

## COMPOSITION OPERATORS: HYPERINVARIANT SUBSPACES, QUASI-NORMALS AND ISOMETRIES

BRUCE A. CLOAD

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ABSTRACT. We exhibit hyperinvariant subspaces of some composition operators. We also consider quasi-normal composition operators and discuss the commutant of isometric composition operators.

### 1. INTRODUCTION

Let  $D$  be the unit disc in the complex plane. The Hardy space on  $D$ ,  $H^2(D)$ , is defined to be the set of analytic functions on  $D$  which have square summable power series coefficients. Given an analytic self map of the disc,  $\phi$ , we may define a composition operator  $C_\phi$  on  $H^2$  by  $C_\phi(f) = f \circ \phi$ , for all  $f$  in  $H^2$ . These bounded operators have been studied extensively (see [18] or [10]).

We say an operator  $B$  commutes with an operator  $A$  if  $AB = BA$ . Recently, we have been interested in which operators  $B \in B(H^2)$  commute with a given composition operator,  $C_\phi$  (see [6] and [7]). In the second section we address the question of what are the hyperinvariant subspaces for a composition operator,  $C_\phi$ , induced by a particular type of function,  $\phi$ . The question was of interest in that a solution may provide a tool for classifying the commutant of  $C_\phi$ ; that is, the algebra of all operators which commute with  $C_\phi$ . In particular, in the second section, we show (Corollary 2) that if  $C_\phi$  is Riesz, it has a triangularizing chain of hyperinvariant subspaces. In the third section, we develop some tools for classifying quasi-normal composition operators, while in the fourth section, we discuss the commutant of composition operators which are also isometries and pose some research questions.

### 2. HYPERINVARIANT SUBSPACES

Let  $\phi$  be an analytic self map of the disc and define

$$\phi^{[n]} = \underbrace{\phi \circ \phi \circ \dots \circ \phi}_{n \text{ times}}$$

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the  $n^{th}$  iterate of  $\phi$  under composition. Also, suppose  $\phi(0) = 0$  and  $0 < |\phi'(0)| < 1$ . In 1884, Koenigs showed that the sequence  $\{\sigma_k\}$  with

$$\sigma_k(z) = \frac{\phi^{[k]}(z)}{(\phi'(0))^k}$$

converges uniformly on compact subsets of  $D$  to a non-constant function  $\sigma$ , which is known as the Koenigs' function for  $\phi$  (see [18] or [10]). Paul Bourbon proved the following theorem when  $\phi$  was univalent, and shortly afterwards Pietro Poggi-Corradini was able to remove that hypothesis (see [1] or [14]).

**Theorem 1.** *Let  $\phi$  be an analytic self map of the disc with  $\phi(0) = 0$  and  $0 < |\phi'(0)| < 1$ . Let  $\sigma$  be the Koenigs' function of  $\phi$  and  $q$  a natural number. If  $(\sigma)^q$  is in  $H^2$ , then the sequence  $\{(\sigma_k)^q\}$  converges to  $(\sigma)^q$  in the  $H^2$  norm.*

In [2], Bourdon and Shapiro proved a sufficient condition for the Koenigs' function to belong to  $H^p$  and showed the condition to be necessary in the case that the function  $\phi$  is analytic on the closed unit disc. In [15], Pietro Poggi-Corradini was able to prove the necessity of the condition without any additional conditions on  $\phi$ . These results together lead to the following theorem.

**Theorem 2.** *Let  $\phi$  be an analytic self map of the disc with  $\phi(0) = 0$  and  $0 < |\phi'(0)| < 1$ . Let  $\sigma$  be the Koenigs function. Then  $(\sigma)^q$  is in  $H^2$  if and only if  $|\phi'(0)|^q$  exceeds the essential spectral radius of  $C_\phi$ .*

A subspace  $M$  is hyperinvariant for an operator  $A$  if it is invariant for every operator which commutes with  $A$ .

