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ON THE EXCESS OF SETS OF COMPLEX EXPONENTIALS

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ABSTRACT. For $-\infty < n < \infty$ let μ_n be complex numbers such that $\mu_n - n$ is bounded. For $n > 0$ define $\lambda_n = \mu_n + a$, $\lambda_{-n} = \mu_{-n} - b$ where $a, b \geq 0$. Then the excesses E in the sense of Paley and Wiener satisfy $E(\{\lambda_n\}) \leq E(\{\mu_n\})$.

The original sources underlying the problem treated here are [1], [2], [4], and all the results used can be found in the excellent expository account [5], supplemented in a few cases by [3]. In view of this, we do not interrupt the flow by constant reference to the literature.

We use a, b, c for constants and we denote by $\mu = \{\mu_n\}$ a complex sequence defined for $-\infty < n < \infty$. The L^2 excess of the set μ in the sense of Paley and Wiener is denoted by $E(\mu)$. If $|\mu_n - n| \leq c$, it is well known that the completeness interval is $[-\pi, \pi]$ and that $E(\mu)$ on this interval is finite. Our object is to prove:

Theorem 1. *Let $|\mu_n - n| \leq c$ for $-\infty < n < \infty$ and $\lambda_0 = \mu_0$,*

$$\lambda_n = \mu_n + a, \quad \lambda_{-n} = \mu_{-n} - b, \qquad n > 0,$$

where $a \geq 0$ and $b \geq 0$. Then $E(\lambda) \leq E(\mu)$ on the interval $[-\pi, \pi]$.

For example, suppose $E(\mu) = 0$, so μ is exact on $[-\pi, \pi]$; that is, $\{\exp(i\mu_n x)\}$ is complete as it stands but becomes incomplete if any term is removed. Then Theorem 1 shows that $\{\exp(i\lambda_n x)\}$ is either exact or incomplete. Hence it is *free* in the sense of Schwartz; that is, no exponential $\exp(i\lambda_n x)$ is in the subspace spanned by the others. This always holds if $c < 1/4$, since the set $\exp(i\mu_n x)\}$ forms a basis in that case.

We precede the proof by the following simplifications:

(i) It is easily checked that the completeness properties are unchanged when λ_n is replaced by $\lambda_n - d$, where d is any real constant. Hence there is no loss of generality in taking $b = 0$ and replacing a by $a_0 = a + b$. We write a_0 instead of $a + b$ because we want a to be a variable, $0 \leq a \leq a_0$. This interval describes the *relevant values* of a.

(ii) Since any finite number of terms can be changed without altering the completeness, we assume without loss of generality that

$$\text{Re } \mu_n < 0 \text{ for } n < 0, \quad \text{Re } \mu_n \geq 1 \text{ for } n > 0.$$

In particular, $\lambda_n \neq 0$ for $n \neq 0$ and all relevant values of a. This step and (iii) below may involve increasing c, but that does no harm.

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(iii) By adding or removing a finite number of terms μ_n and the λ_n corresponding thereto, we can ensure that the set μ is exact.

1. Proof

Let

$$F(a,x) = \prod_{n=1}^{\infty}\left(1-\frac{x}{\mu_{-n}}\right)\left(1-\frac{x}{\mu_n+a}\right), \tag{1}$$

where the factors have been grouped in such a way that the product converges. Since μ is exact we have $F(0,\,x) \in L^2$, but this would not hold if we had retained the factor corresponding to μ_0. Theorem 1 will follow as soon as we show $F(a,x) \in L^2$.

Let us therefore consider the integral

$$I(a) = \int_{-R}^{R+a} |F(a,\,x)|^2\,dx, \tag{2}$$

where R is a large constant that will later tend to ∞. The integral is decomposed in the form

$$I(a) = \int_{-R}^{-m} + \int_{-m}^{m+a} + \int_{m+a}^{R+a}, \tag{3}$$

where $m \geq 2c + c^2$ is an integer. With $\mu_{-n} = -p - iq$ and $\mu_n = r + is$ for $n \geq m$, the constraint is

$$(p-n)^2 + q^2 \leq c^2, \quad (r-n)^2 + s^2 \leq c^2. \tag{4}$$

The magnitude squared of the general term in the product is

$$K(a) = \left|1-\frac{x}{\mu_{-n}}\right|^2 \frac{(r+a-x)^2+s^2}{(r+a)^2+s^2}.$$

In the first integral (3) we have $x \leq -m$. Differentiation shows that $K'(a)$ has the same sign as $s^2 - (r+a)^2 + x(r+a)$, which is at most $s^2 - m$ and hence ≤ 0. Therefore the first integral is dominated by the corresponding integral with μ, which converges when $R \to \infty$.

