

## ON THE TANGENTIAL INTERPOLATION PROBLEM FOR MATRIX-VALUED $H_2$ -FUNCTIONS OF TWO VARIABLES

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ABSTRACT. All solutions of a general tangential interpolation problem for matrix-valued Hardy functions of two variables are described. The minimal norm solution is explicitly expressed in terms of the interpolation data.

### 1. INTRODUCTION

Interpolation theory of matrix-valued functions analytic and contractive in the open unit disk  $\mathbb{D}$  is a well developed topic, and quite a number of approaches have flourished in the recent years; see for instance [5], [8], [10]. For a history of the subject (which originates at the beginning of the century with the work of Schur [14]) see the review paper [9]. The case of functions defined in the polydisk seems to be much less studied; we refer to the preprints [1], [6] and to [13] for the general theory of analytic functions in polydisks.

Here we focus on the tangential interpolation problem for Hardy functions in the bidisk (the case of Hardy functions of one variable was studied in [2] and [4]). Let  $\mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  denote the Hardy space of the bidisk  $\mathbb{D}^2$ , which consists of all  $\mathbb{C}^{p \times q}$ -valued functions  $H(z_1, z_2)$  of two complex variables analytic inside  $\mathbb{D}^2$  and with expansions

$$(1.1) \quad H(z_1, z_2) = \sum_{j, \ell=0}^{\infty} H_{j\ell} z_1^j z_2^\ell, \quad H_{j\ell} \in \mathbb{C}^{p \times q},$$

with square summable coefficients:

$$(1.2) \quad \|H\|_{\mathbf{H}_2^{p \times q}}^2 := \sum_{j, \ell=0}^{\infty} \text{Trace } H_{j\ell}^* H_{j\ell} < \infty.$$

The space  $\mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  is a Hilbert space with respect to the inner product

$$(1.3) \quad \langle H, G \rangle_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)} := \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \text{Trace } H(e^{it}, e^{i\tau})^* G(e^{it}, e^{i\tau}) dt d\tau$$

and it is a Hilbert module with respect to the Hermitian matrix-valued form

$$(1.4) \quad [H, G]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)} := \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} H(e^{it}, e^{i\tau})^* G(e^{it}, e^{i\tau}) dt d\tau.$$

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In this paper we consider the following interpolation problem.

**Problem 1.1.** *Given matrices  $A_1, A_2 \in \mathbb{C}^{r \times r}$  such that*

$$(1.5) \quad \text{spec } A_1 \cup \text{spec } A_2 \subset \mathbb{D}$$

*and matrices  $B_+ \in \mathbb{C}^{r \times p}, B_- \in \mathbb{C}^{r \times q}, \Upsilon \in \mathbb{C}^{q \times q}$ , find all functions  $H \in \mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  satisfying the interpolation condition*

$$(1.6) \quad \frac{1}{(2\pi i)^2} \int_{|\zeta|=1} (\zeta I_r - A_2)^{-1} \left( \int_{|\xi|=1} (\xi I_r - A_1)^{-1} B_+ H(\xi, \zeta) d\xi \right) d\zeta = B_-$$

*and the matrix norm constraint*

$$(1.7) \quad [H, H]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)} \leq \Upsilon.$$

Note that the operator  $\mathbf{R}_{A_1, A_2} : \mathbf{H}_2^{r \times q}(\mathbb{D}^2) \rightarrow \mathbb{C}^{r \times q}$  specified by the rule

$$(1.8) \quad \mathbf{R}_{A_1, A_2}(F(z_1, z_2)) = \frac{1}{(2\pi i)^2} \int_{|\zeta|=1} (\zeta I_r - A_2)^{-1} \left( \int_{|\xi|=1} (\xi I_r - A_1)^{-1} F(\xi, \zeta) d\xi \right) d\zeta,$$

is well defined for the ordered pair  $(A_1, A_2)$  satisfying the spectral condition (1.5). Using this operator one can rewrite the interpolation condition (1.6) in a more compact form

$$(1.9) \quad \mathbf{R}_{A_1, A_2}(B_+ H(z_1, z_2)) = B_-.$$

Making use of the expansion (1.1) leads to the explicit expression of integrals in the left-hand side of (1.6) in terms of Taylor coefficients of the function  $H$

$$\sum_{j, \ell=0}^{\infty} A_2^\ell A_1^j B_+ H_{j\ell} = B_-.$$

From the last equality it follows that a necessary condition for the problem to be solvable is

$$(1.10) \quad \text{Ran } B_- \subset \text{span} \left\{ \text{Ran } A_2^\ell A_1^j B_+; \ell, j = 0, 1, \dots \right\}$$

We will always assume that this condition is in force. Note that the set of interpolation points (*the spectra of the interpolation problem*) is specified by the matrices  $A_1$  and  $A_2$  (and the spectral condition (1.6) provides all these points are inside the bidisk  $\mathbb{D}^2$ ) while the directions at which the interpolant  $H$  has the preassigned values are determined by the matrix  $B_+$ . We illustrate the latter remark by the following example.

