

## THE FURUTA INEQUALITY WITH NEGATIVE POWERS

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ABSTRACT. Let  $A, B \in B(H)$  be bounded linear operators on a Hilbert space  $H$  satisfying  $O \leq B \leq A$ . Furuta showed the operator inequality  $(A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}}$  as long as positive real numbers  $p, q, r$  satisfy  $p + 2r \leq (1 + 2r)q$  and  $1 \leq q$ . In this paper, we show this inequality is valid if negative real numbers  $p, q, r$  satisfy a certain condition. Also, we investigate the optimality of that condition.

### 1. INTRODUCTION

Let  $A, B$  be bounded linear operators on a Hilbert space  $H$ . The following operator inequality is well-known as the Heinz inequality. (In the case where  $p = \frac{1}{2}$ , the inequality is called the Löwner inequality, and it is known that there are counterexamples in the case where  $p > 1$ . See [1] for  $p = 2$  and [12, p. 465] for  $p > 1$ .)

**Proposition 1** ([10],[11]).  $O \leq B \leq A$  implies  $B^p \leq A^p$  for all  $p \in [0, 1]$ .

Concerning the Heinz inequality, Chan and Kwong [1] conjectured that  $O \leq B \leq A$  will imply  $B^2 \leq (BA^2B)^{\frac{1}{2}}$  and Furuta gave an affirmative answer as follows.

**Proposition 2** ([4]).  $O \leq B \leq A$  implies

$$(1) \quad (A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}}$$

and

$$(2) \quad B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}}$$

as long as positive real numbers  $p, q, r$  satisfy

$$(3) \quad p + 2r \leq (1 + 2r)q \quad \text{and} \quad 1 \leq q.$$

The range of  $p, q, r$  which satisfy the condition (3) is as in Figure 1 and, in [13], the author proved that the condition (3) is optimal for the validity of the Furuta inequality by using  $2 \times 2$  matrices.

The Furuta inequality has many applications. In the operator mean theory, the Furuta inequality was generalized to the grand Furuta inequality [8] and it extends the Ando-Hiai log-majorization theory. Also, the Furuta inequality was used in the  $p$ -hyponormal operator theory ([2], [9]) and the relative entropy theory ([6], [7]).

Recently, it is known that the Furuta inequality is valid for some negative real numbers  $p, q, r$  by Yoshino [14] and Fujii, Furuta, Kamei [3]. And, in this paper,

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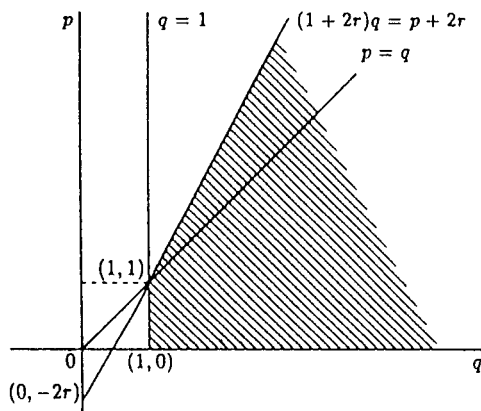


FIGURE 1

we shall consider the range of  $p, q, r$  for which the Furuta inequality is valid and investigate its optimality by using the same technique as in [13].

## 2. RESULTS

Let  $A, B \in B(H)$  be invertible operators on  $H$  satisfying  $O \leq B \leq A$ . In the Furuta inequality, if  $p = 0$ , then  $A^0 = B^0 = I$  and the inequalities (1) and (2) become the equalities. Also, if  $r = 0$ , then the inequalities (1) and (2) reduce to the Löwner-Heinz inequality. Hence we may assume  $p \neq 0$  and  $r \neq 0$  in this paper. Since  $O \leq B \leq A$  implies  $O \leq A^{-1} \leq B^{-1}$ , the validity of (1) is equivalent to the validity of (2). By a simple observation, the problem to find all real numbers  $p, q, r$  guaranteeing the validity of the Furuta inequality is reduced to the case where  $0 < p, 0 < q$  and  $r < 0$ .

