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COMPOSITION OPERATORS: HYPERINVARIANT SUBSPACES, QUASI-NORMALS AND ISOMETRIES

BRUCE A. CLOAD

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Abstract. We exhibit hyperinvariant subspaces of some composition operators. We also consider quasi-normal composition operators and discuss the commutant of isometric composition operators.

1. Introduction

Let D be the unit disc in the complex plane. The Hardy space on D, $H^2(D)$, is defined to be the set of analytic functions on D which have square summable power series coefficients. Given an analytic self map of the disc, ϕ, we may define a composition operator C_ϕ on H^2 by $C_\phi(f) = f \circ \phi$, for all f in H^2. These bounded operators have been studied extensively (see [18] or [10]).

We say an operator B commutes with an operator A if $AB = BA$. Recently, we have been interested in which operators $B \in B(H^2)$ commute with a given composition operator, C_ϕ (see [6] and [7]). In the second section we address the question of what are the hyperinvariant subspaces for a composition operator, C_ϕ, induced by a particular type of function, ϕ. The question was of interest in that a solution may provide a tool for classifying the commutant of C_ϕ; that is, the algebra of all operators which commute with C_ϕ. In particular, in the second section, we show (Corollary 2) that if C_ϕ is Riesz, it has a triangularizing chain of hyperinvariant subspaces. In the third section, we develop some tools for classifying quasi-normal composition operators, while in the fourth section, we discuss the commutant of composition operators which are also isometries and pose some research questions.

2. Hyperinvariant subspaces

Let ϕ be an analytic self map of the disc and define

$$\phi^{[n]} = \underbrace{\phi \circ \phi \circ \ldots \circ \phi}_{n\ times},$$

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the n^{th} iterate of ϕ under composition. Also, suppose $\phi(0) = 0$ and $0 < |\phi'(0)| < 1$. In 1884, Koenigs showed that the sequence $\{\sigma_k\}$ with

$$\sigma_k(z) = \frac{\phi^{[k]}(z)}{(\phi'(0))^k}$$

converges uniformly on compact subsets of D to a non-constant function σ, which is known as the Koenigs' function for ϕ (see [18] or [10]). Paul Bourbon proved the following theorem when ϕ was univalent, and shortly afterwards Pietro Poggi-Corradini was able to remove that hypothesis (see [1] or [14]).

Theorem 1. *Let ϕ be an analytic self map of the disc with $\phi(0) = 0$ and $0 < |\phi'(0)| < 1$. Let σ be the Koenigs' function of ϕ and q a natural number. If $(\sigma)^q$ is in H^2, then the sequence $\{(\sigma_k)^q\}$ converges to $(\sigma)^q$ in the H^2 norm.*

In [2], Bourdon and Shapiro proved a sufficient condition for the Koenigs' function to belong to H^p and showed the condition to be necessary in the case that the function ϕ is analytic on the closed unit disc. In [15], Pietro Poggi-Corradini was able to prove the necessity of the condition without any additional conditions on ϕ. These results together lead to the following theorem.

Theorem 2. *Let ϕ be an analytic self map of the disc with $\phi(0) = 0$ and $0 < |\phi'(0)| < 1$. Let σ be the Koenigs function. Then $(\sigma)^q$ is in H^2 if and only if $|\phi'(0)|^q$ exceeds the essential spectral radius of C_ϕ.*

A subspace M is hyperinvariant for an operator A if it is invariant for every operator which commutes with A.

Theorem 3. *Let ϕ be an analytic self map of the disc with $\phi(0) = 0$ and $0 < |\phi'(0)| < 1$. Suppose $(\sigma)^q$ is in H^2 where σ is the Koenigs' function of ϕ. Then for each natural number k, $1 \le k \le q$, the subspaces $z^k H^2$ are hyperinvariant for C_ϕ.*

Proof. In [7], we showed that zH^2 is a hyperinvariant subspace for such a C_ϕ. This covers the case $k = 1$. Proceeding by induction on $n < q$, we will assume, for $k \le n$, that $z^k H^2$ is hyperinvariant for C_ϕ. Let A be an operator which commutes with C_ϕ and let $z^{n+1}p$ be a function in $z^{n+1}H^2$ where p is a polynomial. We wish to show

$$\langle A(z^{n+1}p), z^l \rangle = 0$$

for $l \le n$. Since $z^{n+1}p$ is in $z^n H^2$, we have by the induction hypothesis that

$$\langle A(z^{n+1}p), z^l \rangle = 0$$

for $l < n$. Now

$$\langle AC_\phi^m(z^{n+1}p), z^n \rangle = \langle A(z^{n+1}p), (C_\phi^*)^m(z^n) \rangle$$

$$= \langle A(z^{n+1}p), (\overline{\phi'(0)})^{mn} z^n \rangle.$$

The last equality follows from the induction hypothesis and the fact that C_ϕ^* is upper triangular as a matrix when represented in the standard basis. Dividing both sides of the equation by $(\phi'(0))^{mn}$, it follows that

$$\langle A((\sigma_m)^n \phi^{[m]}(p \circ \phi^{[m]})), z^n \rangle = \langle A(z^{n+1}p), z^n \rangle.$$

