

INVARIANT MANIFOLDS OF HYPERCYCLIC VECTORS FOR THE REAL SCALAR CASE

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ABSTRACT. We show that every hypercyclic operator on a real locally convex vector space admits a dense invariant linear manifold of hypercyclic vectors.

Given a locally convex vector space X and a continuous operator $T : X \rightarrow X$, we say that T is **hypercyclic** provided there exists some $x \in X$ whose orbit

$$\text{Orb}\{x, T\} = \{x, Tx, T^2x, \dots\}$$

is dense in X . Such an x is said to be a hypercyclic vector for T . A motivation for this definition comes from the invariant subset problem: T has no non-trivial closed invariant subset if and only if every non-zero vector in X is hypercyclic for T .

In 1990, B. Beauzamy constructed an example of an operator on a separable complex Hilbert space admitting a dense, invariant linear manifold whose non-zero vectors were all hypercyclic [3, Thm. A]. Using different techniques, G. Godefroy and J. Shapiro provided a large class of examples on Fréchet spaces of entire functions with the same property [5, Thm 5.1].

Soon after, D. Herrero [6, Proposition 4.1] and P. Bourdon [4] independently showed that *every* hypercyclic operator on a complex Hilbert space admits such a dense, invariant linear manifold of hypercyclic vectors (in fact, Bourdon's proof works for arbitrary complex locally convex spaces as well). We'd like to show here that the same holds for the real scalar case, by presenting a positive answer to the following question, raised by S. Ansari [1, Problem 1]:

“Suppose X is a locally convex real vector space and $T : X \rightarrow X$ is a continuous linear operator with a hypercyclic vector x . Is it true that $P(T)x$ is a hypercyclic vector for T whenever P is a non-zero polynomial with real coefficients?” (If so,

$$\mathcal{M} = \{P(T)x : P \text{ polynomial with real coefficients}\}$$

is a dense, T -invariant manifold of hypercyclic vectors.)

Notice that since $P(T)$ and T commute,

$$\text{Orb}\{P(T)x, T\} = P(T)(\text{Orb}\{x, T\}).$$

That is, given x a hypercyclic vector for T , $P(T)x$ will also be hypercyclic if and only if $P(T)$ has dense range. So it will suffice for us to show the following.

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Theorem. *Let X be a locally convex real vector space, and $T \in L(X)$ be hypercyclic. If P is a non-zero polynomial with real coefficients, then $P(T)$ has dense range.*

We'll make use of two results. The first one is due to C.Kitai [6, Theorem 2.3]:

Lemma 1. *Let X be a locally convex real (complex) vector space, and $T \in L(X)$ be hypercyclic. Then the adjoint T^* of T has no eigenvalues. In other words, for any scalar b , the operator $T + bI$ must have dense range.*

Proof. Suppose T^* admits an eigenvector x with eigenvalue λ , and let $z \in X$ be a hypercyclic vector for T . Since $x \neq 0$, the set

$$\{\langle x, T^n z \rangle\}_{n \geq 1} = \{\langle T^{*n} x, z \rangle\}_{n \geq 1} = \{\lambda^n \langle x, z \rangle\}_{n \geq 1}$$

must be dense in the real (complex) scalar field, a contradiction. \square

The statement of the second result may be traced back at least to S. Rolewicz [8, p. 17]. We are grateful to P. Bourdon for suggesting the following argument to us.

Lemma 2. *R^n admits no hypercyclic operators.*

Proof. Suppose there exists $A : R^n \rightarrow R^n$ linear with hypercyclic vector z . Notice that $B = \{z, Az, \dots, A^{n-1}z\}$ must be linearly independent. Now, since z is a hypercyclic vector for A , there exist sequences of positive integers (n_k) and (\tilde{n}_k) so that

$$\begin{aligned} A^{n_k} z &\rightarrow 0, \\ A^{\tilde{n}_k} z &\rightarrow z. \end{aligned}$$

Because B is a basis of R^n , we have in fact that

$$\begin{aligned} A^{n_k} x &\rightarrow 0, \\ A^{\tilde{n}_k} x &\rightarrow x \end{aligned}$$

for all $x \in R^n$. Hence, if $|A|$ denotes the determinant of A , the above lines imply the contradictory fact that

$$\begin{aligned} |A|^{n_k} = |A^{n_k}| &\rightarrow 0, \\ |A|^{\tilde{n}_k} = |A^{\tilde{n}_k}| &\rightarrow 1. \end{aligned}$$

So Lemma 2 holds. \square

Now, let's show the Theorem.