**Example 1.2** (The left-sided Nevanlinna–Pick problem). *Take  $\mathbf{z}_k = (z_{1k}, z_{2k}) \in \mathbb{D}^2, b_k \in \mathbb{C}^{r_k \times q}, c_k \in \mathbb{C}^{r_k \times p}$  ( $k = 1, \dots, n$ ) and set*

$$(1.11) \quad B_+ = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad B_- = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad A_j = \begin{pmatrix} z_{j1} I_{r_1} & & \\ & \ddots & \\ & & z_{jn} I_{r_n} \end{pmatrix} \quad (j = 1, 2).$$

It is easily seen that condition (1.9) reduces then to the left-sided Nevanlinna–Pick conditions.

$$(1.12) \quad b_k H(z_{1k}, z_{2k}) = c_k \quad (k = 1, \dots, n).$$

Setting  $A_1$  and  $A_2$  to be general matrices (say, in Jordan form) one can deduce from (1.9) more general conditions than (1.12) involving partial derivatives of  $H$  of higher order at different prescribed points in the bidisk  $\mathbb{D}^2$ . The main result of the paper is:

**Theorem 1.3.** *Let  $\mathbb{P}^{[-1]}$  be the Moore–Penrose pseudoinverse of the matrix*

$$(1.13) \quad \mathbb{P} = \sum_{k,m=0}^{\infty} A_2^k A_1^m B_+ B_+^* A_1^{*m} A_2^{*k}.$$

*Problem 1.1 has a solution if and only if the matrix  $\Upsilon - B_-^* \mathbb{P}^{[-1]} B_-$  is nonnegative. When this is the case, all solutions  $H$  of Problem 1.1 are parametrized by the formula*

$$(1.14) \quad H(z_1, z_2) = H_{\min}(z_1, z_2) + \left( \Theta \widehat{H} \right) (z_1, z_2)$$

where  $H_{\min}$  is the minimal norm solution given by

$$(1.15) \quad H_{\min}(z_1, z_2) = B_+^* (I_r - z_1 A_1^*)^{-1} (I_r - z_2 A_2^*)^{-1} \mathbb{P}^{[-1]} B_-,$$

where  $\Theta$  is the isometric operator in  $\mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  given in (3.10) and where  $\widehat{H}$  is a free parameter from  $\mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  satisfying the norm constraint

$$(1.16) \quad \left[ \widehat{H}, \widehat{H} \right]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)} \leq \Upsilon - B_-^* \mathbb{P}^{[-1]} B_-.$$

For the case of the left-sided Nevanlinna–Pick problem, let us make some remarks on this result. Since  $\mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  is the reproducing kernel Hilbert space with reproducing kernel

$$(1.17) \quad K(z_1, \omega_1; z_2, \omega_2) = \frac{1}{(1 - z_1 \bar{\omega}_1)(1 - z_2 \bar{\omega}_2)},$$

the general interpolation theory in reproducing kernel Hilbert spaces (see e.g. [11, p. 114–115] and [7, Problème 4, p. 123–124] for the scalar case and [3] for the vector case) allows to say that the minimal norm solution of the problem is the function

$$(1.18) \quad H_{\min}(z_1, z_2) = (K(z_1, z_{11}; z_2, z_{21})b_1^*, \dots, K(z_1, z_{1n}; z_2, z_{2n})b_n^*) \mathbb{P}^{[-1]} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

where  $\mathbb{P}^{[-1]}$  is the Moore–Penrose pseudoinverse of the matrix

$$(1.19) \quad \mathbb{P} = (b_j b_\ell^* K(z_{1j}, z_{1\ell}; z_{2j}, z_{2\ell}))_{j,\ell=1}^n.$$

Every function  $H \in \mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  satisfying the interpolation conditions (1.12) is of the form

$$(1.20) \quad H(z_1, z_2) = H_{\min}(z_1, z_2) + \Psi(z_1, z_2),$$

where  $\Psi$  satisfies the homogeneous condition

$$(1.21) \quad \mathbf{R}_{A_1, A_2}(B_+ \Psi(z_1, z_2)) = 0$$

and the representation (1.20) is orthogonal with respect to the inner product in the Hilbert space.

