PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 6, Pages 1837–1845
S 0002-9939(99)04665-1
Article electronically published on February 23, 1999

A CENTRAL LIMIT THEOREM FOR MARKOV CHAINS AND APPLICATIONS TO HYPERGROUPS

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(Communicated by Stanley Sawyer)

ABSTRACT. Let (X_n) be a homogeneous Markov chain on an unbounded Borel subset of $\mathbb{R}$ with a drift function d which tends to a limit m_1 at infinity. Under a very simple hypothesis on the chain we prove that $n^{-1/2}(X_n - \sum_{k=1}^{n} d(X_{k-1}))$ converges in distribution to a normal law $N(0,\sigma^2)$ where the variance σ^2 depends on the asymptotic behaviour of (X_n). When $d-m_1$ goes to zero quickly enough and $m_1 \neq 0$, the random centering may be replaced by nm_1. These results are applied to the case of random walks on some hypergroups.

1. INTRODUCTION

In the sequel, (X_n) is a homogeneous Markov chain on an unbounded Borel subset K of $\mathbb{R}$. Let $\alpha > 0$. We consider the following functions:

$$d(t) = \mathbb{E}(X_n - X_{n-1}|X_{n-1} = t), \tag{1.1}$$

$$c_\alpha(t) = \mathbb{E}(|X_n - X_{n-1}|^\alpha|X_{n-1} = t), \tag{1.2}$$

for $t \in K$. The first function d is called the drift of (X_n). When (X_n) is a classical (non-deterministic) random walk on $\mathbb{R}$ with finite second moment, $d(t) \equiv m_1$ and $c_2(t) \equiv m_2$ are constant functions. Then the Central Limit Theorem (CLT) asserts that $n^{-1/2}(X_n - nm_1)$ converges in distribution to the centered normal law $N(0,\sigma^2)$ with variance $\sigma^2 = m_2 - m_1^2$. In this paper we will generalize this result to certain Markov chains such that c_2 is a bounded function on K and the limits $\lim_{|t|\to\infty} d(t) = m_1$ and $\lim_{|t|\to\infty} c_2(t) = m_2$ exist.

As in [G1] and [G3], where we established a law of large numbers, our results can be applied very naturally to get a CLT for random walks on one-dimensional hypergroups with asymptotic drift of the convolution (see section 4). Central limit theorems for random walks have already been studied in some particular commutative hypergroups using Fourier methods ([B.H]). The results of this paper are of a different nature. We emphasize the asymptotic behaviour of the Markov chain and derive a CLT using only tools from martingale theory and ergodic theory.

Received by the editors April 14, 1997 and, in revised form, September 22, 1997.
1991 *Mathematics Subject Classification.* Primary 60J10, 60F05, 60J15.

2. Generalities and hypotheses

2.1. For the Markov chain (X_n) we will make use of some of the following hypotheses:

$H_0(\alpha) =$ "function c_α exists and is measurable and bounded on K".

$H_1 =$"$H_0(1)$ is satisfied and $\lim_{|t|\to\infty} d(t) = m_1(\in \mathbb{R})$ exists".

$H_2 =$ "$H_0(2)$ is satisfied and $\lim_{|t|\to\infty} c_2(t) = m_2(\in \mathbb{R}_+)$ exists".

H_1 and H_2 are often verified in applications. In the case of random walks on hypergroups, a different condition is satisfied:

$H_3 =$"There exists a sequence (ξ_n) of I.I.D random variables with finite second moment such that

$$|X_n - X_{n-1}| \leq \xi_n \quad a.s.$$

for all $n \in \mathbb{N}^*$."

Note that if $H_0(2)$ is satisfied, we have $c_2(t) - d^2(t) \geq 0 \quad (\forall t \in K)$. We will suppose that this function never vanishes, i.e., (X_n) has no absorbing state.

