

## SUPPORT FUNCTIONALS AND SMOOTH POINTS IN ABSTRACT $M$ SPACES

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**ABSTRACT.** By presenting some properties of support functionals in abstract  $M$  spaces, we get some sufficient and necessary conditions for smooth points in abstract  $M$  (function) spaces. Moreover, the notion of the smallest support semi-norm is introduced and an explicit form for this functional in abstract  $M$  function spaces is also given.

### 1. INTRODUCTION

There are some recent papers devoted to the study of support functionals and their applications in a class of Banach function spaces. Hudzik and Ye in [3] gave a characteristic of support functionals in Musielak-Orlicz sequence spaces equipped with the Luxemburg norm, where they introduced a generalized Banach limit to describe singular support functionals. Some characteristics of support functionals in Orlicz spaces endowed with the Luxemburg norm and with the Orlicz norm have been obtained by Wang in [9] and Chen, Hudzik and Kaminska in [1], respectively. Moreover, some criteria of smooth points and exposed points in these spaces were also considered in [1], [3] and [9].

The aim of the present paper is to study the support functionals and their applications in abstract  $M$  spaces and in abstract  $M$  function spaces. This paper is organized as follows. The first part is an introduction. The second part consists of some results concerning the characteristic of support functionals in abstract  $M$  spaces. Section 3 is devoted to some applications of support functionals in these spaces. Some criteria of smooth points are obtained. In section 4 we further discuss the support functionals at a point in abstract  $M$  function spaces. The notion of the smallest support semi-norm at a point is introduced and an explicit form for the smallest support semi-norm at any point in such spaces is given.

Let  $X$  be a normed linear space and  $X^*$  be its dual space.  $f \in X^*$  is said to be a *support functional* at  $x \in X \setminus \{0\}$  if  $\langle f, x \rangle = \|x\|$  and  $\|f\| = 1$ . For convenience, we denote by  $S_x^*$  the set of all support functionals at  $x$ . We say that  $x \in X \setminus \{0\}$  is a *smooth point* if the support functional at  $x$  is unique.

Let  $X$  be a normed Riesz space. The norm on  $X$  is said to be  $\infty$ -additive if  $x, y \in X$  with  $x \wedge y = 0$  implies that  $\|x + y\| = \max(\|x\|, \|y\|)$ . A normed

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Riesz space  $X$  equipped with the  $\infty$ -additive norm is called an *abstract  $M$  space* ( $X \in AM$ )<sup>1</sup> (cf. [7] and [13]).

Let  $(\Omega, \Sigma, \mu)$  be a complete  $\sigma$ -finite measure space. A normed space  $X$  consisting of equivalence classes, modulo equality almost everywhere, of real valued measurable functions on  $\Sigma$  is called a *normed Köthe function space*, if  $|x(t)| \leq |y(t)|$  a.e. on  $\Omega$ , with  $x$  measurable and  $y \in X$  implies that  $x \in X$  and  $\|x\| \leq \|y\|$  (cf. [6]). A normed Köthe function space equipped with the  $\infty$ -additive norm is called an *abstract  $M$  function space* ( $X \in AMF$ ).

## 2. SOME PROPERTIES OF SUPPORT FUNCTIONALS IN $AM$ AND $AMF$ SPACES

**Proposition 2.1.** *Let  $X \in AM$  and  $x \in X \setminus \{0\}$ . Let  $f \in S_x^*$ . Then we have*

- (i)  $\langle f, y \rangle \geq 0$  if  $x^- \wedge y = 0$  or  $x^+ \wedge (-y) = 0$ ;
- (ii)  $\langle f, y \rangle \leq 0$  if  $x^- \wedge (-y) = 0$  or  $x^+ \wedge y = 0$ ;
- (iii)  $\langle f, y \rangle = 0$  if  $|x| \wedge |y| = 0$ .