It is easily seen that for the special choice (1.17) of  $K$ , the formulas (1.18) and (1.19) define the same  $H_{\min}$  and  $\mathbb{P}$  as the formulas (1.15) and (1.13) for the choice (1.11) of  $B_+$ ,  $B_-$ ,  $A_1$  and  $A_2$ . Nevertheless, the reproducing kernel method does not seem to be an adequate tool to study Problem 1.1 in its full generality. Another point which seems to us new (even in the case of the Nevanlinna–Pick problem) is the characterization of the set  $\Psi$  of all solutions  $\Psi$  of the homogeneous problem (1.21) as  $\Psi = \Theta \mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  for some isometric operator depending only on interpolation data. Note that for the one variable problem,  $\Theta$  is the operator of multiplication by an inner function.

## 2. REDUCTION TO A ONE-VARIABLE PROBLEM

In this section we reduce Problem 1.1 to a left–sided Nevanlinna–Pick problem for operator–valued Hardy functions of one variable. We denote by  $\mathbf{H}_2^{\infty \times q}(\mathbb{D})$  the space of  $\mathcal{L}(\ell_2; \mathbb{C}^q)$ –valued functions of one variable analytic in  $\mathbb{D}$ :

$$(2.1) \quad \mathbf{H}_2^{\infty \times q}(\mathbb{D}) := \left\{ F(z) = \sum_{j=0}^{\infty} F_j z^j; \ F_j \in \mathcal{L}(\mathbb{C}^q; \ell_2) \text{ and } \text{Trace} \sum_{j=0}^{\infty} F_j^* F_j < \infty \right\}.$$

This space is the Hilbert space module with respect to the Hermitian form

$$(2.2) \quad [F, G]_{\mathbf{H}_2^{\infty \times q}(\mathbb{D})} := \frac{1}{2\pi} \int_0^{2\pi} G(e^{it})^* F(e^{it}) dt.$$

To reduce Problem 1.1 to a “one variable” problem we introduce the function

$$(2.3) \quad E(z) = \begin{pmatrix} I_p & zI_p & z^2I_p & \cdots \end{pmatrix}$$

which is analytic and takes values in  $\ell_2^{p \times p}$  for  $z \in \mathbb{D}$  and rewrite the function  $H(z_1, z_2)$  given by (1.1) as

$$(2.4) \quad H(z_1, z_2) = \sum_{j=0}^{\infty} z_1^j F_j(z_2) = E(z_1)F(z_2),$$

where

$$F_j(z) = \sum_{\ell=0}^{\infty} H_{j\ell} z^\ell$$

and

$$(2.5) \quad F(z) = \begin{pmatrix} F_0(z) \\ F_1(z) \\ \vdots \end{pmatrix} = \sum_{\ell=0}^{\infty} \begin{pmatrix} H_{0\ell} \\ H_{1\ell} \\ \vdots \end{pmatrix} z^\ell.$$

For a given  $j$ , the function  $F_j$  belongs to  $\mathbf{H}_2^{p \times q}(\mathbb{D})$  whereas the function  $F$  clearly belongs to  $\mathbf{H}_2^{\infty \times q}(\mathbb{D})$ .

*Remark 2.1.* Let  $H^{(1)}$  and  $H^{(2)}$  be two functions from  $\mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  and let

$$H^{(1)}(z_1, z_2) = E(z_1)F^{(1)}(z_2), \quad H^{(2)}(z_1, z_2) = E(z_1)F^{(2)}(z_2)$$

for some (uniquely defined)  $F_1, F_2 \in \mathbf{H}_2^{\infty \times q}(\mathbb{D})$ . Then

$$[H^{(1)}, H^{(2)}]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)} = \sum_{j, \ell=0}^{\infty} H_{j\ell}^{(2*)} H_{j\ell}^{(1)} = [F^{(1)}, F^{(2)}]_{\mathbf{H}_2^{\infty \times q}(\mathbb{D})}.$$

Let  $\mathbf{B}_+$  be an element of  $\mathcal{L}(\ell_2^p; \mathbb{C}^r)$  defined by

$$\mathbf{B}_+ = \frac{1}{2\pi i} \int_{|\xi|=1} (\xi I_r - A_1)^{-1} B_+ E(\xi) d\xi,$$

which in turn, can be written in the residue form as

$$(2.6) \quad \mathbf{B}_+ = \sum_{z_1 \in \mathbb{D}} \text{Res } (z_1 I_r - A_1)^{-1} B_+ E(z_1).$$