The left-hand side of this equation is equal to

$$\langle (\sigma_m)^n \phi^{[m]}(p \circ \phi^{[m]}), A^*(z^n) \rangle.$$

Since $H^{2q} \subset H^{2n}$ when $q \geq n$, $(\sigma)^n$ is in H^2. By Theorem 1, $(\sigma_m)^n$ is a bounded sequence in the H^2 norm. Since p is in H^∞,

$$\{(\sigma_m)^n \phi^{[m]}(p \circ \phi^{[m]})\}$$

is also a bounded sequence in the H^2 norm. This sequence converges pointwise to 0 as m goes to ∞, and thus it converges weakly to 0. It follows that $\langle A(z^{n+1}p), z^n \rangle = 0$.

If $z^{n+1}f$ is in $z^{n+1}H^2$, we may find a sequence of polynomials $\{p_l\}$ which converges strongly to f as l goes to ∞. In particular, $\langle A(z^{n+1}p_l), z^n \rangle$ will converge to $\langle A(z^{n+1}f), z^n \rangle$ as l goes to infinity. Hence the result holds. $\square$

Let $\|A\|_e$ denote the essential norm of an operator A. An operator A is a Riesz operator if $\|A^n\|_e^{\frac{1}{n}}$ tends to 0 as n tends to ∞. This implies that A is Riesz if and only if the essential spectrum of A is $\{0\}$. In [3], Bourdon and Shapiro show that a Riesz composition operator must be induced by a function which has an interior fixed point in D. For basic details of Riesz operators see [11].

Corollary 1. *Let ϕ be an analytic self map of D with $\phi(b) = b$ where b is in D and $0 < |\phi'(b)| < 1$. If C_ϕ is a Riesz operator, then C_ϕ has a triangularizing chain of hyperinvariant subspaces.*

Proof. Let $\alpha(z) = \frac{b-z}{1-\bar{b}z}$. Define $\psi = \alpha \circ \phi \circ \alpha$. Note $\psi(0) = 0$ and $0 < |\psi'(0)| < 1$. Since $C_\psi^n = C_\alpha C_\phi^n C_\alpha$, it follows that C_ψ is a Riesz operator. Now by Theorem 2 and Theorem 3, C_ψ has the $z^q H^2$ as hyperinvariant subspaces for all $q = 0, 1, 2....$ Thus C_ϕ has $(\alpha)^q H^2$, $q = 0, 1, 2...$ as a chain of hyperinvariant subspaces. Moreover, any operator which holds these subspaces invariant must be lower triangular with respect to the orthonormal basis $\{(\alpha)^n \frac{1-|b|^2}{1-\bar{b}z} : n = 0, 1, 2...\}$. $\square$

Let $f^{(k)}$ be the k^{th} derivative of f.

Corollary 2. *Let ϕ be an analytic self map of the disc with $\phi(0) = 0$ and $0 < |\phi'(0)| < 1$. Let k be a natural number greater than or equal to 1. Suppose $(\sigma)^{k+1}$ is in H^2 and let A commute with C_ϕ. Then, for all f in H^2, $\frac{d^k}{dz^k}(A(f))(0) = k! \sum_{n=1}^{k} \frac{f^{(n)}(0)}{n!} \langle A(z^n), z^k \rangle$.*

Proof. Let $f = \sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of f. In [7], we showed that if A commutes with C_ϕ, then $(A(f))(0) = A(1)f(0)$ where $A(1)$ is a constant. This implies that $\langle A(a_0), z^k \rangle = 0$. Now

$$\langle A(f), z^k \rangle = \langle A(a_0), z^k \rangle + \sum_{n=1}^{k} a_n \langle A(z^n), z^k \rangle + \langle A(\sum_{n=k+1}^{\infty} a_n z^n), z^k \rangle.$$

The last term is 0 since by Theorem 3, $z^{k+1}H^2$ is hyperinvariant for C_ϕ. Since $a_n = \frac{f^{(n)}(0)}{n!}$, the result follows. $\square$

The next example shows that there are analytic self maps of D with $\phi(0) = 0$ and $\phi'(0) = 0$ such that the $z^q H^2$ are not hyperinvariant for C_ϕ for $q \geq 2$.

