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EXISTENCE OF MANY POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS ON AN ANNULUS

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Abstract. This paper is concerned with multiplicity of positive nonradial solutions of a nonlinear eigenvalue problem on an expanding annulus domain with a fixed width in $\mathbf{R}^N$ with $N \geq 4$. For $0 < \lambda < \pi^2$, we show that the number of nonrotationally equivalent nonradial solutions tends to infinity as the inner radius of the domain tends to infinity.

In this paper, we study the existence of positive solutions of semilinear elliptic equations on an annulus in $\mathbf{R}^N$, $N \geq 4$:

$$\Omega_r := \{x \in \mathbf{R}^N \ : \ r < |x| < r+1\},$$

where $r > 0$.

We consider the problem

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1}, u > 0 & \text{in } \Omega_r, \\ u = 0 & \text{on } \partial\Omega_r \end{cases} \tag{1}$$

where $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent.

Our main result is the following.

Theorem 1. *For every $0 < \lambda < \pi^2$ and $n \geq 1$, there exists $R(\lambda, n)$ such that for $r > R(\lambda, n)$ equation (1) has at least n nonrotationally equivalent nonradial solutions.*

For the corresponding subcritical problem (i.e., 2^* is replaced by some p with $2 < p < 2^*$) the following result, which generalizes an earlier result of Coffman for $N = 2$ ([C]), is due to Y. Y. Li ([Ly]) and Suzuki ([S]) (see also [K] and [Ls]): when $\lambda = 0$, the number of nonrotationally equivalent nonradial solutions of the equation tends to infinity as r tends to infinity. For the critical problem (1) with $\lambda = 0$, it is also proved in [Ly] (see also [Ls] for some extensions) that there exists $r_0 > 0$ such that, for $r > r_0$, (1) has at least $[\frac{N}{2}] - 1$ nonrotationally equivalent nonradial solutions.

Following the method introduced by Coffman in [C], we shall minimize the Rayleigh quotient on some spaces of invariant functions in $H_0^1(\Omega_r)$.

Let G be a closed subgroup of $\mathbf{O(N)}$. The action of G on $H_0^1(\Omega_r)$ is defined by

$$gu(x) := u(g^{-1}x).$$

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The subspace of invariant functions is defined by

$$H^1_{0,G}(\Omega_r) := \{u \in H^1_0(\Omega_r) \ : \ gu = u, \forall u \in G\}.$$

For any $k \geq 1$, let H_k be a representation of $\mathbf{Z}_k$ in $\mathbf{O(2)}$ defined by

$$gz := e^{\frac{i2\pi}{k}} z, \ \text{ for } \ z \in \mathbf{C} \simeq \mathbf{R}^2.$$

Then $H_k = \{g^i \ : \ i = 0, 1, ..., k-1.\}$. Define

$$G_k := H_k \times \mathbf{O(N\text{-}2)},$$

$$G_\infty := \mathbf{O(2)} \times \mathbf{O(N\text{-}2)}.$$

Hence, G_k has the canonical product action on $\mathbf{R}^N$.

We define also, for $k = 1, 2, ..., \infty$,

$$S_\lambda(r,k) := \inf_{u \in H^1_{0,G_k}(\Omega_r), |u|_{2^*}=1} \int_{\Omega_r} |\nabla u|^2 - \lambda u^2,$$

where $|u|_{2^*}$ is the L^{2^*}-norm of u.

The approach of [Ly] and [Ls] in the critical case is to choose G in such a way that the embedding $H^1_{0,G}(\Omega_r) \subset L^{2^*}(\Omega_r)$ is compact. However, it seems impossible to obtain many solutions in this way. In order to overcome the difficulty, we use an idea in [BN] by Brezis and Nirenberg and a concentration-compactness result proved by P.L. Lions in [Lp].

Lemma 1. *For every* $0 < \lambda < \pi^2$,

$$\lim_{r \to \infty} S_\lambda(r, \infty) = \infty.$$

Proof. For every $r > 0$ the Poincaré inequality

$$\int_{\Omega_r} |\nabla u|^2 \geq \lambda_0(r) \int_{\Omega_r} u^2$$

holds on $H^1_0(\Omega_r)$ and

$$\lim_{r \to \infty} \lambda_0(r) = \pi^2.$$

Hence there exists $\delta > 0$ and $r_0 > 0$ such that, for $r > r_0$ and $u \in H^1_0(\Omega_r)$,

$$\int_{\Omega_r} |\nabla u|^2 - \lambda u^2 \geq (1 - \frac{\lambda}{\lambda_0(r)}) \int_{\Omega_r} |\nabla u|^2 \geq \delta \int_{\Omega_r} |\nabla u|^2.$$