Using the structure (2.3) of  $E$  one can represent  $\mathbf{B}_+$  in the block matrix form as

$$(2.7) \quad \mathbf{B}_+ = (B_+ \quad A_1 B_+ \quad A_1^2 B_+ \quad \dots)$$

which leads in particular to the following useful relation:

$$(2.8) \quad E(z) \mathbf{B}_+^* = \sum_{j=0}^{\infty} z^j B_+^* A_1^{*j} = B_+^* (I_r - z A_1^*)^{-1}.$$

Substituting (2.4) into the left-hand side of the interpolation condition (1.9) and taking into account (2.7) we obtain the condition

$$\frac{1}{2\pi i} \int_{|\zeta|=1} (\zeta I_r - A_2)^{-1} \mathbf{B}_+ F(\zeta) d\zeta = B_-$$

which can be written in the residue form as

$$(2.9) \quad \sum_{z_2 \in \mathbb{D}} \text{Res } (z_2 I_r - A_2)^{-1} \mathbf{B}_+ F(z_2) = B_-$$

and is equivalent to (1.9). It is easily seen that in order for such a function  $F$  to exist, it is necessary that

$$(2.10) \quad \text{Ran } B_- \subset \text{Ran } \{ \text{Ran } A_2^\ell \mathbf{B}_+; \ell = 0, 1, \dots \}.$$

In view of the definition (2.7) of  $\mathbf{B}_+$  this condition is equivalent to (1.10).

Making use of Remark 2.1 we conclude that Problem 1.1 reduces to the following interpolation problem in one variable (but with infinite dimensional coefficient space):

**Problem 2.2.** *Given matrices  $A_2 \in \mathbb{C}^{r \times r}$  (spec  $A_2 \in \mathbb{D}$ ),  $B_- \in \mathbb{C}^{r \times q}$ ,  $\Upsilon \in \mathbb{C}^{q \times q}$  and the operator  $\mathbf{B}_+ \in \mathcal{L}(\ell_2^q; \mathbb{C}^r)$ , find all functions  $F \in \mathbf{H}_2^{\infty \times q}(\mathbb{D})$  satisfying the interpolation condition (2.9) and the matrix norm constraint*

$$(2.11) \quad [F, F]_{\mathbf{H}_2^{\infty \times q}(\mathbb{D})} \leq \Upsilon.$$

## 3. DESCRIPTION OF ALL SOLUTIONS

A general bitangential residue interpolation problem, of which Problem 2.2 is a special case, has been solved in [4]. We now recall results from that paper suitably adapted to the present setting. Since spectra of  $A_1$  and  $A_2$  are inside the unit disk, the double series in the right-hand side of (1.13) converges and defines a nonnegative matrix  $\mathbb{P} \in \mathbb{C}^{r \times r}$ . Keeping on mind the application of results from [4] and using (2.7) we represent  $\mathbb{P}$  in a less “symmetric” form

$$(3.1) \quad \mathbb{P} = \sum_{k=0}^{\infty} A_2^k \mathbf{B}_+ \mathbf{B}_+^* A_2^{*k}$$

and note that it satisfies the Stein equation

$$(3.2) \quad \mathbb{P} - A_2 \mathbb{P} A_2^* = \mathbf{B}_+ \mathbf{B}_+^*.$$

By  $\mathbb{P}^{[-1]}$  we denote the Moore–Penrose pseudoinverse matrix uniquely defined by the conditions

$$\mathbb{P}^{[-1]} \mathbf{P}_{\text{Ran } \mathbb{P}} = \mathbf{P}_{\text{Ran } \mathbb{P}} \mathbb{P}^{[-1]} = \mathbb{P}^{[-1]}, \quad \mathbb{P}^{[-1]} \mathbb{P} = \mathbb{P} \mathbb{P}^{[-1]} = \mathbf{P}_{\text{Ran } \mathbb{P}}$$

(see [12]) where  $\mathbf{P}_{\text{Ran } \mathbb{P}}$  stands for the orthogonal projection onto  $\text{Ran } \mathbb{P}$ . We now present two lemmas and a theorem which were proved in [4] for finite dimensional  $\mathbf{B}_+$ . The same arguments are still valid for the present situation.