Example 1. Let n be a natural number greater than 1. For C_{z^n}, the subspaces $z^q H^2$ for $q \geq 2$ are not hyperinvariant.

Proof. In [6], we discuss the commutant of C_{z^n} in more detail. Let q be a natural number greater than or equal to 2. If we define an operator A on H^2 by $A(z^{q\cdot n^l}) = z^{n^l}$ and $A(z^k) = 0$ when $k \neq q \cdot n^l$, then A commutes with C_ϕ. Now $A(z^q) = z$ and hence $z^q H^2$ is not a hyperinvariant subspace for C_ϕ. □

3. Quasi-normals

We now turn to quasi-normal operators. An operator A is quasi-normal if A commutes with A^*A. For more information, see [8].

Lemma 1. *If C_ϕ is quasi-normal, then $\phi(0) = 0$.*

Proof. $C_\phi C_\phi^* C_\phi(1) = C_\phi^* C_\phi C_\phi(1)$ implies that

$$\frac{1}{1 - \overline{\phi(0)}\phi(z)} = \frac{1}{1 - \overline{\phi(0)}z}.$$

It follows that $\phi(0) = 0$. □

Lemma 2. *Let ϕ belong to $L^2(\partial D)$ with ϕ non-zero and not a characteristic function of a proper subset of the unit circle. Also suppose that $\|\phi\|_{L^2} = \mu$ and $\|(\phi)^{k_l}\|_{L^2} = \mu$ where k_l is a sequence of natural numbers which diverges to ∞ and μ is a non-zero real number. Then $|\phi(e^{i\theta})| = 1$ almost everywhere on ∂D.*

Proof. Let $\lambda = \mu^2$. Let $m(A)$ be the normalized standard Lebesgue measure of a set $A \subset \partial D$. Suppose that $|\phi(e^{i\theta})| > 1$ on a set of positive measure of ∂D. Then, in particular, there exists $\epsilon > 0$ such that the set $A_\epsilon = \{e^{i\theta} : |\phi(e^{i\theta})| > 1 + \epsilon\}$ has positive measure. It follows that

$$\lambda = \|\phi^{k_l}\|_{L^2}^2 \geq \frac{1}{2\pi}\int_{A_\epsilon} |\phi(z)|^{2k_l} dz \geq (1+\epsilon)^{2k_l} m(A_\epsilon).$$

As k_l tends to infinity, the right-hand side also tends to infinity. This is a contradiction; thus $|\phi(e^{i\theta})| \leq 1$ almost everywhere.

Similarly, suppose that $|\phi(e^{i\theta})| < 1$ on a set of positive measure of ∂D. Then, in particular, there exists $\epsilon > 0$ such that the set $B_\epsilon = \{e^{i\theta} : |\phi(e^{i\theta})| < 1 - \epsilon\}$ has positive measure. Then

$$\lambda = \frac{1}{2\pi}\int_{B_\epsilon} |\phi(z)|^2 dz + \frac{1}{2\pi}\int_{\partial D \setminus B_\epsilon} |\phi(z)|^2 dz.$$

Let $\int_{\partial D \setminus B_\epsilon} |\phi(z)|^2 dz = \kappa$. We note κ is strictly less than λ. Now

$$\frac{1}{2\pi}\int_{B_\epsilon} |\phi(z)|^{2k_l} dz \leq (1-\epsilon)^{2k_l} m(B_\epsilon).$$

Choose a natural number N such that $k_l \geq N$ implies that $(1-\epsilon)^{2k_l} m(B_\epsilon)$ is strictly less than $\lambda - \kappa$. Since $|\phi(z)| \leq 1$ almost everywhere, we note that $|\phi(z)|^{k_l} \leq |\phi(z)|$. Then with $k_l \geq N$,

$$\lambda = \|(\phi)^{k_l}\|_{L^2}^2 = \frac{1}{2\pi}\int_{B_\epsilon} |\phi(z)|^{2k_l} dz + \frac{1}{2\pi}\int_{\partial D \setminus B_\epsilon} |\phi(z)|^{2k_l} dz$$

$$\leq \frac{1}{2\pi}\int_{B_\epsilon} |\phi(z)|^{2k_l} dz + \frac{1}{2\pi}\int_{\partial D \setminus B_\epsilon} |\phi(z)|^2 dz$$

$$< \lambda - \kappa + \frac{1}{2\pi}\int_{\partial D \setminus B_\epsilon} |\phi(z)|^2 dz = \lambda$$

which is a contradiction; thus $|\phi(z)| = 1$ almost everywhere. □

Theorem 4. *Suppose C_ϕ is quasi-normal. If $C_\phi^* C_\phi$ is a diagonal matrix in the standard basis, then either $\phi(z) = cz$ for some constant c of modulus less than 1 or ϕ is an inner function.*