*Proof.* First, we consider case (ii). If  $x^- \wedge (-y) = 0$ , then  $y \leq 0$ . Suppose that  $Q \leq -1$  satisfying  $\|y/Q\| \leq \|x\|$ . Noticing that  $y/Q \geq 0$  and

$$\begin{aligned} |x - y/Q| &= |x^+ - x^- - y/Q| \\ &= x^+ \vee (x^- + y/Q) - x^+ \wedge (x^- + y/Q) \leq x^+ \vee (x^- + y/Q), \end{aligned}$$

we have

$$\|x - y/Q\| \leq \max(\|x^+\|, \|x^-\|, \|y/Q\|) = \|x\|.$$

Thus

$$\langle f, x - y/Q \rangle \leq \|f\| \|x\| = \langle f, x \rangle.$$

It follows that  $\langle f, y/Q \rangle \geq 0$ . So  $\langle f, y \rangle \leq 0$ .

Similarly, if  $x^+ \wedge y = 0$ , we can choose  $R \geq 1$  such that  $\|y/R\| \leq \|x\|$ . We have  $|x + y/R| \leq x^- \vee (x^+ + y/R)$ . It follows that  $\|x + y/R\| \leq \|x\|$ . So we have  $\langle f, x + y/R \rangle \leq \langle f, x \rangle$ , which implies that  $\langle f, y \rangle \leq 0$ .

Next, if we replace  $y$  with  $-y$  in the above procedure, one can easily verify that (i) holds.

Finally, (iii) is a direct consequence of (i) and (ii).

**Proposition 2.2.** *Let  $X \in AM$  and  $x \in X \setminus \{0\}$ . Let  $f \in S_x^*$ . Suppose that  $|x| = x_1 \vee x_2$  and  $x_1 \wedge x_2 = 0$ . If  $\|x_1\| < \|x\|$ , then  $\langle f, y \rangle = 0$  for all  $y \in X$  satisfying  $|y| \wedge x_2 = 0$ .*

*Proof.* First, we prove that  $\langle f, x_1 \wedge x^+ \rangle = \langle f, x_1 \wedge x^- \rangle = 0$ . It follows from Proposition 2.1 that  $\langle f, x_1 \wedge x^+ \rangle \geq 0$  and  $\langle f, x_1 \wedge x^- \rangle \leq 0$ . We now show that  $\langle f, x_1 \wedge x^+ \rangle = 0$ . Assume for the contrary that  $\langle f, x_1 \wedge x^+ \rangle > 0$ . Let  $a > 1$  satisfy  $\|ax_1\| = \|x\|$ . Put

$$z = x + (a - 1)(x^+ \wedge x_1).$$

Noticing that  $x^+ = |x| \wedge x^+ = x^+ \wedge x_1 + x^+ \wedge x_2$ , we have

$$z = x^+ - x^- + (a - 1)(x^+ \wedge x_1) = a(x^+ \wedge x_1) + x^+ \wedge x_2 - x^-.$$

<sup>1</sup>We do not need that  $X$  is a Banach lattice.

It follows that  $\|z\| \leq \max(\|a(x^+ \wedge x_1)\|, \|x^+ \wedge x_2\|, \|x^-\|) \leq \|x\|$ . On the other hand,

$$\langle f, z \rangle = \langle f, x \rangle + (a-1)\langle f, x^+ \wedge x_1 \rangle > \langle f, x \rangle.$$

This contradicts the inequality  $\langle f, z \rangle \leq \|f\| \|z\| \leq \|x\| = \langle f, x \rangle$ . Similarly, we can prove that  $\langle f, x_1 \wedge x^- \rangle = 0$ . Hence we have

$$\langle f, x \rangle = \langle f, x^+ \wedge x_2 - x^- \wedge x_2 \rangle = \|x\|.$$

Next, assume that  $|y| \wedge x_2 = 0$ . We can choose  $Q \geq 1$  such that  $\|y/Q\| \leq \|x\|$ . Let

$$z = x_2 \wedge x^+ - x_2 \wedge x^- + y/Q.$$

It is easy to see that  $\|z\| \leq \|x\|$ . We have

$$\begin{aligned} \langle f, z \rangle &= \langle f, x_2 \wedge x^+ - x_2 \wedge x^- \rangle + \langle f, y/Q \rangle \\ &= \langle f, x \rangle + \langle f, y/Q \rangle = \|x\| + \langle f, y/Q \rangle. \end{aligned}$$

This implies that  $\langle f, y \rangle \leq 0$ . Replacing  $y$  with  $-y$  in the above procedure, we see that  $\langle f, -y \rangle \leq 0$ . So we have  $\langle f, y \rangle = 0$ .