**Lemma 3.1.** *Let  $\mu$  be an arbitrary point on the unit circle, let  $\{A_2, \mathbf{B}_+, B_-\}$  be any triple of bounded operators satisfying the Stein equation (3.2) and let  $\Theta$  be the  $\mathcal{L}(\ell_2)$ -valued function defined by*

$$(3.3) \quad \Theta(z) = I_{\ell_2} + (z - \mu) \mathbf{B}_+^* (I_r - z A_2^*)^{-1} \mathbb{P}^{[-1]} (\mu I_r - A_2)^{-1} \mathbf{B}_+.$$

Then  $\Theta$  is inner in  $\mathbb{D}$  and moreover,

$$I_{\ell_2} - \Theta(z) \Theta(\omega)^* = (1 - z \bar{\omega}) \mathbf{B}_+^* (I_r - z A_2^*)^{-1} \mathbb{P}^{[-1]} (I - \bar{\omega} A_2)^{-1} \mathbf{B}_+$$

for every choice of points  $z$  and  $\omega$  at which  $\Theta$  is analytic.

**Lemma 3.2.** *Let  $F_{\min}$  be the  $\mathcal{L}(\ell_2; \mathbb{C}^q)$ -valued function given by*

$$(3.4) \quad F_{\min}(z) = \mathbf{B}_+^* (I_r - z A_2^*)^{-1} \mathbb{P}^{[-1]} B_-.$$

Then  $F_{\min}$  belongs to the space  $\mathbf{H}_2^{\infty \times q}(\mathbb{D})$  defined in (2.1), satisfies the condition (1.9) and

$$(3.5) \quad [F_{\min}, F_{\min}]_{\mathbf{H}_2^{\infty \times q}(\mathbb{D})} = B_-^* \mathbb{P}^{[-1]} B_-.$$

**Theorem 3.3.** *All solutions  $F$  of Problem 2.2 are parametrized by the formula*

$$(3.6) \quad F(z) = F_{\min}(z) + \Theta(z) \widehat{F}(z)$$

where  $\Theta$  and  $F_{\min}$  are the functions given by (3.3) and (3.4) respectively and where  $\widehat{F}$  is a free parameter from  $\mathbf{H}_2^{\infty \times q}(\mathbb{D})$  satisfying the norm constraint

$$(3.7) \quad \left[ \widehat{F}, \widehat{F} \right]_{\mathbf{H}_2^{\infty \times q}(\mathbb{D})} \leq \Upsilon - B_-^* \mathbb{P}^{[-1]} B_-.$$

The representation (3.6) is orthogonal with respect to the form (2.2) and therefore,  $F_{\min}$  is the minimal norm solution of Problem 2.2.

Let  $\mathbf{T}_{A_1} : \mathbf{H}_2^{r \times m}(\mathbb{D}^2) \rightarrow \mathbf{H}_2^{r \times m}(\mathbb{D})$  be the operator defined by the rule

$$(3.8) \quad (\mathbf{T}_{A_1} H)(z_2) = \frac{1}{2\pi i} \int_{|\xi|=1} (\xi I_r - A_1)^{-1} H(\xi, z_2) d\xi \quad (H \in \mathbf{H}_2^{r \times m}(\mathbb{D}^2)),$$

or, in terms of the Taylor expansion (1.1) of  $H$ ,

$$(\mathbf{T}_{A_1} H)(z_2) = \sum_{j, \ell=0}^{\infty} A_1^j H_{j\ell} z_2^\ell.$$

Since  $\text{spec } A_1 \subset \mathbb{D}$ , the operator  $\mathbf{T}_{A_1}$  is bounded and moreover, it is a projection as an operator from  $\mathbf{H}_2^{r \times m}(\mathbb{D}^2)$  into itself. We also need such an operator for “ $m = \infty$ ”, i.e., when the  $H_{j\ell} \in \mathcal{L}(\ell_2; \mathbb{C}^r)$ . It is easily seen that also in this case the operator  $\mathbf{T}_{A_1}$  is bounded.

Using this remark and the representation (2.4) of  $H$  we get

$$(3.9) \quad (\mathbf{T}_{A_1} H)(z_2) = (\mathbf{T}_{A_1} E) F(z_2) \quad (H(z_1, z_2) = E(z_1)F(z_2)).$$

**Lemma 3.4.** *Let  $\mathbf{I}$ ,  $\mathbf{M}_{z_1}$  and  $\mathbf{M}_{z_2}$  denote the identity operator and the operators of multiplication by  $z_1$  and  $z_2$ , respectively, in the space  $\mathbf{H}_2^{r \times m}(\mathbb{D}^2)$  and let  $\Theta : \mathbf{H}_2^{r \times m}(\mathbb{D}^2) \rightarrow \mathbf{H}_2^{r \times m}(\mathbb{D}^2)$  be the operator given by*

$$(3.10) \quad \Theta = \mathbf{I} + (\mathbf{M}_{z_2} - \mu \mathbf{I}) \mathbf{B}_+^* (\mathbf{I} - \mathbf{M}_{z_1} A_1^*)^{-1} (\mathbf{I} - \mathbf{M}_{z_2} A_2^*)^{-1} \mathbb{P}^{[-1]} (\mu I_r - A_2)^{-1} \mathbf{T}_{A_1} \mathbf{B}_+.$$

Then  $\Theta$  is an isometric operator and

$$(3.11) \quad \Theta E(z_1) = E(z_1) \Theta(z_2),$$

where  $E(z_1)$  and  $\Theta(z_2)$  are the functions defined via (2.3) and (3.3), respectively.