**Theorem 3.** *Let  $\phi$  be an analytic self map of the disc with  $\phi(0) = 0$  and  $0 < |\phi'(0)| < 1$ . Suppose  $(\sigma)^q$  is in  $H^2$  where  $\sigma$  is the Koenigs' function of  $\phi$ . Then for each natural number  $k$ ,  $1 \leq k \leq q$ , the subspaces  $z^k H^2$  are hyperinvariant for  $C_\phi$ .*

*Proof.* In [7], we showed that  $zH^2$  is a hyperinvariant subspace for such a  $C_\phi$ . This covers the case  $k = 1$ . Proceeding by induction on  $n < q$ , we will assume, for  $k \leq n$ , that  $z^k H^2$  is hyperinvariant for  $C_\phi$ . Let  $A$  be an operator which commutes with  $C_\phi$  and let  $z^{n+1}p$  be a function in  $z^{n+1}H^2$  where  $p$  is a polynomial. We wish to show

$$\langle A(z^{n+1}p), z^l \rangle = 0$$

for  $l \leq n$ . Since  $z^{n+1}p$  is in  $z^n H^2$ , we have by the induction hypothesis that

$$\langle A(z^{n+1}p), z^l \rangle = 0$$

for  $l < n$ . Now

$$\begin{aligned} \langle AC_\phi^m(z^{n+1}p), z^n \rangle &= \langle A(z^{n+1}p), (C_\phi^*)^m(z^n) \rangle \\ &= \langle A(z^{n+1}p), \overline{(\phi'(0))^{mn}} z^n \rangle. \end{aligned}$$

The last equality follows from the induction hypothesis and the fact that  $C_\phi^*$  is upper triangular as a matrix when represented in the standard basis. Dividing both sides of the equation by  $(\phi'(0))^{mn}$ , it follows that

$$\langle A((\sigma_m)^n \phi^{[m]}(p \circ \phi^{[m]})), z^n \rangle = \langle A(z^{n+1}p), z^n \rangle.$$

The left-hand side of this equation is equal to

$$\langle (\sigma_m)^n \phi^{[m]}(p \circ \phi^{[m]}), A^*(z^n) \rangle.$$

Since  $H^{2q} \subset H^{2n}$  when  $q \geq n$ ,  $(\sigma)^n$  is in  $H^2$ . By Theorem 1,  $(\sigma_m)^n$  is a bounded sequence in the  $H^2$  norm. Since  $p$  is in  $H^\infty$ ,

$$\{(\sigma_m)^n \phi^{[m]}(p \circ \phi^{[m]})\}$$

is also a bounded sequence in the  $H^2$  norm. This sequence converges pointwise to 0 as  $m$  goes to  $\infty$ , and thus it converges weakly to 0. It follows that  $\langle A(z^{n+1}p), z^n \rangle = 0$ .

If  $z^{n+1}f$  is in  $z^{n+1}H^2$ , we may find a sequence of polynomials  $\{p_l\}$  which converges strongly to  $f$  as  $l$  goes to  $\infty$ . In particular,  $\langle A(z^{n+1}p_l), z^n \rangle$  will converge to  $\langle A(z^{n+1}f), z^n \rangle$  as  $l$  goes to infinity. Hence the result holds.  $\square$

Let  $\|A\|_e$  denote the essential norm of an operator  $A$ . An operator  $A$  is a Riesz operator if  $\|A^n\|_e^{\frac{1}{n}}$  tends to 0 as  $n$  tends to  $\infty$ . This implies that  $A$  is Riesz if and only if the essential spectrum of  $A$  is  $\{0\}$ . In [3], Bourdon and Shapiro show that a Riesz composition operator must be induced by a function which has an interior fixed point in  $D$ . For basic details of Riesz operators see [11].

**Corollary 1.** *Let  $\phi$  be an analytic self map of  $D$  with  $\phi(b) = b$  where  $b$  is in  $D$  and  $0 < |\phi'(b)| < 1$ . If  $C_\phi$  is a Riesz operator, then  $C_\phi$  has a triangularizing chain of hyperinvariant subspaces.*

*Proof.* Let  $\alpha(z) = \frac{b-z}{1-\bar{b}z}$ . Define  $\psi = \alpha \circ \phi \circ \alpha$ . Note  $\psi(0) = 0$  and  $0 < |\psi'(0)| < 1$ . Since  $C_\psi^n = C_\alpha C_\phi^n C_\alpha$ , it follows that  $C_\psi$  is a Riesz operator. Now by Theorem 2 and Theorem 3,  $C_\psi$  has the  $z^q H^2$  as hyperinvariant subspaces for all  $q = 0, 1, 2, \dots$ . Thus  $C_\phi$  has  $(\alpha)^q H^2$ ,  $q = 0, 1, 2, \dots$  as a chain of hyperinvariant subspaces. Moreover, any operator which holds these subspaces invariant must be lower triangular with respect to the orthonormal basis  $\{(\alpha)^n \frac{1-|b|^2}{1-\bar{b}z} : n = 0, 1, 2, \dots\}$ .  $\square$

Let  $f^{(k)}$  be the  $k^{\text{th}}$  derivative of  $f$ .

