\, x \neq 0$),*
(2) *$|\mu x| = |\mu| x$,*
(3) *$supp(\mu\, x) = supp(\mu)\, x$.*

Proof. Clearly, C is a subalgebra of $UC(G)$ since x is multiplicative. Furthermore, since x is right cancellative, we have $y\,x \neq z\,x$ in UG whenever $y \neq z$ in UG. So there is $f \in UC(G)$ such that $\tilde{f}(y\,x) \neq \tilde{f}(z\,x)$. It follows that

$$\widetilde{f_x}(y) = y(f_x) = (y\,x)(f) = \tilde{f}(y\,x) \neq \tilde{f}(z\,x) = (z\,x)(f) = z(f_x) = \widetilde{f_x}(z).$$

Therefore $\widetilde{C} = \{\widetilde{f_x} \,:\, f \in UC(G)\}$ separates the points in UG, which implies that $\widetilde{C}$ is norm-dense in $C(UG)$. Equivalently, C is norm-dense in $UC(G)$. This implies that for a nonzero μ in $UC(G)^*$, there must be a function $f \in UC(G)$ with $\mu\, x(f) = \mu(f_x) \neq 0$, and so Statement (1) follows.

For simplicity of notation we assume now that $\|\mu\| = 1$. It is known and not difficult to check that $|\mu\nu| \leq |\mu||\nu|$ for all μ and ν in $UC(G)^*$, in particular $|\mu x| \leq |\mu| x$. So we only need to show that $|\mu x| \geq |\mu| x$. Let f be a nonnegative function in $UC(G)$, and let $\epsilon > 0$ be fixed. Then there exists $h \in UC(G)$ such that $|h| \leq f_x$ and $|\mu(h)| \geq |\mu|(f_x) - \frac{\epsilon}{2}$. Since $\{g_x : g \in UC(G)\}$ is norm-dense in $UC(G)$, we pick $g \in UC(G)$ such that $\| g_x - h \| < \frac{\epsilon}{2}$, and so

$$|\mu x(g)| = |\mu(g_x)| \geq |\mu(h)| - \frac{\epsilon}{2} \geq |\mu|(f_x) - \epsilon.$$

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If $|g| \leq f$, then $|\mu x|(f) \geq |\mu|x(f)$ follows. However this is not necessarily true. So we define the following function on UG,

$$\tilde{g'}(x) = \begin{cases} (\tilde{f}(y) + \frac{\epsilon}{2})\frac{\tilde{g}(y)}{|\tilde{g}(y)|}, & \text{if } |\tilde{g}(y)| \geq \tilde{f}(y) + \frac{\epsilon}{2} \\ \tilde{g}(y), & \text{if } |\tilde{g}(y)| < \tilde{f}(y) + \frac{\epsilon}{2}. \end{cases}$$

Then $\tilde{g'}$ is continuous on UG, $|g'| \leq f + \frac{\epsilon}{2}$, and

$$|\tilde{g}(sx)| = |g_x(s)| < |h(s)| + \frac{\epsilon}{2} \leq f_x(s) + \frac{\epsilon}{2} = \tilde{f}(sx) + \frac{\epsilon}{2} \text{ for all } s \in G.$$

Therefore $g'_x(s) = \tilde{g'}(sx) = \tilde{g}(sx) = g_x(s)$ for all s in G, and so the functions g'_x and g_x are equal. It follows that

$$\begin{aligned} |\mu x|(f + \frac{\epsilon}{2}) &\geq |\mu x(g')| = |\mu(g'_x)| = |\mu(g_x)| \\ &\geq |\mu|(f_x) - \epsilon = |\mu|x(f) - \epsilon. \end{aligned}$$

Thus $|\mu x|(f) \geq |\mu|x(f)$, which completes the proof of Statement (2).

Because of Statement (2) we may assume in this last statement that μ is nonnegative, i.e., $\mu = |\mu|$. Let $y \in supp(\mu)\, x$, and let f be a nonnegative function in $UC(G)$ with $\tilde{f}(y) \neq 0$. We need to verify that $\mu\, x(f) \neq 0$. Write $y = z\, x$ with $z \in supp(\mu)$. Then f_x is clearly nonnegative and $\tilde{f}_x(z) = z(f_x) = (z\, x)(f) = \tilde{f}(z\, x) = \tilde{f}(y) \neq 0$. Since $z \in supp(\mu)$, this implies that $(\mu\, x)(f) = \mu(f_x) = |\mu|(f_x) \neq 0$, and so $y \in supp(\mu\, x)$.