*Proof.* By (2.6) and by definition (3.8) of  $\mathbf{T}_{A_1}$ , it follows that

$$(3.12) \quad \mathbf{T}_{A_1} \mathbf{B}_+ E(z_1) = \mathbf{B}_+.$$

Using (3.12) and (2.8) and taking into account (3.3) and (3.10) we obtain that

$$\begin{aligned} \Theta E(z_1) &= E(z_1) + (z_2 - \mu) \mathbf{B}_+^* (I_r - z_1 A_1^*)^{-1} (I_r - z_2 A_2^*)^{-1} \mathbb{P}^{[-1]} (\mu I_r - A_2)^{-1} \mathbf{B}_+ \\ &= E(z_1) + (z_2 - \mu) E(z_1) \mathbf{B}_+^* (I_r - z_2 A_2^*)^{-1} \mathbb{P}^{[-1]} (\mu I_r - A_2)^{-1} \mathbf{B}_+ \\ &= E(z_1) \Theta(z_2), \end{aligned}$$

which proves (3.11). Next, taking  $\widehat{H} \in \mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  in the form

$$(3.13) \quad \widehat{H}(z_1, z_2) = E(z_1) \widehat{F}(z_2)$$

for some (uniquely defined)  $\widehat{F} \in \mathbf{H}_2^{\infty \times q}(\mathbb{D})$  such that

$$(3.14) \quad [\widehat{F}, \widehat{F}]_{\mathbf{H}_2^{\infty \times q}(\mathbb{D})} = [\widehat{H}, \widehat{H}]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)},$$

and taking advantage of the relation (3.11) we get

$$(3.15) \quad (\Theta \widehat{H})(z_1, z_2) = E(z_1) \Theta(z_2) \widehat{F}(z_2).$$

By Remark 2.1 and since  $\Theta$  is inner in  $\mathbb{D}$ , it follows that

$$\begin{aligned} &[E(z_1) \Theta(z_2) \widehat{F}(z_2), E(z_1) \Theta(z_2) \widehat{F}(z_2)]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)} \\ &= [\Theta \widehat{F}, \Theta \widehat{F}]_{\mathbf{H}_2^{\infty \times q}(\mathbb{D})} = [\widehat{F}, \widehat{F}]_{\mathbf{H}_2^{\infty \times q}(\mathbb{D})} \end{aligned}$$

which being compared with (3.14) leads to

$$(3.16) \quad [\Theta \widehat{H}, \Theta \widehat{H}]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)} = [\widehat{H}, \widehat{H}]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)}.$$

The latter equality holds for every  $\widehat{H} \in \mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  and therefore, the operator  $\Theta$  is isometric.  $\square$

Now we turn to the proof of Theorem 1.3.

*Proof of Theorem 1.3.* It was shown in Section 2 that  $H$  of the form (2.4) is a solution of Problem 1.1 if and only if the corresponding function  $F \in \mathbf{H}_2^{\infty \times q}(\mathbb{D})$  solves Problem 2.2. Substituting the formula (3.6) parametrizing all solutions  $F$  of Problem 2.2 into (2.4) we conclude that all solutions  $H$  of Problem 1.1 are parametrized by the formula

$$(3.17) \quad H(z_1, z_2) = E(z_1)F_{\min}(z_2) + E(z_1)\Theta(z_2)\widehat{F}(z_2),$$

where  $\widehat{F}$  is a parameter from  $\mathbf{H}_2^{\infty \times q}(\mathbb{D})$  satisfying the norm constraint (3.7). It follows immediately from (2.8), (3.4) and (1.15) that

$$(3.18) \quad E(z_1)F_{\min}(z_2) = H_{\min}(z_1, z_2).$$

Substituting (3.18) and (3.15) into (3.17) we get the parametrization (1.14) for the parameter  $\widehat{H}$  represented in the form (3.13). It remains to note that  $\widehat{H}(z_1, z_2)$  of the form (3.13) varies on  $\mathbf{H}_2^{p \times q}(\mathbb{D})$  when  $\widehat{F}(z_2)$  runs through  $\mathbf{H}_2^{\infty \times q}(\mathbb{D})$  and it satisfies the norm constraint (1.16) if and only if  $F$  is subject to (3.7). Applying Remark 2.1 and taking into account that the representation (3.6) is orthogonal with respect to the form (2.2), we get