Proof. Let the power series of ϕ be $\sum_{n=1}^\infty a_n z^n$ and let the k^{th} diagonal entry of $C_\phi^* C_\phi$ be λ_k. First of all, if ϕ is identically zero or if only one of the a_n is non-zero, then we are done.

Next suppose that only a finite number of the coefficients a_n are non-zero but more than one of them is non-zero. Thus let $\phi(z) = z^k a_k + \cdots + a_l z^l$ where a_l and a_k are non-zero and k is the smallest power of z with a non-zero coefficient and l is the greatest. Now,

$$\langle (\phi(z))^l, (\phi(z))^k \rangle = (a_k)^l (a_l)^k,$$

which by hypothesis must be zero. Thus if only a finite number of coefficients are non-zero, we have $\phi(z) = cz^k$ where c is a constant of modulus less than or equal to 1. If $k \geq 2$ and c is less than 1 in modulus, then C_ϕ is not quasi-normal, so in this case $k = 1$.

Suppose an infinite number of the coefficients a_n are non-zero. Then $\phi(z) = \sum_{n=k}^\infty a_n z^n$ where a_k is the first non-zero coefficient. Now $C_\phi^* C_\phi C_\phi(z) = C_\phi C_\phi^* C_\phi(z)$ implies that

$$\sum_{n=k}^\infty \lambda_n a_n z^n = \sum_{n=k}^\infty \lambda_1 a_n z^n.$$

It follows that $\lambda_n = \lambda_1$ for infinitely many n. Since $\lambda_1 = \langle \phi(z), \phi(z) \rangle$ and $\lambda_n = \langle (\phi(z))^n, (\phi(z))^n \rangle$, we may apply Lemma 2 to conclude that ϕ must be an inner function. □

Corollary 3. *If C_ϕ has $z^q H^2$, $1 \leq q < \infty$, as hyperinvariant subspaces, and C_ϕ is quasi-normal, then either $\phi(z) = cz$ for some c of modulus less than 1 or ϕ is inner.*

Proof. If C_ϕ is quasi-normal, then C_ϕ commutes with $C_\phi^* C_\phi$, as does C_ϕ^*. Thus $C_\phi^* C_\phi$ is both upper and lower triangular with respect to the standard basis and thus diagonal. Apply Theorem 4. □

We could apply Corollary 1 to conclude that if C_ϕ is quasi-normal and Riesz, then $\phi(z) = cz$ for $\|c\| < 1$. This would say that if C_ϕ is Riesz and quasi-normal, then it is normal. As the next theorem shows, this is true in greater generality.

Theorem 5. *If A is an operator on a Hilbert space H, and A is Riesz and quasi-normal, then A is normal.*

Proof. If A is identically 0, then we are done. Assume A is not equal to 0. By [4], $ker(A)$ is reducing, so we may decompose A as $A_1 \bigoplus 0$ with A_1 quasi-normal and injective. Moreover, we may decompose A_1 as BV with B self-adjoint and injective, V isometric, and B commuting with V. Since $A_1^n = V^n B^n$, we have $\|(V^*)^n A_1^n\| = \|B^n\|$. Thus

$$\|B^n\|_e^{\frac{1}{n}} \leq \|(V^*)^n\|_e^{\frac{1}{n}} \|A_1^n\|_e^{\frac{1}{n}} \leq \|A_1^n\|_e^{\frac{1}{n}}.$$

The last term goes to 0 as n goes to ∞, and thus B is also Riesz. By Theorem 3.7 in [19], since B is self-adjoint and Riesz, it must be compact. Hence by the spectral theorem, B is unitarily equivalent to $\sum \bigoplus \lambda_k I_k$ where λ_k are the non-zero eigenvalues and the I_k are identity operators on finite-dimensional spaces. Since eigenspaces are hyperinvariant, and V commutes with B, V is unitarily equivalent to $\sum \bigoplus V_k$, where the V_k are isometries on finite-dimensional spaces and hence unitary. Thus V is a unitary operator and A_1 is a product of commuting normals and thus normal. $\square$

4. Isometries

An operator A on H^2 is an isometry if $A^*A = I$. In [17], Schwarz proves the following theorem.