In view of Propositions 2.1 and 2.2, we have

**Corollary 2.3.** *Let  $X \in AMF$ , and let  $x \in X \setminus \{0\}$  and  $f \in S_x^*$ . Then for any  $y \in X$ , we have*

- (i)  $\langle f, y \rangle \geq 0$  if  $x(t)y(t) \geq 0$  a.e. on  $\Omega$ ;
- (ii)  $\langle f, y \rangle \leq 0$  if  $x(t)y(t) \leq 0$  a.e. on  $\Omega$ ;
- (iii)  $\langle f, y \rangle = 0$  if  $x(t)y(t) = 0$  a.e. on  $\Omega$ .

**Corollary 2.4.** *Let  $X \in AMF$ , and let  $x \in X \setminus \{0\}$  and  $f \in S_x^*$ . Suppose that  $e \in \Sigma$  and  $\|x\chi_e\| < \|x\|$ . Then we have  $\langle f, y\chi_e \rangle = 0$  for any  $y \in X$ .*

### 3. SMOOTH POINTS IN $AM$ AND $AMF$ SPACES

Let  $X$  be a normed Riesz space. We say that the element  $x \neq 0$  in  $X$  is an *atom* if it follows from  $x = y + z$  and  $|y| \wedge |z| = 0$  that  $y = 0$  or  $z = 0$ .

**Lemma 3.1.** *Let  $x \in AM$  and let  $x \in X \setminus \{0\}$ . If there exist  $x_1, x_2 \in X$  such that  $|x_1| \wedge |x_2| = 0$ ,  $x_1 + x_2 = x$  and  $\|x_1\| = \|x_2\| = \|x\|$ , then  $S_x^*$  contains at least two elements.*

*Proof.* Let  $f_i$  be a support functional at  $x_i$  for  $i = 1, 2$ . In view of Propositions 2.1 and 2.2, we have  $\langle f_i, x_j \rangle = 0$  for  $i, j = 1, 2$ ,  $i \neq j$ . It follows that

$$\langle f_i, x \rangle = \langle f_i, x_i \rangle = \|x_i\| = \|x\|.$$

for  $i = 1, 2$ . Thus  $f_1$  and  $f_2$  are both the support functionals at  $x$ . Clearly,  $f_1 \neq f_2$ . The proof is complete.

**Theorem 3.2.** *Let  $X \in AM$  have an order complete and order semi-continuous norm. Let  $x \in X \setminus \{0\}$ . Then  $x$  is a smooth point if and only if there exists an atom  $y \in X$  such that  $x = y + z$ ,  $|y| \wedge |z| = 0$ ,  $\|y\| = \|x\|$  and  $\|z\| < \|x\|$ .*

*Proof. Necessity.* Let  $x \in X$  be a smooth point. It follows that  $\min(\|x^+\|, \|x^-\|) < \|x\|$ . Indeed, if  $\|x^+\| = \|x^-\| = \|x\|$ , in view of Lemma 3.1 there exist at least two support functionals at  $x$ . This contradicts that  $x$  is smooth.  $X \in AM$  implies that  $\max(\|x^+\|, \|x^-\|) = \|x\|$ . We may assume without loss of generality that  $\|x^+\| = \|x\|$ . Put

$$A = \{w_\alpha : 0 \leq w_\alpha \leq x^+, \|w_\alpha\| < \|x\| \text{ and } (x^+ - w_\alpha) \wedge w_\alpha = 0\}.$$