2.2. **Definitions and notations.**

2.2.1. As usual, for every $x \in K$, we will denote by $\mathbb{P}_x$ (resp. $\mathbb{E}_x$) the probability (resp. the expectation) given $X_0 = x$. In the same way, if ν is any probability measure on $K, \mathbb{P}_\nu$ (resp. $\mathbb{E}_\nu$) is the probability (resp. the expectation) given that the initial state X_0 is distributed according to ν.

2.2.2. In the sequel when we suppose that (X_n) is a recurrent chain, this means that:

i) If K is a countable set, (X_n) is an irreducible and recurrent chain on K in the classical sense. In this case we will denote by λ its invariant measure.

ii) If K is an uncountable Borel subset of $\mathbb{R}, (X_n)$ is Harris-recurrent with respect to an invariant Radon measure λ supported by K ([R]).

In both cases, (X_n) is said to be null recurrent if $\lambda(K) = +\infty$. Otherwise it is called positive recurrent and we can always assume that λ is a probability measure.

2.2.3. The chain will be called transient if for some $x \in K, |X_n| \to +\infty \quad \mathbb{P}_x \quad a.s.$

3. The Central Limit Theorem

We will break the result into two statements according to the asymptotic behaviour of the chain.

3.1. Theorem A. *If (X_n) is positive recurrent with invariant measure λ and satisfies hypothesis $H_0(2)$, then*

$$n^{-1/2}\left(X_n - \sum_{k=1}^{n} d(X_{k-1})\right) \xrightarrow{\mathcal{L}} N(0,\sigma^2) \quad (n \to +\infty),$$

where $\sigma^2 = \int_K \left(c_2(t) - d^2(t)\right) \lambda(dt).$

3.2. Theorem B. *If (X_n) is null recurrent or transient, satisfies H_1 and H_2 with $\sigma^2 = m_2 - m_1^2 > 0$ and moreover satisfies either hypothesis $H_0(2+\epsilon)$ (for some $\epsilon > 0$) or hypothesis H_3, then*

$$n^{-1/2}\left(X_n - \sum_{k=1}^{n} d(X_{k-1})\right) \xrightarrow{\mathcal{L}} N(0,\sigma^2) \quad (n \to +\infty).$$

(In both results, $\xrightarrow{\mathcal{L}}$ means convergence in distribution.)

Proof. The beginning of the proof is common to (3.1) and (3.2). It is not difficult to verify that the sequence

$$M_n = X_n - \sum_{k=1}^{n} d(X_{k-1}), \quad n \geq 1, \tag{3.3}$$

is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_k; k \leq n)$. Let $Z_k = M_k - M_{k-1}$ be the increment of this martingale. We easily get

$$\mathbb{E}(Z_k^2|\mathcal{F}_{k-1}) = g(X_{k-1}) \quad a.s., \tag{3.4}$$

where $g(t) = c_2(t) - d^2(t)$. For every $n \in \mathbb{N}, j \in \mathbb{N}, a > 0$ and $\epsilon > 0$, let us consider the following expressions:

$$V_n^2 = \sum_{k=1}^{n} g(X_{k-1}), \tag{3.5}$$

$$s_n^2 = \mathbb{E}(V_n^2) = \sum_{k=1}^{n} \mathbb{E}(Z_k^2), \tag{3.6}$$

$$W_j(a) = \mathbb{E}(Z_j^2 \mathbb{I}_{[|Z_j|>a]}|\mathcal{F}_{j-1}), \tag{3.7}$$

$$H_n^\epsilon = \frac{1}{n}\sum_{j=1}^{n} W_j(\epsilon\sigma\sqrt{j}). \tag{3.8}$$

The theorems will be proved if we show that the CLT for martingales can be applied to (M_n). For this purpose it is sufficient to prove the three following assertions ([H.H] or [Br]):

$$\lim_{n\to\infty} n^{-1}V_n^2 = \sigma^2 \qquad \text{(in probability)}, \tag{3.9}$$

$$\lim_{n\to\infty} n^{-1}s_n^2 = \sigma^2, \tag{3.10}$$

$$\forall \epsilon > 0, \lim_{n\to\infty} H_n^\epsilon = 0 \qquad \text{(in probability)}. \tag{3.11}$$