**Theorem 3.** *Let  $A, B \in B(H)$  be invertible operators satisfying  $O \leq B \leq A$ . Let  $0 < p \leq 1, 0 < q \leq 1$  and  $-1 \leq 2r < 0$ . Then*

$$(1) \quad (A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}}$$

as long as real numbers  $p, q, r$  satisfy

$$(4) \quad -2r(1-q) \leq p \leq q - 2r(1-q)$$

and

$$(5) \quad \frac{-2r(1-q) - q}{1-2q} \leq p \leq \frac{-2r(1-q)}{1-2q}.$$

We need the following lemma to show Theorem 3.

**Lemma 4** ([5]). *Let  $A, B \in B(H)$  be positive operators and let  $\lambda$  be a real number. Then*

$$(ABA)^\lambda = AB^{\frac{1}{2}} \left( B^{\frac{1}{2}} A^2 B^{\frac{1}{2}} \right)^{\lambda-1} B^{\frac{1}{2}} A.$$

*Proof of Theorem 3.* Let  $0 < p \leq 1, 0 < q \leq 1$  and  $-1 \leq 2r < 0$ .

First, assume  $\frac{1}{2} \leq q \leq 1$  and  $p, q, r$  satisfy (4). ( We remark that  $p, q, r$  satisfy (5) automatically in this case.) Then  $0 \leq A^{2r} \leq B^{2r}$  by the Heinz inequality. Since

$$0 \leq \frac{1}{q} - 1 \leq 1$$

and

$$0 \leq \frac{p+2r}{q} - 2r \leq 1,$$

we have

$$\begin{aligned} (A^r B^p A^r)^{\frac{1}{q}} &= A^r B^{\frac{p}{2}} \left( B^{\frac{p}{2}} A^{2r} B^{\frac{p}{2}} \right)^{\frac{1}{q}-1} B^{\frac{p}{2}} A^r \\ &\leq A^r B^{\frac{p}{2}} \left( B^{\frac{p}{2}} B^{2r} B^{\frac{p}{2}} \right)^{\frac{1}{q}-1} B^{\frac{p}{2}} A^r = A^r B^{\frac{p+2r}{q}-2r} A^r \\ &\leq A^r A^{\frac{p+2r}{q}-2r} A^r = A^{\frac{p+2r}{q}} \end{aligned}$$

by Lemma 4 and the Heinz inequality.

Second, assume  $\frac{1}{3} \leq q < \frac{1}{2}$  and  $p, q, r$  satisfy (4) and (5). Let

$$p' = 2r, q' = \frac{q}{1-q}, r' = \frac{p}{2}$$

and let

$$p'' = -p', q'' = q', r'' = -r'.$$

Then

$$0 < p'' \leq 1, \frac{1}{2} \leq q'' < 1, -\frac{1}{2} \leq r'' < 0$$

and  $p'', q'', r''$  satisfy

$$-2r''(1+q'') - q'' \leq p'' \leq -2r''(1+q'')$$

and

$$-2r''(1-q'') \leq p'' \leq q'' - 2r''(1-q'').$$

Hence, by the same arguments as in the case where  $\frac{1}{2} \leq q \leq 1$ , we have

$$\left( A^{r''} B^{p''} A^{r''} \right)^{\frac{1}{q''}} \leq A^{\frac{p''+2r''}{q''}},$$

and

$$(6) \quad A^{\frac{p'+2r'}{q'}} \leq \left( A^{r'} B^{p'} A^{r'} \right)^{\frac{1}{q'}}.$$

It is known that (6) implies

$$\left( B^{r'} A^{p'} B^{r'} \right)^{\frac{1}{q'}} \leq B^{\frac{p'+2r'}{q'}}$$

and hence

$$\left( B^{\frac{p}{2}} A^{2r} B^{\frac{p}{2}} \right)^{\frac{1}{q}-1} \leq \left( B^{\frac{p}{2}} B^{2r} B^{\frac{p}{2}} \right)^{\frac{1}{q}-1}.$$

Since  $0 \leq \frac{p+2r}{q} - 2r \leq 1$ , we have

$$(A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}}$$

by the same arguments as in the case where  $\frac{1}{2} \leq q \leq 1$ .