$$(3.19) \quad [H_{\min}, \Theta \widehat{H}]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)} = [F_{\min}, \Theta \widehat{F}]_{\mathbf{H}_2^{\infty \times q}(\mathbb{D})} = 0$$

and thus, the representation (1.14) is orthogonal with respect to the form (1.4). Using (3.5) and (3.18) we get by Remark 2.1

$$(3.20) \quad [H_{\min}, H_{\min}]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)} = [F_{\min}, F_{\min}]_{\mathbf{H}_2^{\infty \times q}(\mathbb{D})} = B_-^* \mathbb{P}^{[-1]} B_-.$$

Finally, in view of (3.16), (3.19) and (3.20)

$$[H, H]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)} = B_-^* \mathbb{P}^{[-1]} B_- + [\widehat{H}, \widehat{H}]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)}.$$

We obtain from this equality the fact that  $\Upsilon - B_-^* \mathbb{P}^{[-1]} B_- \geq 0$  is a necessary and sufficient condition for the problem to have a solution. Furthermore, we also deduce from it that  $H_{\min}$  is indeed the minimal norm solution of Problem 1.1.  $\square$

As an illustration, let us consider the homogeneous Nevanlinna–Pick example, supposing moreover that the functions are scalar valued. Then,  $b_i = 1$  for  $i = 1, \dots, n$ . We have

$$T_{A_1}(B_+ \widehat{H}) = \begin{pmatrix} \widehat{H}(z_{11}, z_2) \\ \vdots \\ \widehat{H}(z_{1n}, z_2) \end{pmatrix}.$$

When  $n = 1$  and setting  $\mu = 1$ , we have

(3.21)

$$(\Theta\widehat{H})(z_1, z_2) = \widehat{H}(z_1, z_2) + \frac{(z_2 - 1)(1 - |z_{11}|^2)(1 - |z_{21}|^2)}{(1 - z_1\bar{z}_{11})(1 - z_2\bar{z}_{21})(1 - z_{21})} \widehat{H}(z_{11}, z_2),$$

which gives an explicit formula for all functions in  $\mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  vanishing at the point  $(z_{11}, z_{12})$  when  $\widehat{H}$  varies in  $\mathbf{H}_2^{p \times q}(\mathbb{D}^2)$ . Because of the isometry property of the operator  $\Theta$ , we have (3.16).

In the case where  $n > 1$ , the presence of  $\mathbb{P}^{[-1]}$  prevents to have such an explicit expression.

One could try to extend the methods described here to the case of more than two variables.  $\mathbb{P}$  is then an operator, and new phenomena seem to appear when 0 is in the continuous spectrum of  $\mathbb{P}$ . One could also look for a more general bitangential interpolation problem, of the kind studied in [5] in the one-variable case. Here too, the methods developed in the present paper do not seem applicable. On the other hand, the present approach allows to solve the following problem, which amounts to preassign the inner integral in (1.6). Note that in this example, one obtains a Beurling-Lax type representation of the set of solutions.

**Problem 3.5.** *Given matrices  $A \in \mathbb{C}^{r \times r}$  (spec  $A \subset \mathbb{D}$ ),  $B \in \mathbb{C}^{r \times p}$ ,  $\Upsilon \in \mathbb{C}^{q \times q}$  and given a function*

$$(3.22) \quad C(z_1) = \sum_{j=0}^{\infty} z_1^j C_j \in \mathbf{H}_2^{r \times q}(\mathbb{D}),$$

*find all functions  $H \in \mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  satisfying the condition*

$$(3.23) \quad \frac{1}{2\pi i} \int_{|\zeta|=1} (\zeta I_r - A)^{-1} B H(z_1, \zeta) d\zeta = C(z_1) \quad (\forall z_1 \in \mathbb{D})$$

*and the matrix norm constraint (1.7).*

Note that for the special choice

$$A = \begin{pmatrix} z_{21} I_{r_1} & & \\ & \ddots & \\ & & z_{2n} I_{r_n} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad C(z_1) = \begin{pmatrix} c_1(z_1) \\ \vdots \\ c_n(z_1) \end{pmatrix},$$

where  $z_{2j}$  are preassigned points from  $\mathbb{D}$ , the condition (3.23) reduces to Nevanlinna-Pick-like conditions

$$b_j H(z_1, z_{2j}) = c_j(z_1) \quad (j = 1, \dots, n; \forall z_1 \in \mathbb{D}).$$