It is easy to see that  $w_{\alpha_i} \in A$  ( $i = 1, 2$ ) implies  $w_{\alpha_1} \vee w_{\alpha_2} \in A$ . It follows that  $A$  is an upwards directed set.  $X$  is order complete, so we have  $w_\alpha \uparrow \bigvee_\alpha w_\alpha$ . For convenience, we denote  $w = \bigvee_\alpha w_\alpha$ . Since  $X$  has the order semi-continuous norm, we have  $\|w\| = \sup_\alpha \|w_\alpha\|$ . Therefore, there exists a sequence  $\{w_{\alpha_n}\} \subset A$  such that  $\|w_{\alpha_n}\| \geq \|w\| - 1/n$ . We may assume without loss of generality that  $w_{\alpha_n} \uparrow$ . Indeed, we can select  $\bigvee_{i \leq n} w_{\alpha_i}$  instead of  $w_{\alpha_n}$ . It follows that  $\|w_{\alpha_n}\| \uparrow \|w\|$ . This implies that  $\|w\| < \|x^+\|$ . In fact, if  $\|w\| = \|x^+\| = \|x\|$ , then we have  $\|w_{\alpha_n}\| \uparrow \|x^+\|$ . Put  $v_0 = w_{\alpha_1}$  and  $v_n = w_{\alpha_{n+1}} - w_{\alpha_n}$  ( $n \geq 1$ ). We can easily see that  $w_{\alpha_n} = \bigvee_{0 \leq i \leq n} v_i$  and  $v_i \wedge v_j = 0$  ( $i \neq j$ ). It follows from  $X \in AM$  that there exists a subsequence of  $\{v_n\}$ , still denoted by  $\{v_n\}$  such that  $\|v_n\| \uparrow \|x^+\|$ . Let  $w_1 = \bigvee_{k \geq 1} v_{2k+1}$  and  $w_2 = \bigvee_{k \geq 1} v_{2k}$ . We have  $\|w_1\| = \|w_2\| = \|x^+\|$  and  $w_1 \wedge w_2 = 0$ . In virtue of Lemma 3.1 we can see that there exist at least two support functionals at  $x$ . This contradicts that  $x$  is smooth.

Let  $y = x^+ - w$ . It follows from  $w_\alpha \wedge (x - w_\alpha) = 0$  that  $w \wedge (x^+ - w) = 0$  (cf. [8], §15).  $X \in AM$  and  $\|w\| < \|x^+\|$  imply that  $\|y\| = \|x^+\| = \|x\|$ . Now we show that  $y$  is an atom. If not, then  $y = y_1 + y_2$ ,  $y_1 \wedge y_2 = 0$  and  $y_1, y_2 > 0$ . By Lemma 3.1 we have  $\min(\|y_1\|, \|y_2\|) < \|x^+\|$ . Assume that  $\|y_1\| < \|x^+\|$ . One can easily see that  $w \vee y_1 \in A$ . This contradicts the fact  $w = \bigvee_\alpha w_\alpha$ . Let  $z = w - x^-$ . It follows that  $x = y + z$ ,  $|y| \wedge |z| = 0$  and  $\|z\| < \|x\|$ .

*Sufficiency.* Let  $x = y + z$ ,  $y$  be an atom satisfying  $\|y\| = \|x\|$ ,  $\|z\| < \|x\|$  and  $|y| \wedge |z| = 0$ . Let  $f$  be a support functional at  $x$ . It follows from Proposition 2.2 that  $\langle f, z \rangle = 0$ . So  $f$  is also a support functional at  $y$ . Clearly,  $y$  has a unique support functional. Indeed, we first have  $y < 0$  or  $y > 0$  (cf. [8], §26). Assume without loss of generality that  $y > 0$ . For any  $x \in X^+$ , let

$$B = \{u : 0 \leq u \leq x, u \wedge (x - u) = 0, \text{ and } u \wedge y = 0\}.$$

Similarly to the above, we see that  $B$  is an upwards directed set and so,  $u_0 \stackrel{\text{def}}{=} \sup_{u \in B} u$  exists. If  $u_0 \neq x$ , we claim that  $x - u_0$  is also an atom. If not, then there exist  $u_1$  and  $u_2$  such that  $x - u_0 = u_1 \vee u_2$  and  $u_1 \wedge u_2 > 0$ . Since  $y$  is an atom, we have  $u_1 \wedge y = 0$  or  $u_2 \wedge y = 0$ . We may assume that  $u_1 \wedge y = 0$ . One can easily verify that  $u_0 \vee u_1 \in B$ . A contradiction. It follows from Theorem 26.4 in [8] that if  $u_0 \neq x$ , then either  $(x - u_0) \wedge y = 0$  or  $(x - u_0) = ay$  for some  $a \in (0, 1]$ . Thus, for any  $f_i \in S_y^*$  ( $i = 1, 2$ ), from the equality