Similarly we can show the inequality (1) in the case where  $\frac{1}{n+1} \leq q < \frac{1}{n}$  ( $n = 3, 4, 5, \dots$ ) and  $p, q, r$  satisfy (4) and (5) because

$$\frac{-2r(1-q)-q}{1-2q} \leq -2r(1+q)-q < -2r(1+q) \leq \frac{-2r(1-q)}{1-2q}$$

if  $0 < 2q \leq 1$  and  $-1 \leq 2r < 0$ . Thus the proof is complete.

Theorem 3 shows the validity of the Furuta inequality as long as  $p, q, r$  satisfy any one of the following conditions (I), (II), (III), (IV).

Condition (I).  $-\frac{1}{4} \leq r < 0$  and

$$\begin{cases} q \geq \frac{p+2r}{2r}, & \text{if } 0 < p < -2r, \\ q > 0, & \text{if } p = -2r, \\ q \geq \frac{p+2r}{2p+2r}, & \text{if } -2r < p \leq \frac{1}{2}, \\ q \geq \frac{p+2r}{1+2r}, & \text{if } \frac{1}{2} < p \leq 1. \end{cases}$$

Condition (II).  $-\frac{1}{2} < r < -\frac{1}{4}$  and

$$\begin{cases} q \geq \frac{p+2r}{2r}, & \text{if } 0 < p < \frac{1}{2}, \\ q \geq \frac{p+2r}{2p+2r-1}, & \text{if } \frac{1}{2} \leq p < -2r, \\ q > 0, & \text{if } p = -2r, \\ q \geq \frac{p+2r}{1+2r}, & \text{if } -2r < p \leq 1. \end{cases}$$

Condition (III).  $r = -\frac{1}{2}$  and

$$\begin{cases} q \geq \frac{p-1}{-1}, & \text{if } 0 < p < \frac{1}{2}, \\ q \geq \frac{1}{2}, & \text{if } \frac{1}{2} \leq p < 1, \\ q > 0, & \text{if } p = 1. \end{cases}$$

Condition (IV).  $r < -\frac{1}{2}$  and

$$q \geq 1, \quad \text{if } 0 < p \leq 1.$$

*Remark.* Yoshino [14] showed the validity of the inequality (1) as long as  $p, q, r$  satisfy  $-1 < 2r < 0, 1 \leq 2q, (1+2r)q \leq p+2r, 0 < p \leq 1$  or the condition (IV). Moreover, Fujii, Furuta, Kamei [3] showed the validity of the inequality (1) as long as  $p, q, r$  satisfy  $-1 < 2r < 0, 1 \leq 2p \leq 2, 0 \leq p+2r, (1+2r)q \leq p+2r$ .

In the sequel, we shall show that the condition (I) with  $0 < p \leq -2r, \frac{1}{2} \leq p \leq 1$  and the conditions (II), (III), (IV) are optimal. Unfortunately, we remark that this is a partial answer for our problem and it is still an open question whether the Furuta inequality is valid as long as  $p, q, r$  satisfy

$$-\frac{1}{4} < r < 0, \quad -2r < p < \frac{1}{2}, \quad \frac{p+2r}{1+2r} \leq q < \frac{p+2r}{2p+2r}.$$

For our purpose, we need the following.

**Lemma 5** ([13]). *Let  $a, b, d, \theta \in \mathbf{R}$  satisfy  $0 < a+b, ab = d^2$  where  $\mathbf{R}$  denotes the set of all real numbers and let*

$$S = \begin{pmatrix} a & de^{-i\theta} \\ de^{i\theta} & b \end{pmatrix}.$$

Then

$$S^p = (a + b)^{p-1}S \quad \text{for } 0 < p.$$

Now we investigate the optimality of the conditions (I) with  $0 < p \leq -2r, \frac{1}{2} \leq p \leq 1$ , (II), (III), (IV) by 6 steps.