**Corollary 2.** *Let  $\phi$  be an analytic self map of the disc with  $\phi(0) = 0$  and  $0 < |\phi'(0)| < 1$ . Let  $k$  be a natural number greater than or equal to 1. Suppose  $(\sigma)^{k+1}$  is in  $H^2$  and let  $A$  commute with  $C_\phi$ . Then, for all  $f$  in  $H^2$ ,  $\frac{d^k}{dz^k}(A(f))(0) = k! \sum_{n=1}^k \frac{f^{(n)}(0)}{n!} \langle A(z^n), z^k \rangle$ .*

*Proof.* Let  $f = \sum_{n=0}^{\infty} a_n z^n$  be the power series expansion of  $f$ . In [7], we showed that if  $A$  commutes with  $C_\phi$ , then  $(A(f))(0) = A(1)f(0)$  where  $A(1)$  is a constant. This implies that  $\langle A(a_0), z^k \rangle = 0$ . Now

$$\langle A(f), z^k \rangle = \langle A(a_0), z^k \rangle + \sum_{n=1}^k a_n \langle A(z^n), z^k \rangle + \langle A(\sum_{n=k+1}^{\infty} a_n z^n), z^k \rangle.$$

The last term is 0 since by Theorem 3,  $z^{k+1}H^2$  is hyperinvariant for  $C_\phi$ . Since  $a_n = \frac{f^{(n)}(0)}{n!}$ , the result follows.  $\square$

The next example shows that there are analytic self maps of  $D$  with  $\phi(0) = 0$  and  $\phi'(0) = 0$  such that the  $z^q H^2$  are not hyperinvariant for  $C_\phi$  for  $q \geq 2$ .

**Example 1.** Let  $n$  be a natural number greater than 1. For  $C_{z^n}$ , the subspaces  $z^q H^2$  for  $q \geq 2$  are not hyperinvariant.

*Proof.* In [6], we discuss the commutant of  $C_{z^n}$  in more detail. Let  $q$  be a natural number greater than or equal to 2. If we define an operator  $A$  on  $H^2$  by  $A(z^{q \cdot n^l}) = z^{n^l}$  and  $A(z^k) = 0$  when  $k \neq q \cdot n^l$ , then  $A$  commutes with  $C_\phi$ . Now  $A(z^q) = z$  and hence  $z^q H^2$  is not a hyperinvariant subspace for  $C_\phi$ .  $\square$

### 3. QUASI-NORMALS

We now turn to quasi-normal operators. An operator  $A$  is quasi-normal if  $A$  commutes with  $A^*A$ . For more information, see [8].

**Lemma 1.** *If  $C_\phi$  is quasi-normal, then  $\phi(0) = 0$ .*

*Proof.*  $C_\phi C_\phi^* C_\phi(1) = C_\phi^* C_\phi C_\phi(1)$  implies that

$$\frac{1}{1 - \overline{\phi(0)}\phi(z)} = \frac{1}{1 - \overline{\phi(0)}z}.$$

It follows that  $\phi(0) = 0$ .  $\square$

**Lemma 2.** *Let  $\phi$  belong to  $L^2(\partial D)$  with  $\phi$  non-zero and not a characteristic function of a proper subset of the unit circle. Also suppose that  $\|\phi\|_{L^2} = \mu$  and  $\|(\phi)^{k_l}\|_{L^2} = \mu$  where  $k_l$  is a sequence of natural numbers which diverges to  $\infty$  and  $\mu$  is a non-zero real number. Then  $|\phi(e^{i\theta})| = 1$  almost everywhere on  $\partial D$ .*

*Proof.* Let  $\lambda = \mu^2$ . Let  $m(A)$  be the normalized standard Lebesgue measure of a set  $A \subset \partial D$ . Suppose that  $|\phi(e^{i\theta})| > 1$  on a set of positive measure of  $\partial D$ . Then, in particular, there exists  $\epsilon > 0$  such that the set  $A_\epsilon = \{e^{i\theta} : |\phi(e^{i\theta})| > 1 + \epsilon\}$  has positive measure. It follows that

$$\lambda = \|\phi^{k_l}\|_{L^2}^2 \geq \frac{1}{2\pi} \int_{A_\epsilon} |\phi(z)|^{2k_l} dz \geq (1 + \epsilon)^{2k_l} m(A_\epsilon).$$

As  $k_l$  tends to infinity, the right-hand side also tends to infinity. This is a contradiction; thus  $|\phi(e^{i\theta})| \leq 1$  almost everywhere.