Proof of the Theorem. Since scalar multiples and compositions of operators having dense range have dense range, we may assume P is irreducible and monic. Moreover, by Lemma 1 we may assume P is of the form

$$P(t) = t^2 - 2\operatorname{Re}(w)t + |w|^2, \quad \text{for some non-real complex number } w.$$

Now, suppose that $P(T)$ does *not* have dense range. Let $0 \neq x \in \operatorname{Ker}(P(T)^*)$. Then

$$(1) \quad T^{*2}x = a_2 T^*x + b_2 x, \quad \text{where} \quad \begin{cases} a_2 &= w + \bar{w}, \\ b_2 &= -|w|^2. \end{cases}$$

By Lemma 1, $\{T^*x, x\} \subset Ker(P(T)^*)$ must be linearly independent. So there exist unique scalars a_n and b_n satisfying

$$(2) \quad T^{*n}x = a_nT^*x + b_nx \quad (n = 1, 2, \dots).$$

Notice that by (2)

$$\begin{aligned} T^{*n+1}x &= T^*(a_nT^*x + b_nx) \\ &= a_n(a_2T^*x + b_2x) + b_nT^*x \\ &= (a_na_2 + b_n)T^*x + a_nb_2x. \end{aligned}$$

That is,

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = A \begin{bmatrix} a_n \\ b_n \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} a_2 & 1 \\ b_2 & 0 \end{bmatrix} = \begin{bmatrix} w + \bar{w} & 1 \\ -|w|^2 & 0 \end{bmatrix}.$$

So for all $n \geq 2$,

$$(3) \quad \begin{bmatrix} a_n \\ b_n \end{bmatrix} = A^{n-2} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

and

$$(4) \quad \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} (A^{n-2})^t = \begin{bmatrix} a_n & b_n \\ a_{n+1} & b_{n+1} \end{bmatrix}.$$

Next, notice that since $\{T^*x, x\}$ is linearly independent, the operator

$$\begin{aligned} X &\xrightarrow{\Pi} R^2 \\ y &\mapsto (\langle x, y \rangle; \langle T^*x, y \rangle) \end{aligned}$$

is onto. In particular, if $z \in X$ is a hypercyclic vector for T , then

$$(5) \quad \{ \Pi(T^n z) \}_{n \geq 1}$$

must be dense in R^2 .

Hence, by (2) and (4) we have for each $n \geq 2$

$$\begin{aligned} \Pi(T^n z) &= (\langle x, T^n z \rangle; \langle T^*x, T^n z \rangle) \\ &= (\langle T^{*n}x, z \rangle; \langle T^{*n+1}x, z \rangle) \\ &= (\langle a_nT^*x + b_nx, z \rangle; \langle a_{n+1}T^*x + b_{n+1}x, z \rangle) \\ &= \begin{bmatrix} a_n & b_n \\ a_{n+1} & b_{n+1} \end{bmatrix} \begin{bmatrix} \langle T^*x, z \rangle \\ \langle x, z \rangle \end{bmatrix} \\ (6) \quad &= \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} (A^{n-2})^t \begin{bmatrix} \langle T^*x, z \rangle \\ \langle x, z \rangle \end{bmatrix}. \end{aligned}$$

Thus, (5) and (6) force A^t to be hypercyclic on R^2 , which contradicts Lemma 2. So $P(T)$ must have dense range. \square

Remark. As a consequence of the theorem, S. Ansari's proof that every operator on a complex Banach Space shares with its powers the *same* hypercyclic vectors [2, Thm. 1] works for the real scalar case as well (see also [1, Note 3]).

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