The last assertion is the Lindeberg condition. We now have to consider separately (3.1) and (3.2).

i) Proof of Theorem A. The bounded function g is λ integrable and the ergodic theorem gives immediately (3.9):

$$n^{-1}V_n^2 = \frac{1}{n}\sum_{k=1}^{n} g(X_{k-1}) \longrightarrow \langle \lambda, g\rangle = \sigma^2 \quad a.s.$$

But we have dominated convergence and (3.10) is also satisfied. For all fixed $a > 0$, consider the function defined on K by $h(x) = \mathbb{E}_x(Z_1^2 \mathbb{I}_{[|Z_1|>a]})$. By the

Markov property and using notation (3.7), we have

$$K_n(a) := \frac{1}{n}\sum_{j=1}^{n} h(X_{j-1}) = \frac{1}{n}\sum_{j=1}^{n} W_j(a). \tag{3.12}$$

But h is also a bounded function on K (indeed $h(x) \le \mathbb{E}_x(Z_1^2) \le 2c_2(x) + 2d^2(x) \le 2(\|c_2\|_\infty + \|d^2\|_\infty)$) and the ergodic theorem applied to h gives

$$\lim_{n\to\infty} K_n(a) = \langle \lambda, h\rangle = \mathbb{E}_\lambda(Z_1^2 \mathbb{I}_{[|Z_1|>a]}).$$

For every $a > 0, \epsilon > 0$, it is clear that we have

$$\limsup_{n\to\infty} H_n^\epsilon \le \lim_{n\to\infty} K_n(a). \tag{3.13}$$

This implies that the left-hand side of (3.13) is zero because $\lim_{a\to\infty} \mathbb{E}_\lambda(Z_1^2 \mathbb{I}_{[|Z_1|>a]}) = 0$ and assertion (3.11) follows. □

ii) Proof of Theorem B. If we set $f(t) = g(t) - \sigma^2$, assertion (3.9) will follow immediately from the lemma: □

3.14. Lemma. *If (X_n) is a null recurrent or transient chain on K and if f is a measurable and bounded function on K such that $\lim_{|t|\to\infty} f(t) = 0$, then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i) = 0 \ \ a.s.$$

Proof of the lemma. If (X_n) is transient, the result is trivial. If (X_n) is null recurrent, let λ be its invariant measure and let E_m be an increasing sequence of subsets of K such that $\bigcup_m E_m = K$ and $\lambda(E_m) < +\infty$ for all $m \in \mathbb{N}$. For all $k \in \mathbb{N}$, let us consider the function $f_k = \mathbb{I}_{[-k,k]} f$. By the Chacon-Ornstein theorem ([R], p. 123), we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n |f_k(X_i)|}{\sum_{i=1}^n \mathbb{I}_{E_m}(X_i)} = \frac{\langle \lambda, |f_k|\rangle}{\lambda(E_m)} \quad a.s., \tag{3.15}$$

for all fixed k and $m \in \mathbb{N}^*$. But $\lim_{m\to\infty} \lambda(E_m) = +\infty$, and we immediately deduce from (3.15) that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f_k(X_i) = 0 \quad a.s.$$

for every $k \in \mathbb{N}^*$. The result of the lemma follows from the uniform convergence of f_k to f on K as $k \to +\infty$. □