Similarly, suppose that  $|\phi(e^{i\theta})| < 1$  on a set of positive measure of  $\partial D$ . Then, in particular, there exists  $\epsilon > 0$  such that the set  $B_\epsilon = \{e^{i\theta} : |\phi(e^{i\theta})| < 1 - \epsilon\}$  has positive measure. Then

$$\lambda = \frac{1}{2\pi} \int_{B_\epsilon} |\phi(z)|^2 dz + \frac{1}{2\pi} \int_{\partial D \setminus B_\epsilon} |\phi(z)|^2 dz.$$

Let  $\int_{\partial D \setminus B_\epsilon} |\phi(z)|^2 dz = \kappa$ . We note  $\kappa$  is strictly less than  $\lambda$ . Now

$$\frac{1}{2\pi} \int_{B_\epsilon} |\phi(z)|^{2k_l} dz \leq (1 - \epsilon)^{2k_l} m(B_\epsilon).$$

Choose a natural number  $N$  such that  $k_l \geq N$  implies that  $(1 - \epsilon)^{2k_l} m(B_\epsilon)$  is strictly less than  $\lambda - \kappa$ . Since  $|\phi(z)| \leq 1$  almost everywhere, we note that  $|\phi(z)|^{k_l} \leq |\phi(z)|$ . Then with  $k_l \geq N$ ,

$$\begin{aligned} \lambda = \|(\phi)^{k_l}\|_{L^2}^2 &= \frac{1}{2\pi} \int_{B_\epsilon} |\phi(z)|^{2k_l} dz + \frac{1}{2\pi} \int_{\partial D \setminus B_\epsilon} |\phi(z)|^{2k_l} dz \\ &\leq \frac{1}{2\pi} \int_{B_\epsilon} |\phi(z)|^{2k_l} dz + \frac{1}{2\pi} \int_{\partial D \setminus B_\epsilon} |\phi(z)|^2 dz \end{aligned}$$

$$< \lambda - \kappa + \frac{1}{2\pi} \int_{\partial D \setminus B_\epsilon} |\phi(z)|^2 dz = \lambda$$

which is a contradiction; thus  $|\phi(z)| = 1$  almost everywhere.  $\square$

**Theorem 4.** *Suppose  $C_\phi$  is quasi-normal. If  $C_\phi^* C_\phi$  is a diagonal matrix in the standard basis, then either  $\phi(z) = cz$  for some constant  $c$  of modulus less than 1 or  $\phi$  is an inner function.*

*Proof.* Let the power series of  $\phi$  be  $\sum_{n=1}^{\infty} a_n z^n$  and let the  $k^{\text{th}}$  diagonal entry of  $C_\phi^* C_\phi$  be  $\lambda_k$ . First of all, if  $\phi$  is identically zero or if only one of the  $a_n$  is non-zero, then we are done.

Next suppose that only a finite number of the coefficients  $a_n$  are non-zero but more than one of them is non-zero. Thus let  $\phi(z) = z^k a_k + \cdots + a_l z^l$  where  $a_l$  and  $a_k$  are non-zero and  $k$  is the smallest power of  $z$  with a non-zero coefficient and  $l$  is the greatest. Now,

$$\langle (\phi(z))^l, (\phi(z))^k \rangle = (a_k)^l (a_l)^k,$$

which by hypothesis must be zero. Thus if only a finite number of coefficients are non-zero, we have  $\phi(z) = cz^k$  where  $c$  is a constant of modulus less than or equal to 1. If  $k \geq 2$  and  $c$  is less than 1 in modulus, then  $C_\phi$  is not quasi-normal, so in this case  $k = 1$ .