$$\langle f_i, x \rangle = \langle f_i, u_0 \rangle + \langle f_i, x - u_0 \rangle = a, \quad i = 1, 2,$$

we see that  $\langle f_i, x \rangle = 0$  if  $u_0 = x$  or  $(x - u_0) \wedge y = 0$ , and  $\langle f_i, x \rangle = a$  if  $(x - u_0) = ay$ . This leads to  $f_1 \equiv f_2$ . Thus, the support functional at  $y$  is unique. The same result also holds true for  $x \in X^-$  and so, for  $x \in X$ . The proof is complete.

**Theorem 3.3.** *Let  $X \in AMF$  and let  $x \in X \setminus \{0\}$ . Then  $x$  is a smooth point if and only if for any disjoint  $e_1, e_2 \in \Sigma$ ,  $\min(\|x\chi_{e_1}\|, \|x\chi_{e_2}\|) < \|x\|$ .*

*Proof. Sufficiency.* We assume without loss of generality that  $\|x\| = 1$ . If  $x$  is not a smooth point, then there exist two elements  $f_1$  and  $f_2$  in  $S_x^*$  and  $y \in X$ ,  $\|y\| \leq 1$  such that

$$\langle f_1, y \rangle \neq \langle f_2, y \rangle.$$

Let

$$\Omega_0 = \{t \in \Omega : x(t)y(t) = 0\},$$

$$\Omega_1 = \{t \in \Omega : x(t)y(t) > 0\},$$

$$\Omega_2 = \{t \in \Omega : x(t)y(t) < 0\}.$$

It follows from the hypothesis that there exists only one  $\Omega_j$  such that  $\|x\chi_{\Omega_j}\| = 1$  for some  $j \in \{0, 1, 2\}$ . We shall show that this yields a contradiction. Let us consider for this purpose three separate cases.

*Case I.*  $\|x\chi_{\Omega_0}\| = 1$ . We have  $\|x\chi_{\Omega_i}\| < 1$  for  $i = 1, 2$ . In virtue of Corollaries 2.3 and 2.4, we have  $\langle f, y \rangle = 0$  for any  $f \in S_x^*$ , which implies that  $\langle f_1, y \rangle = \langle f_2, y \rangle = 0$ . A contradiction.

*Case II.*  $\|x\chi_{\Omega_1}\| = 1$ . We have  $\|x\chi_{\Omega_i}\| < 1$  for  $i = 0, 2$ . First, we assume that  $x(t), y(t) > 0$  for all  $t \in \Omega_1$ . Take an arbitrary  $\varepsilon > 0$  and let

$$\Omega_{1\varepsilon} = \{t \in \Omega_1 : y(t) < (1 + \varepsilon)x(t)\}.$$

We have  $\|x\chi_{\Omega \setminus \Omega_{1\varepsilon}}\| < 1$ . Indeed, if this inequality does not hold, then  $\|y\chi_{\Omega \setminus \Omega_{1\varepsilon}}\| \geq 1 + \varepsilon$ . A contradiction. So we have  $\|x\chi_{\Omega_{1\varepsilon}}\| = 1$ . Let

$$\Omega_{>} = \left\{ t \in \Omega_{1\varepsilon} : 1 \leq \frac{y(t)}{x(t)} < 1 + \varepsilon \right\},$$

$$\Omega_{<} = \left\{ t \in \Omega_{1\varepsilon} : 0 \leq \frac{y(t)}{x(t)} < 1 \right\}.$$

It follows that  $\max(\|x\chi_{\Omega_{>}}\|, \|x\chi_{\Omega_{<}}\|) = 1$ . We will divide Case II into the following two cases.