The dominated convergence in (3.9) also implies (3.10). Now in order to prove (3.11) let us denote $b_j = \epsilon\sigma\sqrt{j}$ to simplify notations. We have to consider separately the two cases:

i) Suppose H_3 is satisfied. For every $j \ge 1$, we have

$$Z_j^2 = [(X_j - X_{j-1}) - d(X_{j-1})]^2 \le 2(\xi_j^2 + d^2(X_{j-1})).$$

Then

$$W_j(b_j) \le 2(\xi_j^2 + d^2(X_{j-1}))\mathbb{E}(\mathbb{I}_{[b_j,+\infty[}(|Z_j|)|\mathcal{F}_{j-1}) \tag{3.16}$$

and observe that

$$\begin{aligned}\mathbb{E}(\mathbb{I}_{[b_j,+\infty[}(|Z_j|)|\mathcal{F}_{j-1}) &= \mathbb{E}(\mathbb{I}_{[b_j^2,+\infty[}(Z_j^2)|\mathcal{F}_{j-1}) \\ &\le \mathbb{E}(h_j(Z_j^2)|\mathcal{F}_{j-1}) \\ &\le h_j(\mathbb{E}(Z_j^2|\mathcal{F}_{j-1})) \le h_j(\|g\|_\infty),\end{aligned} \tag{3.17}$$

where h_j is the continuous concave function on $\mathbb{R}_+$ such that $h_j(0) = 0, h_j(x) = 1$ if $x \ge b_j^2$ and h_j linear on $[0, b_j^2]$.

Now (3.11) follows easily from (3.16), (3.17) and $\lim_{j\to\infty} h_j(\|g\|_\infty) = 0$.

ii) Suppose $H_0(2+\epsilon)$ is satisfied for some $\epsilon > 0$. Replacing ϵ by $\frac{\epsilon}{2}$, we can suppose that $H_0(2(1+\epsilon))$ is satisfied. Then let $p = 1+\epsilon$ and let q be such that $1/p + 1/q = 1$. By Hölder's inequality for conditional expectations, we get

$$W_j(b_j) \le \mathbb{E}(Z_j^{2(1+\epsilon)}|\mathcal{F}_{j-1})^{1/p}\mathbb{E}(\mathbb{I}_{[b_j,+\epsilon[}(|Z_j|)|\mathcal{F}_{j-1})^{1/q}. \tag{3.18}$$

The second factor in the right-hand side of (3.18) has already been considered in (3.17); it is dominated by $(h_j(\|g\|_\infty))^{1/q}$. The first factor is bounded (by a constant). Indeed

$$\begin{aligned}\mathbb{E}(Z_j^{2(1+\epsilon)}|\mathcal{F}_{j-1})^{1/p} &\le \mathbb{E}((|X_j - X_{j-1}| + |d(X_{j-1})|)^{2(1+\epsilon)}|\mathcal{F}_{j-1})^{1/p} \\ &\le (2^{1+2\epsilon}(\|c_{2(1+\epsilon)}\|_\infty + \|d\|_\infty^{2(1+\epsilon)}))^{1/p},\end{aligned}$$

where we have used the trivial inequality $(a+b)^{2(1+\epsilon)} \le 2^{1+2\epsilon}(a^{2(1+\epsilon)} + b^{2(1+\epsilon)})$ for $a, b > 0$. Then we obtain (3.11) in the same way as in i). □

Now if $m_1 \neq 0$ and if $d(t)$ converges to m_1 quickly enough, we can restate the CLT in a more usual form.

3.19. Corollary. *Under the hypothesis of Theorem B, if $K = \mathbb{R}_+$ or $\mathbb{N}$, if $m_1 \neq 0$ and if $d(t) - m_1 = 0(|t|^{-\alpha})$ with $\alpha > 1/2$, we have*

$$n^{-1/2}(X_n - nm_1) \xrightarrow{\mathcal{L}} N(0, \sigma^2) \qquad (n \to +\infty).$$