Suppose an infinite number of the coefficients  $a_n$  are non-zero. Then  $\phi(z) = \sum_{n=k}^{\infty} a_n z^n$  where  $a_k$  is the first non-zero coefficient. Now  $C_\phi^* C_\phi C_\phi(z) = C_\phi C_\phi^* C_\phi(z)$  implies that

$$\sum_{n=k}^{\infty} \lambda_n a_n z^n = \sum_{n=k}^{\infty} \lambda_1 a_n z^n.$$

It follows that  $\lambda_n = \lambda_1$  for infinitely many  $n$ . Since  $\lambda_1 = \langle \phi(z), \phi(z) \rangle$  and  $\lambda_n = \langle (\phi(z))^n, (\phi(z))^n \rangle$ , we may apply Lemma 2 to conclude that  $\phi$  must be an inner function.  $\square$

**Corollary 3.** *If  $C_\phi$  has  $z^q H^2$ ,  $1 \leq q < \infty$ , as hyperinvariant subspaces, and  $C_\phi$  is quasi-normal, then either  $\phi(z) = cz$  for some  $c$  of modulus less than 1 or  $\phi$  is inner.*

*Proof.* If  $C_\phi$  is quasi-normal, then  $C_\phi$  commutes with  $C_\phi^* C_\phi$ , as does  $C_\phi^*$ . Thus  $C_\phi^* C_\phi$  is both upper and lower triangular with respect to the standard basis and thus diagonal. Apply Theorem 4.  $\square$

We could apply Corollary 1 to conclude that if  $C_\phi$  is quasi-normal and Riesz, then  $\phi(z) = cz$  for  $\|c\| < 1$ . This would say that if  $C_\phi$  is Riesz and quasi-normal, then it is normal. As the next theorem shows, this is true in greater generality.

**Theorem 5.** *If  $A$  is an operator on a Hilbert space  $H$ , and  $A$  is Riesz and quasi-normal, then  $A$  is normal.*

*Proof.* If  $A$  is identically 0, then we are done. Assume  $A$  is not equal to 0. By [4],  $\ker(A)$  is reducing, so we may decompose  $A$  as  $A_1 \oplus 0$  with  $A_1$  quasi-normal and injective. Moreover, we may decompose  $A_1$  as  $BV$  with  $B$  self-adjoint and injective,  $V$  isometric, and  $B$  commuting with  $V$ . Since  $A_1^n = V^n B^n$ , we have  $\|(V^*)^n A_1^n\| = \|B^n\|$ . Thus

$$\|B^n\|_e^{\frac{1}{n}} \leq \|(V^*)^n\|_e^{\frac{1}{n}} \|A_1^n\|_e^{\frac{1}{n}} \leq \|A_1^n\|_e^{\frac{1}{n}}.$$

The last term goes to 0 as  $n$  goes to  $\infty$ , and thus  $B$  is also Riesz. By Theorem 3.7 in [19], since  $B$  is self-adjoint and Riesz, it must be compact. Hence by the spectral theorem,  $B$  is unitarily equivalent to  $\sum \bigoplus \lambda_k I_k$  where  $\lambda_k$  are the non-zero eigenvalues and the  $I_k$  are identity operators on finite-dimensional spaces. Since eigenspaces are hyperinvariant, and  $V$  commutes with  $B$ ,  $V$  is unitarily equivalent to  $\sum \bigoplus V_k$ , where the  $V_k$  are isometries on finite-dimensional spaces and hence unitary. Thus  $V$  is a unitary operator and  $A_1$  is a product of commuting normals and thus normal.  $\square$

#### 4. ISOMETRIES

An operator  $A$  on  $H^2$  is an isometry if  $A^*A = I$ . In [17], Schwarz proves the following theorem.

**Theorem 6.**  $C_\phi$  is an isometry on  $H^2$  if and only if  $\phi(0) = 0$  and  $\phi$  is an inner function.

If  $\phi$  is an elliptic disc automorphism, then the commutant of  $C_\phi$  is well understood (see [6] or [7]). If  $\phi$  is not an elliptic disc automorphism and  $C_\phi$  is an isometry, Nordgren ([13]) shows that  $C_\phi$ , restricted to the constants, and  $C_\phi$ , restricted to  $zH^2$ , are the unitary and purely isometric parts, respectively. If  $A$  commutes with  $C_\phi$ , then the constants are a reducing subspace for  $A$  (see [7]). Hence we may consider the commutant of  $C_\phi$  as the direct sum of the commutant of the unitary part and the commutant of the purely isometric part. The purely isometric part is similar to a unilateral shift of infinite multiplicity on the wandering subspace,  $M$ , of  $C_\phi$  (see [16]). The commutant of such a unilateral shift is given in terms of multiplication operators on  $H^2(M)$  (see [16]).