*Case IIa.*  $\|x\chi_{\Omega_{>}}\| = 1$ . We have  $\|x\chi_{\Omega_{<}}\| < 1$ . Noticing that

$$\Omega = (\Omega_1 \setminus \Omega_{1\varepsilon}) \cup \Omega_{<} \cup (\Omega \setminus \Omega_1) \cup \Omega_{>} \quad \text{and} \quad \|x\chi_{(\Omega \setminus \Omega_1) \cup (\Omega_1 \setminus \Omega_{1\varepsilon}) \cup \Omega_{<}}\| < 1,$$

we obtain that for any  $f \in S_x^*$ ,

$$\langle f, y \rangle = \langle f, y\chi_{\Omega_{>}} \rangle + \langle f, y\chi_{(\Omega \setminus \Omega_1) \cup (\Omega_1 \setminus \Omega_{1\varepsilon}) \cup \Omega_{<}} \rangle = \langle f, y\chi_{\Omega_{>}} \rangle$$

and

$$\langle f, x \rangle = \langle f, x\chi_{\Omega_{>}} \rangle.$$

Since  $t \in \Omega_{>}$  implies that  $x(t) \leq y(t) < (1 + \varepsilon)x(t)$ , we have

$$\begin{aligned} 1 &= \langle f, x\chi_{\Omega_{>}} \rangle \leq \langle f, y\chi_{\Omega_{>}} \rangle \\ &= \langle f, y \rangle \leq (1 + \varepsilon)\langle f, x\chi_{\Omega_{>}} \rangle = 1 + \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, this means that  $\langle f, y \rangle = 1$ . Thus  $\langle f_1, y \rangle = \langle f_2, y \rangle = 1$ . This is a contradiction.

*Case IIb.*  $\|x\chi_{\Omega_{<}}\| = 1$ . We have  $\|x\chi_{\Omega_{>}}\| < 1$ . Put

$$e_i = \left\{ t \in \Omega_{<} : \frac{i-1}{2^n} \leq \frac{y(t)}{x(t)} < \frac{i}{2^n} \right\}, \quad i = 1, \dots, 2^n.$$

It is easy to see that there exists only one  $e_{i_n} \subset \Omega_{<}$ ,  $i \in \{1, 2, \dots, 2^n\}$ , such that  $\|x\chi_{e_{i_n}}\| = 1$ , i.e.  $\|x\chi_{e_i}\| < 1$  for any  $i \neq i_n$ ,  $1 \leq i \leq 2^n$ . Moreover, we have

$e_{i_{n+1}} \subset e_{i_n}$ . In fact, if  $e_{i_{n+1}} \subset e_{i'_n}$  for some  $i'_n \in \{1, \dots, 2^n\}$  and  $i'_n \neq i_n$ , then  $\|x\chi_{e_{i'_n}}\| = 1$ . This contradicts the equality  $\|x\chi_{e_{i_n}}\| = 1$ . So we have obtained that

$$\langle f, z \rangle = \langle f, z\chi_{e_{i_n}} \rangle$$

for all  $f \in S_x$ ,  $z \in X$  and  $n \in \mathbb{N}$ . Therefore

$$\begin{aligned} \frac{i_n - 1}{2^n} &= \frac{i_n - 1}{2^n} \langle f, x\chi_{e_{i_n}} \rangle \leq \langle f, y\chi_{e_{i_n}} \rangle \\ &= \langle f, y \rangle \leq \frac{i_n}{2^n} \langle f, x\chi_{e_{i_n}} \rangle = \frac{i_n}{2^n}. \end{aligned}$$

Clearly,  $\{i_n/2^n\}$  is a Cauchy sequence. Suppose that  $\lim_{n \rightarrow \infty} i_n/2^n = a$ . Then we have  $\langle f, y \rangle = a$  for any  $f \in S_x^*$ . This implies that  $\langle f_1, y \rangle = \langle f_2, y \rangle$ . A contradiction.

At the end of Case II, we consider the general case  $x(t)y(t) > 0$  for all  $t \in \Omega_1$ . Let  $f \in S_x^*$  and let  $f'$  satisfy

$$\langle f', z\text{sign}(x) \rangle = \langle f, z \rangle$$

for any  $z \in X$ . One can easily verify that  $f \in S_x^*$  if and only if  $f' \in S_{|x|}^*$ . Similarly to the above discussions, we can prove that  $\langle f'_1, y\text{sign}(x) \rangle = \langle f'_2, y\text{sign}(x) \rangle$ , which implies that  $\langle f_1, y \rangle = \langle f_2, y \rangle$ . A contradiction.

*Case III.*  $\langle x\chi_{\Omega_2} \rangle = 1$ . In exactly the same way as in Case II, we can prove the result.