Step 1. (I)  $0 < p \leq -2r$ , (II)  $0 < p < \frac{1}{2}$ , (III)  $0 < p < \frac{1}{2}$ .

We show that if  $p, q, r$  satisfy

$$-\frac{1}{2} \leq r < 0, 0 < p < -2r, 0 < q < \frac{p+2r}{2r},$$

then there exist invertible operators  $A, B \in B(\mathbf{R}^2)$  with  $O \leq B \leq A$  which do not satisfy the inequality (1).

Let

$$A = \begin{pmatrix} a & \sqrt{\varepsilon(a-b-\delta)} \\ \sqrt{\varepsilon(a-b-\delta)} & b + \varepsilon + \delta \end{pmatrix}$$

and let

$$B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix},$$

where

$$0 < b < 1 < a, 0 < \delta = \frac{1-b}{a-1}\varepsilon$$

as in [13]. Then  $O \leq B \leq A$ . To show the assertion by contradiction, we assume the inequality (1). We remark

$$a^{\frac{p+2r}{q}}, a^{\frac{2r}{q}} \rightarrow 0 \quad (a \rightarrow \infty).$$

Let

$$U = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} \sqrt{a-b-\delta} & \sqrt{\varepsilon} \\ \sqrt{\varepsilon} & -\sqrt{a-b-\delta} \end{pmatrix},$$

where

$$\gamma = a - b + \varepsilon - \delta.$$

Then  $U$  is unitary and

$$U^*AU = \begin{pmatrix} a + \varepsilon & 0 \\ 0 & b + \delta \end{pmatrix}$$

and hence, by (1) (see [13]), we have

$$\gamma^{-\frac{1}{q}} \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix}^{\frac{1}{q}} \leq \begin{pmatrix} (a + \varepsilon)^{\frac{p+2r}{q}} & 0 \\ 0 & (b + \delta)^{\frac{p+2r}{q}} \end{pmatrix},$$

where

$$\begin{aligned} A_1 &= (a + \varepsilon)^{2r}(a - b - \delta + \varepsilon b^p), \\ A_2 &= (b + \delta)^{2r}(\varepsilon + b^p(a - b - \delta)), \\ A_3 &= (a + \varepsilon)^r(b + \delta)^r(1 - b^p)\sqrt{\varepsilon(a - b - \delta)}. \end{aligned}$$

Let

$$V = \frac{1}{\sqrt{A_2 - A_1 + 2\varepsilon_1}} \begin{pmatrix} \sqrt{\varepsilon_1} & \sqrt{A_2 - A_1 + \varepsilon_1} \\ \sqrt{A_2 - A_1 + \varepsilon_1} & -\sqrt{\varepsilon_1} \end{pmatrix},$$

where

$$2\varepsilon_1 = -A_2 + A_1 + \sqrt{(A_2 - A_1)^2 + 4A_3^2}.$$

Then  $V$  is unitary and

$$V^* \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix} V = \begin{pmatrix} A_2 + \varepsilon_1 & 0 \\ 0 & A_1 - \varepsilon_1 \end{pmatrix}.$$

Hence

$$\begin{aligned} & \varepsilon_1 \left( (A_2 + \varepsilon_1)^{\frac{1}{q}} - (a + \varepsilon)^{\frac{p+2r}{q}} \gamma^{\frac{1}{q}} \right) \left( (b + \delta)^{\frac{p+2r}{q}} \gamma^{\frac{1}{q}} - (A_1 - \varepsilon_1)^{\frac{1}{q}} \right) \\ & \leq (A_2 - A_1 + \varepsilon_1) \left( (a + \varepsilon)^{\frac{p+2r}{q}} \gamma^{\frac{1}{q}} - (A_1 - \varepsilon_1)^{\frac{1}{q}} \right) \left( (b + \delta)^{\frac{p+2r}{q}} \gamma^{\frac{1}{q}} - (A_2 + \varepsilon_1)^{\frac{1}{q}} \right) \end{aligned}$$