Proof. We can clearly suppose $\alpha \le 1$. Let $\epsilon(t) = d(t) - m_1$. By [G1] we have $X_k \sim km_1$ a.s. But $\lim_{n\to\infty} n^{-1/2}\sum_{k=a}^{n} k^{-\alpha} = 0$ for all fixed $a \ge 1$. This implies

$$\lim_{n\to\infty} n^{-1/2}\sum_{k=1}^{n}\epsilon(X_{k-1}) = 0 \quad a.s.$$

and the result follows immediately from Theorem B. □

3.20. *Remarks.* a) The result of the corollary is true for a more general state space K if we can prove a law of large numbers in the form $\lim_{n\to\infty} n^{-1}X_n = m_1 \neq 0 \quad a.s.$

b) If $m_1 = 0$, the result of the corollary is clearly false (consider for example the simple random walk on $\mathbb{N}$ with reflecting barrier at 0).

c) If X_n is positive recurrent in $\mathbb{R}$, then $X_n/\sqrt{n} \to 0$ in law and Theorem A is a CLT for the (correlated) random variables $d(X_k)$.

4. Application to random walks

Let $(K,*)$ be a commutative hypergroup in the sense of Jewett ([B.H]) such that $K \subset \mathbb{R}$. If $K = \mathbb{R}$ with the usual topology, there is only one hypergroup structure: it is the usual additive group structure $(\mathbb{R},+)$ ([Z2]). On the other hand, on $\mathbb{R}_+$ there are many hypergroup structures and (modulo isomorphism) we know that the convolution always satisfies the following property ([Z2]):

$$\mathrm{Supp}(\delta_x * \delta_y) \subset [|x-y|, x+y] \qquad (\forall x, y \in K). \tag{4.1}$$

(δ_x is the Dirac measure at point x and *Supp* denotes support.) In most known examples of the discrete case (i.e. $K = \mathbb{N}$) the convolution also satisfies condition (4.1). Chébli-Trimèche (C-T) hypergroups (resp. polynomial hypergroups) are typical examples of the continuous case (resp. the discrete case). Let us recall briefly how they are defined.

4.2. **C-T hypergroups.** Let A be an increasing unbounded differentiable function on $\mathbb{R}_+$ such that $A(0) = 0$. We suppose A'/A decreasing on $\mathbb{R}_+^*$, $\lim_{x\to+\infty}(A'/A)(x) = 2\rho \geq 0$ and $(A'/A)(x) = \alpha x^{-1} + B(x)$ in a neighbourhood of $x = 0$, where $\alpha > 0$ and B is a odd C^∞ function on $\mathbb{R}$. Consider the differential operator $L = \partial_x^2 + (A'/A)(x)\partial_x$. We know ([C]) that there is a unique hypergroup structure $(\mathbb{R}_+,*)$ on $\mathbb{R}_+$ with unit δ_0, involution the identity function and such that for every even C^∞ function f on $\mathbb{R}$, the function $u(x,y) = \langle \delta_x * \delta_y, f\rangle$ is the solution of the hyperbolic Cauchy problem $L_x u = L_y u$ with initial conditions $u(x,0) = f(x)$ and $\partial_y u(x,0) = 0$. Moreover the convolution satisfies property (4.1). This hypergroup is called the C-T hypergroup associated to the function A. Its Haar measure is $\lambda(dx) = A(x)dx$. Particular cases are the Bessel-Kingman hypergroup for which $A(x) = x^{2\alpha+1}(\alpha > -1/2)$ and the Jacobi hypergroup for $A(x) = (shx)^{2\alpha+1}(chx)^{2\beta+1}$ $(\alpha \geq \beta > -1/2)$ (see [B.H] for further details).

4.3. **Polynomial hypergroups.** Let p_n, q_n and r_n be three sequences of real numbers such that $p_n > 0, r_n \geq 0, q_{n+1} > 0, q_0 = 0$ and $p_n + r_n + q_n = 1$ for all $n \in \mathbb{N}$. The polynomials (P_n) defined by $P_0 \equiv 1, P_n(x) = x$ and

$$xP_n(x) = q_n P_{n-1}(x) + r_n P_n(x) + p_n P_{n+1}(x) \quad (n \geq 1),$$

are orthogonal polynomials with respect to some positive Radon measure $d\Pi$ on $[-1,1]$. If they have non negative linearization coefficients (i.e., for all $m, n \in \mathbb{N}, P_m P_n = \sum_{r=|m-n|}^{m+n} c(m,n,r)P_r$ and $c(m,n,r) \geq 0$ for all r), we can define a hypergroup structure $(\mathbb{N},*)$ on $\mathbb{N}$ by