**Theorem 3.6.** *Let  $\mathbb{P}^{[-1]}$  be the Moore-Penrose pseudoinverse of the matrix*

$$\mathbb{P} = \sum_{k=0}^{\infty} A^k B_+ B_+^* A^{*k}.$$

*Problem 3.5 has a solution if and only if the matrix  $\Upsilon - \sum_{j=0}^{\infty} C_j^* \mathbb{P}^{[-1]} C_j$  is nonnegative*

*(where  $C_j$  are the Taylor coefficients from the expansion (3.22)). When this is the case, all solutions  $H$  of Problem 3.5 are parametrized by the formula*

$$(3.24) \quad H(z_1, z_2) = H_{\min}(z_1, z_2) + \Theta(z_2) \widehat{H}(z_1, z_2),$$

where  $H_{\min}$  is the minimal norm solution given by

$$(3.25) \quad H_{\min}(z_1, z_2) = B_+^* (I_r - z_2 A^*)^{-1} \mathbb{P}^{[-1]} C(z_1),$$

where  $\Theta$  is the inner function specified by the formula

$$(3.26) \quad \Theta(z) = I_p + (z - \mu) B_+^* (I_r - z A^*)^{-1} \mathbb{P}^{[-1]} (\mu I_r - A_2)^{-1} B_+ \quad (|\mu| = 1)$$

and where  $\widehat{H}$  is a free parameter from  $\mathbf{H}_2^{p \times q}(\mathbb{D}^2)$  satisfying the norm constraint

$$(3.27) \quad \left[ \widehat{H}, \widehat{H} \right]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)} \leq \Upsilon - \sum_{j=0}^{\infty} C_j^* \mathbb{P}^{[-1]} C_j.$$

*Proof.* Substituting the expansion (2.4) of  $H$  inside the integral and the Taylor expansion (3.22) of  $C$  in the right-hand side of (3.23) and comparing coefficients at  $z_1^j$  in the obtained equality we get

$$(3.28) \quad \frac{1}{2\pi i} \int_{|\zeta|=1} (\zeta I_r - A)^{-1} B F_j(\zeta) d\zeta = C_j \quad (j = 0, 1, \dots).$$

The system of the latter conditions is equivalent to (3.23). Each condition in (3.28) is in fact, the left sided residue interpolation condition for the function  $F_j \in \mathbf{H}_2^{p \times q}(\mathbb{D})$ . By Theorem 3.3, a function  $F_j$  satisfies (3.28) if and only if it is of the form

$$(3.29) \quad F_j(z) = F_{j,\min}(z) + \Theta(z) \widehat{F}_j(z),$$

where  $\Theta$  is the function defined in (3.26), where  $F_{j,\min}$  is given by

$$(3.30) \quad F_{j,\min}(z) = B_+^* (I_r - z A^*)^{-1} \mathbb{P}^{[-1]} C_j$$

and where  $\widehat{F}_j$  is an arbitrary function from  $\mathbf{H}_2^{p \times q}(\mathbb{D})$ . Note that the function  $\Theta$  in (3.29) does not depend on  $j$ . Substituting (3.29) into (2.4) and using (3.22), (3.25) we get

$$\begin{aligned} H(z_1, z_2) &= \sum_{j=0}^{\infty} z_1^j \left( F_{j,\min}(z_2) + \Theta(z_2) \widehat{F}_j(z_2) \right) \\ &= B_+^* (I_r - z_2 A^*)^{-1} \mathbb{P}^{[-1]} \sum_{j=0}^{\infty} z_1^j C_j + \Theta(z_2) \sum_{j=0}^{\infty} z_1^j \widehat{F}_j(z_2) \\ &= H_{\min}(z_1, z_2) + \Theta(z_2) \sum_{j=0}^{\infty} z_1^j \widehat{F}_j(z_2), \end{aligned}$$

which upon setting  $\widehat{H}(z_1, z_2) = \sum_{j=0}^{\infty} z_1^j \widehat{F}_j(z_2)$ , coincides with (3.24). The rest follows from the fact that the representation (3.24) is orthogonal with respect to the form (2.2) and that

$$[H_{\min}, H_{\min}]_{\mathbf{H}_2^{p \times q}(\mathbb{D}^2)} = \sum_{j=0}^{\infty} [F_{j,\min}, F_{j,\min}]_{\mathbf{H}_2^{p \times q}(\mathbb{D})} = \sum_{j=0}^{\infty} C_j^* \mathbb{P}^{[-1]} C_j. \quad \square$$

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