and

$$\begin{aligned} (7) \quad & q(1 - a^{-1})(1 - b^p)^2 \left( 1 - a^{\frac{p+2r}{q}} b^{-\frac{p+2r}{q}} \right) \left( 1 - a^{\frac{2r}{q}} b^{-\frac{p+2r}{q}} \right) \\ & \leq a^{\frac{p+2r}{q} - 2r} b^{-\frac{p+2r}{q} + 2p + 2r - 1} (1 - a^{2r} b^{-p-2r}) \left( 1 - a^{-\frac{p}{q}} \right) \\ & \quad \times \{ p(1 - b)(1 - a^{-1}b)(1 - a^{2r} b^{-p-2r}) \\ & \quad \quad - b^{1-p}(1 - b^p)(1 - a^{-1})(1 - a^{2r} b^{-2r}) \} \end{aligned}$$

by similar arguments as in [13]. Since  $\frac{p+2r}{q} - 2r < 0$ , we have

$$0 < q(1 - b^p)^2 \leq 0$$

by letting  $a \rightarrow \infty$ . This is a contradiction.

Step 2. (I)  $\frac{1}{2} \leq p \leq 1$ , (II)  $-2r < p \leq 1$ .

We show that if  $p, q, r$  satisfy

$$-\frac{1}{2} \leq r < 0, -2r < p \leq 1, 0 < q < \frac{p + 2r}{1 + 2r},$$

then there exist invertible operators  $A, B \in B(\mathbf{R}^2)$  with  $O \leq B \leq A$  which do not satisfy the inequality (1). (We remark that, in the condition (I), if  $-\frac{1}{4} < r < 0$ , then  $\{p \in \mathbf{R} \mid \frac{1}{2} \leq p \leq 1\} \subset \{p \in \mathbf{R} \mid -2r < p \leq 1\}$ , and if  $r = -\frac{1}{4}, p = \frac{1}{2}$ , then the inequality (1) is valid for all  $q > 0$ .) We remark that if the inequality (1) is valid for all invertible operators  $A, B \in B(H)$  satisfying  $O \leq B \leq A$ , then the inequality (1) is valid for all operators  $A, B \in B(H)$  satisfying  $O \leq B \leq A$ .

Let

$$A = \begin{pmatrix} 2 & 2\sqrt{c(1-c)} \\ 2\sqrt{c(1-c)} & 4c \end{pmatrix},$$

where  $0 < c < 1$  and let

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then  $O \leq B \leq A$ . To show the assertion by contradiction, we assume the inequality (1). Let

$$V = \begin{pmatrix} \sqrt{1-c} & \sqrt{c} \\ \sqrt{c} & -\sqrt{1-c} \end{pmatrix}.$$

Then  $V$  is unitary and

$$V^*AV = \begin{pmatrix} 2 + 2c & 0 \\ 0 & 2c \end{pmatrix}.$$

By (1), we have

$$((V^*AV)^r V^*B^pV (V^*AV)^r)^{\frac{1}{q}} \leq (V^*AV)^{\frac{p+2r}{q}}$$

and

$$\begin{aligned} & \left( \begin{array}{cc} (2+2c)^{2r}(1-c) & (2+2c)^r(2c)^r\sqrt{c(1-c)} \\ (2+2c)^r(2c)^r\sqrt{c(1-c)} & (2c)^{2r}c \end{array} \right)^{\frac{1}{q}} \\ & \leq \begin{pmatrix} (2+2c)^{\frac{p+2r}{q}} & 0 \\ 0 & (2c)^{\frac{p+2r}{q}} \end{pmatrix} \end{aligned}$$

and, by Lemma 5, we have

$$\begin{aligned} & \delta^{-1} \left( \begin{array}{cc} (2+2c)^{2r}(1-c) & (2+2c)^r(2c)^r\sqrt{c(1-c)} \\ (2+2c)^r(2c)^r\sqrt{c(1-c)} & (2c)^{2r}c \end{array} \right) \\ & \leq \begin{pmatrix} (2+2c)^{\frac{p+2r}{q}} & 0 \\ 0 & (2c)^{\frac{p+2r}{q}} \end{pmatrix} \end{aligned}$$