$$\delta_m * \delta_n = \sum_{r=|m-n|}^{m+n} c(m,n,r)\delta_r,$$

with unit δ_0, and involution the identity function. It is the polynomial hypergroup with parameters (p_n, q_n, r_n). It has convergent parameters if $\lim_{n\to\infty} p_n = p \in]0,1[$ and $\lim_{n\to\infty} q_n = q \in]0,1[$ exist. Most classical families of orthogonal polynomials on $[-1,1]$ like Jacobi polynomials give rise to a polynomial hypergroup (see [B.H] for further details).

4.4. **ADC hypergroups.** The two previous examples belong to a class of hypergroups we have introduced in [G4]. Let $(K,*)$ satisfy (4.1) and consider the following functions on $K \times K$:

$$d(t;x) = \int_E (u-t)\delta_t * \delta_x(du), \tag{4.4.1}$$

$$c_2(t;x) = \int_E (u-t)^2\delta_t * \delta_x(du). \tag{4.4.2}$$

If for all $x \in K$, the limits $\lim_{|t|\to\infty} d(t;x) = m_1(x)$ and $\lim_{|t|\to\infty} c_2(t;x) = m_2(x)$ exist (in $\mathbb{R}$), we say that $(K,*)$ has the property of Asymptotic Drift for the Convolution (ADC) at infinity. More briefly, we say that $(K,*)$ is a ADC hypergroup. The functions $m_i (i=1,2)$ are the moment functions of the hypergroup. C-T hypergroups (resp. polynomial hypergroups with converging parameters) are ADC hypergroups with moment functions given by ([G4]):

$$m_1(x) = 2\rho \int_0^x (A(t))^{-1} \int_0^t A(\xi)d\xi dt, \tag{4.4.3}$$

$$m_2(x) = \int_0^x (\int_u^x (A(z))^{-1}dz)(2+4\rho m_1(u))A(u)du \tag{4.4.4}$$

(resp.

$$m_1(n) = (p-q)P_n'(1), \tag{4.4.5}$$

$$m_2(n) = (p+q)P_n'(1) + (p-q)P_n''(1)). \tag{4.4.6}$$

4.5. **Random walks.** A random walk with law $\mu \in M_1(K)$ on the hypergroup $(K,*)$ is a homogeneous Markov chain on K with Markovian kernel given by

$$P(x,dy) = (\delta_x * \mu)(dy) \tag{4.5.1}$$

(see [B.H] or [G2]). For example random walks on the Bessel-Kingman hypergroup have been studied in [K].

If μ has finite second moment (i.e., $\int_K x^2\mu(dx) < +\infty$) we will say more briefly that the random walk (X_n) has a finite second moment.

4.5.2. Proposition. *Every random walk with finite second moment on a ADC hypergroup $(K,*)$ satisfies the hypothesis H_1 and H_2 of section 1.*

Proof. With the notations of section 1, the functions d and c_2 are given by

$$d(t) = \int_K (u-t)\delta_t * \mu(du)$$

and

$$c_2(t) = \int_K (u-t)^2\delta_t * \mu(du).$$

Using the bilinearity of the convolution and notations (4.4.1) and (4.4.2), we get

$$d(t) = \int_K d(t;x)\mu(dx) \quad \text{and} \quad c_2(t) = \int_K c_2(t;x)\mu(dx).$$

Note that these functions are well-defined because by condition (4.1) we have $|d(t;x)| \le |x|$ and $c_2(t;x) \le x^2$ for all $t \in K$. These facts allow us to use the dominated convergence theorem to obtain immediately $\lim_{t\to\infty} d(t) = \int_K m_1(x)\mu(dx)$ and $\lim_{t\to\infty} c_2(t) = \int_K m_2(x)\mu(dx)$, and the proof follows. □