Theorem 6. *C_ϕ is an isometry on H^2 if and only if $\phi(0) = 0$ and ϕ is an inner function.*

If ϕ is an elliptic disc automorphism, then the commutant of C_ϕ is well understood (see [6] or [7]). If ϕ is not an elliptic disc automorphism and C_ϕ is an isometry, Nordgren ([13]) shows that C_ϕ, restricted to the constants, and C_ϕ, restricted to zH^2, are the unitary and purely isometric parts, respectively. If A commutes with C_ϕ, then the constants are a reducing subspace for A (see [7]). Hence we may consider the commutant of C_ϕ as the direct sum of the commutant of the unitary part and the commutant of the purely isometric part. The purely isometric part is similar to a unilateral shift of infinite multiplicity on the wandering subspace, M, of C_ϕ (see [16]). The commutant of such a unilateral shift is given in terms of multiplication operators on $H^2(M)$ (see [16]).

Lemma 3. *Let $\phi = zf$ where f is a nonconstant inner function. Let $\{g_n : n = 0, 1, ...\}$ be an orthogonal basis for $(fH^2)^\perp$. Then the following set, S,*

$$\{z(\phi)^k g_n : n = 0, 1, 2..., k = 0, 1, 2...\},$$

is an orthogonal basis for the wandering subspace of C_ϕ on H^2.

Proof. It is easy to see that the vectors in S are orthogonal. The wandering subspace is given by the perp of $C_\phi(zH^2)$ in zH^2 (see [12]). We want to show that the vectors in S span this space. We note that the set of vectors T, given by $\{(\phi)^k : k = 1, 2, ...\}$ span $C_\phi(zH^2)$. Thus it is sufficient to show that the vectors in T and S, along with the set containing the constant function 1, span all of H^2 since these three sets are mutually orthogonal.

Suppose that h is orthongal to the vectors in these three sets. We claim that h is 0. First of all $\langle h, 1\rangle = 0$ implies that $h = zk_1$ for some k_1 in H^2. Now

$$\langle zk_1, zg_n\rangle = \langle k_1, g_n\rangle = 0$$

for all n implies that k_1 belongs to fH^2. Thus $h = zfh_1$ for some h_1 in H^2. Thus $h = \phi h_1$. Now if we proceed by induction and assume that $h = (\phi)^l h_l$ for some h_l in H^2, we have that $\langle(\phi)^l h_l, (\phi)^l\rangle = 0$ implies that $h_l = zk_{l+1}$ for some k_{l+1} in H^2. Thus

$$\langle(\phi)^l zk_{l+1}, (\phi)^l zg_n\rangle = \langle k_{l+1}, g_n\rangle = 0$$

for all n implies that $k_{l+1} = fh_{l+1}$ for some h_{l+1} in H^2. Thus $h = (\phi)^{l+1}h_{l+1}$ and we have by induction that $(\phi)^l$ divides h for arbitrary powers of l and hence $h = 0$. $\square$

In particular, let $\phi = zB$ where B is a Blaschke product with zero set $\{a_n : n = 0, 1, 2...\}$. Let k_λ be the reproducing kernel function for λ in D and let B_l be the Blaschke product with zero set $\{a_n : n = 0, 1, ..., l\}$. Then the following is an orthonormal basis for $(BH^2)^\perp$: $\{g_0 = k_{a_0},\ g_1 = B_0 k_{a_1},\ ...,\ g_n = B_{n-1} k_{a_n}\}$. This gives us a basis for the wandering subspace of C_ϕ and the commutant of C_ϕ can be explicitly interpreted in terms of this basis.

Questions

(1): If ϕ is an analytic self map of D, $\phi(0) = 0$ and $0 < |\phi'(0)| < 1$, are the subspaces $\{z^q H^2\}$ hyperinvariant subspaces for all natural numbers q? If this is true, then the only C_ϕ which are quasi-normal are either isometries or are given by $\phi(z) = cz$ for some constant c of absolute value less than 1.

(2): If ϕ is an analytic self map of D, $\phi(0) = 0$ and $\phi'(0) = 0$, what can be said about the hyperinvariant subspaces of C_ϕ? If C_ϕ is quasi-normal, what form does ϕ have to take?

(3): Given a basis for the wandering subspace of an isometric composition operator, can the hyperinvariant subspaces or the composition operators which commute be explicitly determined?

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Department of Mathematics, Brock University, St. Catharines, Ontario, Canada L2S 3A1

E-mail address: `bcload@spartan.ac.brocku.ca`