In the second integral $-m \leq x \leq m + a$. Since the integrand is a continuous function of $(a,\,x)$ the integral is bounded by a constant C_m for all relevant values of a.

After the substitution $x = y + a$ the third integral is

$$J(a) = \int_m^R \prod_{n=1}^{\infty} \left|1-\frac{y+a}{\mu_{-n}}\right|^2 \left|1-\frac{y+a}{\mu_n+a}\right|^2 dy.$$

The monotonicity of the integrand

$$\left|\frac{\mu_n - y}{\mu_{-n}}\right|^2 \left|\frac{\mu_{-n}-a-y}{\mu_n+a}\right|^2$$

as a function of a is determined by the second factor,

$$L(a) = \left|\frac{p+iq+a+y}{r+is+a}\right|^2 = \frac{(p+a+y)^2+q^2}{(r+a)^2+s^2}.$$

Evidently $L'(a)$ has the same sign as

$$[(r+a)^2+s^2](p+a+y) - [(p+a+y)^2+q^2](r+a).$$

Now comes an important point. Other variables being fixed, $L'(a)$ is largest when $q = 0$. Since $q = 0$ allows the greatest freedom in the choice of p, we see that $q = 0$ is in fact the worst case. Setting $q = 0$, we divide out the factor $p + a + y$ and find that $L'(a) \leq 0$ holds if the function

$$P = (r + a)^2 + s^2 - (p + a + y)(r + a) = (r + a)(r - p - y) + s^2$$

satisfies $P \leq 0$. The two points (p, q) and (r, s) are both in a circle of radius c centered at $(n, 0)$. Hence their distance is at most $2c$, so $|r - p| \leq 2c$. Since also $|s| \leq c,\ r \geq 1$ and $y \geq m \geq 2c + c^2$, it follows that $P \leq 0$. Hence $L'(a) \leq 0$ and the integral is maximized when $a = 0$. In that case it reduces to the corresponding integral with μ, which converges as $R \to \infty$. This completes the proof.

2. Supplementary remarks

The foregoing methods apply to the weighted integral

$$\int_{-\infty}^{\infty} |F(a, x)|^2 w(x)\, dx,$$

where $w(x)$ is any positive continuous function satisfying

$$0 < \liminf_{|x|\to\infty} \frac{w(x + d)}{w(x)} \leq \limsup_{|x|\to\infty} \frac{w(x + d)}{w(x)} < \infty, \qquad -\infty < d < \infty.$$

Namely, if the integral converges for $a = 0$ then it converges for $a \geq 0$. Admissible weights that are often encountered are

$$w(x) = e^{hx}, \quad w(x) = (1 + x^2)^h,$$

where h is a positive or negative constant.

As another generalization, instead of $|\mu_n - n| \leq c$ we could assume that $E(\mu)$ is finite and that

$$n\mathrm{Re}\ \mu_n \geq 0, \quad |\operatorname{Im} \mu_n| \leq c, \quad |\mu_{-n} + \mu_n| \leq c.$$

The question whether $|E(\mu)| < \infty \Rightarrow E(\lambda) \leq E(\mu)$ was suggested to one of us by Prof. Lennart Carleson. It appears to be considerably deeper than Theorem 1 and is left open here. The case $p \neq 2$ is also left open, though we do have

$$E(\lambda) \leq E(\mu) + 1$$

for $1 \leq p \leq \infty$. This follows from the result for $p = 2$ and from the fact that changing p changes $E(\lambda)$ by at most 1.

Theorem 1 is especially plausible when when $\mu = \{n\}$ and $b = 0$. In that case the set λ for $a = 1$ is the same as $\mu - \{1\}$, the set for $a = 2$ is the same as $\mu - \{1, 2\}$ and so on. Further insight into the case $\mu = \{n\}$ is given by [1], Theorem V and its proof. The latter suggests a conjecture that is formulated next.

An exponent p satisfying $1 < p < \infty$ will be called *critical* if

$$1 < r < p < q < \infty \Rightarrow E(\lambda, q) > E(\lambda, r). \tag{5}$$

When $p = 1$ or $p = \infty$ we modify (5) somewhat and instead of L^∞ we use the class of absolutely continuous functions that vanish at $\pm\pi$. As $a + b$ increases in Theorem 1, we conjecture that the critical value p in the excess $E(\lambda, p)$ increases steadily to ∞, passing through all intermediate values on the way. Upon further increase $E(\lambda, p)$ drops by 1 while p changes to 1, and the process repeats.

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