4.5.3. Proposition. *Every random walk (X_n) with finite second moment on a ADC hypergroup $(K,*)$ satisfies condition H_3 of section 1. More precisely we can construct (X_n) on an adequate probability space in order that H_3 be satisfied.*

Proof. We will use the tool of randomized sums introduced in [K] and [B] and formalized in [Z2] for the case of random walks on hypergroups. For simplicity we use the approach of [G2] (p. 142). Let (ξ_i) be a sequence of i.i.d K valued random variables with common law μ (= the law of the random walk) and let (Z_i) be an independent sequence of i.i.d uniformly distributed random variables on [0,1]. Fix $x \in K$ and define inductively (X_n) by $X_0 = x$ and for every $n \geq 1$:

$$X_n(\omega) = inf\{t; \delta_{X_{n-1}(\omega)} * \delta_{\xi_n(\omega)}(]-\infty, t]) \geq Z_n(\omega)\}. \tag{4.5.4}$$

By construction and according to property (4.1) of the convolution, we have that $|X_n - X_{n-1}| \leq \xi_n$ *a.s.* Moreover it follows from [G2] (section 2.2) that the process (X_n) defined by (4.5.4) is a random walk of law μ on $(K,*)$ starting from x. □

In order to apply the CLT to random walks we need some conditions on μ.

4.5.5. Definitions. a) The probability measure μ on $(K,*)$ is *adapted* if the smallest closed subhypergroup generated by *supp*μ is equal to K.

b) The probability measure μ is *spread out* with respect to the Haar measure λ of $(K,*)$ if there exists a p convolution power of μ non singular to λ (i.e., there exists $p \in \mathbb{N}^*$, a constant $c > 0$ and an open set $V \subset K$ such that $\mu^{*p} \geq c\mathbb{I}_V\lambda$).

4.5.6. Proposition (CLT for random walks)**.** *Let (X_n) be a random walk with law μ on a ADC hypergroup $(K,*)$. Suppose μ has a finite second moment and is adapted and spread out on K. Then (X_n) satisfies Theorem B.*

Proof. By [G.G] (Theorem 3.6) we know that (X_n) is either transient or Harris recurrent with respect to Haar measure λ. So by Propositions 4.5.2 and 4.5.3 the hypothesis of the CLT of section 2 are satisfied. To finish the proof we have to verify that (X_n) is never positive recurrent on ADC hypergroups, i.e., Haar measure has always infinite mass. But this is an easy consequence of the structural hypothesis 4.1. □

4.5.7. Remarks. a) The adaptation hypothesis on μ is a sort of irreducibility condition for the Markov chain (X_n); the spread out hypothesis ensures good ergodic properties for (X_n) (see [G.G] for further details).

b) The probability measure μ is spread out in particular when some convolution power of μ has a density with respect to Haar measure. For example, in a C-T hypergroup, if for all x and $y \in \mathbb{R}_+$ the measure $\delta_x * \delta_y$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+$, i.e., if there exists a positive measurable function $W(x,y,t)$ such that

$$\delta_x * \delta_y(dt) = W(x,y,t)dt, \tag{4.5.8}$$

then $\delta_x * \delta_y$ is also absolutely continuous with respect to Haar measure $\lambda(dx) = A(x)dx$ (indeed $A(x) > 0$). It is then easily seen that for every probability measure $\mu \in M_1(\mathbb{R}_+)$, μ^{*2} is absolutely continuous with respect to λ and therefore μ is spread out. Property (4.5.8) is satisfied for Bessel-Kingman hypergroups [K] and Jacobi hypergroups [F.K], and more generally if A satisfies some regularity conditions in a neighbourhood of zero (see [B.S] for details).

c) In the discrete case (i.e., $K = \mathbb{N}$) the spread out hypothesis is automatically satisfied for every probability measure ([G.G]).

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