and hence

$$0 \leq \begin{pmatrix} \delta(2+2c)^{\frac{p+2r}{q}} - (2+2c)^{2r}(1-c) & -(2+2c)^r(2c)^r\sqrt{c(1-c)} \\ -(2+2c)^r(2c)^r\sqrt{c(1-c)} & \delta(2c)^{\frac{p+2r}{q}} - (2c)^{2r}c \end{pmatrix},$$

where

$$\delta^{-1} = ((2+2c)^{2r}(1-c) + (2c)^{2r}c)^{\frac{1}{q}-1}.$$

Therefore

$$0 \leq \delta^2(2+2c)^{\frac{p+2r}{q}}(2c)^{\frac{p+2r}{q}} - \delta(2c)^{\frac{p+2r}{q}}(2+2c)^{2r}(1-c) - \delta(2+2c)^{\frac{p+2r}{q}}(2c)^{2r}c,$$

$$(2c)^{\frac{p+2r}{q}}(2+2c)^{2r}(1-c) + (2+2c)^{\frac{p+2r}{q}}(2c)^{2r}c \leq \delta(2+2c)^{\frac{p+2r}{q}}(2c)^{\frac{p+2r}{q}}$$

and

$$\begin{aligned} & 2^{\frac{p+2r}{q}}(2+2c)^{2r}(1-c)c^{\frac{p+2r}{q}-1-2r} + (2+2c)^{\frac{p+2r}{q}}2^{2r} \\ & \leq \{(2+2c)^{2r}(1-c) + 2^{2r}c^{1+2r}\}^{1-\frac{1}{q}}(2+2c)^{\frac{p+2r}{q}}2^{\frac{p+2r}{q}}c^{\frac{p+2r}{q}-1-2r}. \end{aligned}$$

Since  $0 < \frac{p+2r}{q} - 1 - 2r$ , we have

$$0 < 2^{\frac{p+2r}{q}}2^{2r} \leq 0$$

by letting  $c \rightarrow +0$ . This is a contradiction. (We remark that step 2 is also valid for the condition (I) with  $-2r < p \leq 1$ .)

Step 3. (II)  $\frac{1}{2} \leq p < -2r$ , (III)  $\frac{1}{2} \leq p < 1$ .

We show that if  $p, q, r$  satisfy

$$-\frac{1}{2} \leq r < -\frac{1}{4}, \frac{1}{2} \leq p < -2r, 0 < q < \frac{p+2r}{2p+2r-1},$$

then there exist invertible operators  $A, B \in B(\mathbf{R}^2)$  with  $0 \leq B \leq A$  which do not satisfy the inequality (1).

Define  $A, B$  as in step 1. To show the assertion by contradiction, we assume the inequality (1). Similarly as in step 1, we can show the inequality (7). Since  $0 < -\frac{p+2r}{q} + 2p + 2r - 1$ , we have

$$0 < q(1 - a^{-1}) \leq 0$$

by letting  $b \rightarrow +0$  in (7). This is a contradiction.

Step 4. (I)  $1 < p$ , (II)  $1 < p$ , (III)  $1 < p$ .

We show that if  $p, q, r$  satisfy

$$-\frac{1}{2} \leq r < 0, 1 < p, 0 < q,$$

then there exist invertible operators  $A, B \in B(\mathbf{R}^2)$  with  $O \leq B \leq A$  which do not satisfy the inequality

$$(3) \quad B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}},$$

or, equivalently,

$$(8) \quad I \geq B^{\frac{p+2r}{2q}} (B^{-r} A^{-p} B^{-r})^{\frac{1}{q}} B^{\frac{p+2r}{2q}}.$$