Conversely, we regard (by the Riesz representation theorem) $\tilde{\mu}$ as a bounded, regular, Borel measure on UG, and let y be a point not in $supp(\mu)\, x$. Then $supp(\mu)\, x$ is compact since it is the continuous image of a compact set, and so it is closed. Therefore there is $f \in UC(G)$ such that $\tilde{f}(y) \neq 0$ and $\tilde{f}(supp(\mu)\, x) = \{0\}$. Accordingly,

$$\begin{aligned} \mu\, x(f) =& \mu(f_x) = \tilde{\mu}(\tilde{f}_x) = \int_{UG} \tilde{f}_x(z) d\tilde{\mu}(z) = \int_{supp(\mu)} \tilde{f}_x(z) d\tilde{\mu}(z) \\ =& \int_{supp(\mu)} \tilde{f}(z\, x) d\tilde{\mu}(z) = \int_{supp(\mu)} 0 d\tilde{\mu}(z) = 0. \end{aligned}$$

This means that y does not belong to $supp(\mu\, x)$, and so the proof is complete. □

We are now ready to give the main result of the paper.

Theorem 3. *Let G be either a discrete group, a commutative cancellative discrete semigroup, or a locally compact abelian group. Then finite-dimensional right ideals exist in $UC(G)^*$ if and only if G is compact (and so G is finite in the first two cases).*

Proof. The sufficiency is straightforward. In fact, if G is compact then $UC(G) = C(G)$, so $UC(G)^* = M(G)$ (the algebra of bounded, regular, Borel measures on G), and so the finite-dimensional right ideals are determined in a similar fashion as the left ones, see [7].

For the necessity we suppose that G is not compact. Let R be a right ideal of $UC(G)^*$, and let μ be a nonzero element of R. By Theorems 1 and 2 we can take two elements x_1 and x_2 in $UG \setminus G$ which are right cancellative in UG and with disjoint principal left ideals, i.e., $(UG)\, x_1 \cap (UG)\, x_2 = \emptyset$. Then, by Lemma 2, $\mu\, x_1$

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and μx_2 are nonzero elements of R. Furthermore,

$$supp(\mu x_1) \subseteq supp(\mu)\, x_1 \subseteq (UG)\, x_1 \quad \text{and} \quad supp(\mu x_2) \subseteq supp(\mu)\, x_2 \subseteq (UG)\, x_2.$$

Accordingly, $supp(\mu x_1) \cap supp(\mu x_2) = \emptyset$. Suppose now that R is of dimension n, say. Then, for some complex scalars a_1, a_2, ..., a_n, we must have

$$a_1 \mu x_1 + a_2 \mu x_1 x_2 + ... + a_n \mu x_1 x_2^{n-1} = 0.$$

Such an identity is clearly true if and only if $a_1 = a_2 = ... = a_n = 0$, and so the proof is complete. □

Remark. Let G be a nondiscrete, locally compact, abelian group. As it is well known, $L^\infty(G)^*$ is a Banach algebra with the (first) Arens product. The product of μ and ν in $L^\infty(G)^*$ may be described by $(\mu \circledast \nu)(f) = \mu(f_\nu)$, where $f_\nu(\phi) = \nu(\hat{\phi} * f)$ for $f \in L^\infty(G)$ and $\phi \in L^1(G)$ and where $\hat{\phi}(s) = \phi(s^{-1})$ for $s \in G$. In this algebra, we can only answer partially the question of whether finite-dimensional right ideals exist. In fact, let R be a right ideal, and suppose that $\mu(f) \neq 0$ for some μ in R and $f \in UC(G)$. It can be checked directly that the restriction of the product of $L^\infty(G)^*$ to $UC(G)$ coincides with that of $UC(G)^*$, i.e., $\mu \circledast \nu(f) = \mu\nu(f)$ for all $f \in UC(G)$. Accordingly, we may regard R as a nonzero right ideal of $UC(G)^*$, and so Theorem 3 says that R cannot be of finite dimension unless G is compact. However, it may happen that $\mu(f) = 0$ for all $f \in UC(G)$ and for all $\mu \in R$. Such elements satisfy $L^\infty(G)^* \circledast \mu = \{0\}$, and they exist because $L^\infty(G) \setminus UC(G)$ is even nonseparable, see [8]. We still do not know whether in this situation $\mu \circledast L^\infty(G)^*$ can be of finite dimension.

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