**Lemma 3.** Let  $\phi = zf$  where  $f$  is a nonconstant inner function. Let  $\{g_n : n = 0, 1, \dots\}$  be an orthogonal basis for  $(fH^2)^\perp$ . Then the following set,  $S$ ,

$$\{z(\phi)^k g_n : n = 0, 1, 2, \dots, k = 0, 1, 2, \dots\},$$

is an orthogonal basis for the wandering subspace of  $C_\phi$  on  $H^2$ .

*Proof.* It is easy to see that the vectors in  $S$  are orthogonal. The wandering subspace is given by the perp of  $C_\phi(zH^2)$  in  $zH^2$  (see [12]). We want to show that the vectors in  $S$  span this space. We note that the set of vectors  $T$ , given by  $\{(\phi)^k : k = 1, 2, \dots\}$  span  $C_\phi(zH^2)$ . Thus it is sufficient to show that the vectors in  $T$  and  $S$ , along with the set containing the constant function 1, span all of  $H^2$  since these three sets are mutually orthogonal.

Suppose that  $h$  is orthogonal to the vectors in these three sets. We claim that  $h$  is 0. First of all  $\langle h, 1 \rangle = 0$  implies that  $h = zk_1$  for some  $k_1$  in  $H^2$ . Now

$$\langle zk_1, zg_n \rangle = \langle k_1, g_n \rangle = 0$$

for all  $n$  implies that  $k_1$  belongs to  $fH^2$ . Thus  $h = zfh_1$  for some  $h_1$  in  $H^2$ . Thus  $h = \phi h_1$ . Now if we proceed by induction and assume that  $h = (\phi)^l h_l$  for some  $h_l$  in  $H^2$ , we have that  $\langle (\phi)^l h_l, (\phi)^l \rangle = 0$  implies that  $h_l = zk_{l+1}$  for some  $k_{l+1}$  in  $H^2$ . Thus

$$\langle (\phi)^l zk_{l+1}, (\phi)^l zg_n \rangle = \langle k_{l+1}, g_n \rangle = 0$$

for all  $n$  implies that  $k_{l+1} = fh_{l+1}$  for some  $h_{l+1}$  in  $H^2$ . Thus  $h = (\phi)^{l+1} h_{l+1}$  and we have by induction that  $(\phi)^l$  divides  $h$  for arbitrary powers of  $l$  and hence  $h = 0$ .  $\square$

In particular, let  $\phi = zB$  where  $B$  is a Blaschke product with zero set  $\{a_n : n = 0, 1, 2, \dots\}$ . Let  $k_\lambda$  be the reproducing kernel function for  $\lambda$  in  $D$  and let  $B_l$  be the Blaschke product with zero set  $\{a_n : n = 0, 1, \dots, l\}$ . Then the following is an orthonormal basis for  $(BH^2)^\perp : \{g_0 = k_{a_0}, g_1 = B_0 k_{a_1}, \dots, g_n = B_{n-1} k_{a_n}\}$ . This gives us a basis for the wandering subspace of  $C_\phi$  and the commutant of  $C_\phi$  can be explicitly interpreted in terms of this basis.

#### QUESTIONS

(1): If  $\phi$  is an analytic self map of  $D$ ,  $\phi(0) = 0$  and  $0 < |\phi'(0)| < 1$ , are the subspaces  $\{z^q H^2\}$  hyperinvariant subspaces for all natural numbers  $q$ ? If this is true, then the only  $C_\phi$  which are quasi-normal are either isometries or are given by  $\phi(z) = cz$  for some constant  $c$  of absolute value less than 1.

(2): If  $\phi$  is an analytic self map of  $D$ ,  $\phi(0) = 0$  and  $\phi'(0) = 0$ , what can be said about the hyperinvariant subspaces of  $C_\phi$ ? If  $C_\phi$  is quasi-normal, what form does  $\phi$  have to take?

(3): Given a basis for the wandering subspace of an isometric composition operator, can the hyperinvariant subspaces or the composition operators which commute be explicitly determined?

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DEPARTMENT OF MATHEMATICS, BROCK UNIVERSITY, ST. CATHARINES, ONTARIO, CANADA  
L2S 3A1

*E-mail address:* bcload@spartan.ac.brocku.ca