We remark that if the inequality (8) is valid for all invertible operators  $A, B \in B(H)$  satisfying  $O \leq B \leq A$ , then the inequality (8) is valid for all operators  $A, B \in B(H)$  satisfying  $O \leq B \leq A$ . Define  $A, B, V$  as in step 2. To show the assertion by contradiction, we assume the inequality (8). Since

$$I \geq B^{\frac{p+2r}{2q}} (B^{-r} V V^* A^{-p} V V^* B^{-r})^{\frac{1}{q}} B^{\frac{p+2r}{2q}},$$

we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \geq \{(1-c)(2+2c)^{-p} + c(2c)^{-p}\}^{\frac{1}{q}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence

$$1 \geq (1-c)(2+2c)^{-p} + c(2c)^{-p} = (1-c)(2+2c)^{-p} + 2^{-p}c^{1-p}.$$

Since  $1-p < 0$ , we have  $1 \geq \infty$  by letting  $c \rightarrow +0$ . This is a contradiction.

Step 5. (IV)  $1 < p$ .

We show that if  $p, q, r$  satisfy

$$1 < p, 0 < q, r < 0,$$

then there exist invertible operators  $A, B \in B(\mathbf{R}^2)$  with  $O \leq B \leq A$  which do not satisfy the inequality (1) or (2).

First, let  $0 < p + 2r$ . Then the proof is similar to step 2.

Second, let  $p + 2r = 0$ . Then

$$\begin{aligned} (A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}} &\iff (A^r B^{-2r} A^r)^{\frac{1}{q}} \leq A^0 = I \\ &\iff A^r B^{-2r} A^r \leq I \\ &\iff B^{-2r} \leq A^{-2r}. \end{aligned}$$

The last inequality is valid if and only if  $-1 \leq 2r \leq 0$ .

Last, let  $p + 2r < 0$ . We remark that the inequality (2) is equivalent to

$$(9) \quad B^{\frac{-p-2r}{q}} \geq (B^{-r} A^{-p} B^{-r})^{\frac{1}{q}}.$$

Define  $A, B, V$  as in step 2. To show the assertion by contradiction, we assume the inequality (9). Since

$$B^{-\frac{p-2r}{q}} \geq (B^{-r}VV^*A^{-p}VV^* = B^{-r})^{\frac{1}{q}},$$

we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq \{(1-c)(2+2c)^{-p} + c(2c)^{-p}\}^{\frac{1}{q}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The rest of the proof is similar to step 4.

Step 6. (IV)  $0 < p \leq 1$ .

We show that if  $p, q, r$  satisfy

$$0 < p \leq 1, 0 < q, r < 0,$$

then there exist invertible operators  $A, B \in B(\mathbf{R}^2)$  with  $O \leq B \leq A$  which do not satisfy the inequality (1).

Define  $A, B$  as in step 2. To show the assertion by contradiction, we assume the inequality (1). Then, similarly as in step 2, we have

$$\begin{aligned} & 2^{\frac{p+2r}{q}}(2+2c)^{2r}(1-c) + (2+2c)^{\frac{p+2r}{q}}2^{2r}c^{1+2r-\frac{p+2r}{q}} \\ & \leq \{(2+2c)^{2r}(1-c) + 2^{2r}c^{1+2r}\}^{1-\frac{1}{q}}(2+2c)^{\frac{p+2r}{q}}2^{\frac{p+2r}{q}} \\ & = \{(2+2c)^{2r}(1-c)c^{-1-2r} + 2^{2r}\}^{1-\frac{1}{q}}c^{(1+2r)(1-\frac{1}{q})}(2+2c)^{\frac{p+2r}{q}}2^{\frac{p+2r}{q}}. \end{aligned}$$

Since

$$(1+2r)\left(1-\frac{1}{q}\right) < 0 < 1+2r-\frac{p+2r}{q},$$

we have

$$0 < 2^{\frac{p+2r}{q}}2^{2r} \leq 0$$

by letting  $c \rightarrow +0$ . This is a contradiction.

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and  $B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$  with  $\lambda_i > 0, \lambda_1 \neq \lambda_2$  and  $\lambda_1^{-1} + \lambda_2^{-1} = 2$ .

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