By Lemma 1.2′ of [Ly],

$$\lim_{r \to \infty} \inf_{u \in H^1_{0,G_\infty}(\Omega_r), |u|_{2^*}=1} \int_{\Omega_r} |\nabla u|^2 = \infty.$$

□

Lemma 2. *For every* $0 < \lambda < \pi^2$ *and* $1 \leq k < \infty$,

$$S_\lambda(r,k) = O(1), \quad \text{as } \ r \to \infty.$$

Proof. This result follows directly from Lemma 1.3 in [Ly]. □

Lemma 3. *For every* $0 < \lambda < \pi^2$ *there exists* $r(\lambda) > 0$ *such that for every* $1 \leq k \leq \infty$ *and* $r > r(\lambda)$, $S_\lambda(r,k)$ *is attained by some function* $u \in H^1_{0,G_k}(\Omega_r)$.

Proof. If $k = \infty$, the result is a consequence of the compact embedding $H^1_{0,G_\infty}(\Omega_r) \subset L^\infty(\Omega_r)$. For $1 \le k < \infty$, we combine some arguments from Brezis-Nirenberg [BN] and P.L. Lions [Lp].

As in Lemma 1, there exists $\delta > 0$ and $r(\lambda) > 0$ such that, for $r > r(\lambda)$ and $u \in H^1_0(\Omega_r)$

$$\int_{\Omega_r} |\nabla u|^2 - \lambda u^2 \ge \delta \int_{\Omega_r} |\nabla u|^2.$$

Let $1 \le k < \infty$ and $r > r(\lambda)$ be fixed. There is an open $B \subset \Omega_r$ such that

$$gB \cap hB = \emptyset, \quad \forall \ \ g, h \in G_k, \ g \ne h.$$

By a famous computation due to Brezis and Nirenberg [BN] (see also [Wm], Lemma 1.46) there exists a nonnegative $v \in H^1_0(B)$ such that $|v|_{2^*} = 1$ and

$$\int_B |\nabla v|^2 - \lambda v^2 < S,$$

where S denotes the best Sobolev constant. It is clear that

$$u = \sum_{g \in G_k} gv \in H^1_{0,G_k}(\Omega_r).$$

Moreover,

$$\frac{\int_{\Omega_r} |\nabla u|^2 - \lambda u^2}{(\int_{\Omega_r} u^{2^*})^{\frac{2}{2^*}}} < k^{\frac{2}{N}} S,$$

so that

$$S_\lambda(r,k) < k^{\frac{2}{N}} S.$$

By a result of P. L. Lions ([Lp], see also [Wm], Theorem 8.15, and [Wz]), $S_\lambda(r,k)$ is attained by some $u \in H^1_{0,G_k}(\Omega_r)$. □

Lemma 4. *For every* $0 < \lambda < \pi^2$, $r > r(\lambda)$ *(which is given in Lemma 3),* $k \ge 1$ *and* $m \ge 2$, *if* $S_\lambda(r, km)$ *is achieved and*

$$S_\lambda(r, km) < S_\lambda(r, \infty),$$

then

$$S_\lambda(r,k) < S_\lambda(r, km).$$

Proof. This can be proved with essentially the same proof as of Lemma 1.5 in [Ly]. □

Proof of Theorem 1. Let $0 < \lambda < \pi^2$ and $n \ge 1$ be fixed. According to Lemmas 1 and 2, there exists $R(\lambda, n)$ such that

$$S_\lambda(r, 2^n) < S_\lambda(r, \infty)$$

for $r > R(\lambda, n)$. We can assume that $R(\lambda, n) > r(\lambda)$, so that $S_\lambda(r,k)$ is achieved for $1 \le k \le \infty$ and $r > R(\lambda, n)$ by Lemma 3. Lemma 4 implies that

$$S_\lambda(r,2) < S_\lambda(r,2^2) < \cdots < S_\lambda(r,2^n) < S_\lambda(r,\infty)$$

for $r > R(\lambda, n)$.

Therefore, all minimizers for $S_\lambda(r, 2^j)$ are nonradial and nonrotationally equivalent. We may assume that the minimizers are nonnegative. By the symmetric criticality principle (see e.g. [Wm]) the minimizers, after rescaling, are solutions of

$$-\Delta u - \lambda u = u^{2^*-1}, \quad \text{in } \Omega_r.$$

The maximum principle implies that $u > 0$ in Ω_r. □

We remark that we do not know whether our result holds for $N = 3$. Note that it is also open for the subcritical problem of whether the multiplicity results in [Ly] (also [C], [K], [Ls], [S]) hold for $N = 3$.

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