*Necessity.* It is a direct consequence of Lemma 3.1. The proof is finished.

*Remark 3.4.* By Theorems 3.2 and 3.3 we have seen that  $L^\infty(\Omega, \Sigma, \mu)$  has no smooth point if  $(\Omega, \Sigma, \mu)$  is an atomless measure space and  $x \in l^\infty \setminus \{0\}$  is a smooth point if and only if  $\sup_{i \in \mathbb{N}} |x(i)|$  is attainable at only one coordinate and  $\limsup_{i \rightarrow \infty} |x(i)| < \|x\|_{l^\infty}$ . So we have revised the criteria of smooth points in  $l^\infty$  given by [12].

At the end of this section, we give some counterexamples to show that the order complete and order semi-continuous norm in Theorem 3.2 are necessary for criterion of a smooth point.

**Example 3.5.** Let  $\phi \in l^{\infty*}$  be a positive singular functional and an extreme point of the unit ball of  $l^{\infty*}$ . In view of Theorem 9 in [2] we have  $\langle \phi, x \rangle \langle \phi, y \rangle = 0$  for all  $x, y \in l^\infty$  satisfying  $\text{supp}x \cap \text{supp}y = \emptyset$  (see also [11]). Moreover,  $\phi$  is norm attainable, i.e. there exists  $x_0 \in l^\infty$  such that  $\langle \phi, x_0 \rangle = \|x_0\| = 1$  (cf. [2]). Define

$$\|y\|_1 = \max(\|y\|_{l^\infty}, 2|\langle \phi, y \rangle|), \quad y \in l^\infty.$$

It is easy to see that  $\|\cdot\|_1$  is an equivalent norm on  $l^\infty$ . Clearly,  $(l^\infty, \|\cdot\|_1) \in AMF$  and  $\|\cdot\|_1$  is not order semi-continuous. Noticing that  $\|x_0\|_1 = 2$  and  $\|x_0\|_{l^\infty} = 1$ , in virtue of Theorem 3.3 we get that  $x_0$  is a smooth point in  $(l^\infty, \|\cdot\|_1)$ , but  $x_0$  cannot be decomposed by any atom  $e \in \Sigma$  satisfying  $\|x_0\chi_e\| = 2$  and  $\|x_0\chi_{\Omega \setminus e}\| < 2$ .

**Example 3.6.** Let  $\Omega = \bigcup_{n \geq 1} [1/2^{2n+1}, 1/2^{2n}] \cup \{0\}$ . It is known that  $x \in C(\Omega)$  is a smooth point if and only if  $x$  is a peak function. Let

$$x(t) = \begin{cases} 1, & x = 0, \\ 1 - 1/n, & x \in [1/2^{2n+1}, 1/2^{2n}]. \end{cases}$$

It is easy to see that  $x$  is a peak function. So  $x$  is a smooth point. But  $x$  cannot be decomposed by an atom of norm one and an element of norm less one.

4. THE SMALLEST SUPPORT SEMI-NORM AT A POINT IN  $AMF$  SPACES

**Definition 4.1.** Let  $X$  be a normed linear space and  $x \in X \setminus \{0\}$ . We say that a semi-norm  $p(\cdot)$  is a support semi-norm at  $x$  if  $p(x) = \|x\|$  and  $|\langle f, y \rangle| \leq p(y)$  holds for all  $y \in X$  and  $f \in S_x^*$ . Moreover, if for any support semi-norm  $q(\cdot)$  at  $x$ , we have  $p(y) \leq q(y)$  for all  $y \in X$ , then we say that  $p(\cdot)$  is the smallest support semi-norm at  $x$ .

One can easily verify that if  $p(\cdot)$  is the smallest support semi-norm at  $x$ , then  $p(y) = \sup\{|\langle f, y \rangle| : f \in S_x^*\}$ .

Let  $X \in AMF$ . For any  $x, y \in X$ , let

$$\Omega_{x,y}^+ = \{t \in \Omega : x(t)y(t) > 0\},$$

$$\Omega_{x,y}^- = \{t \in \Omega : x(t)y(t) < 0\}.$$

We denote by  $E_{x,y}$  the set  $\{e_1, \dots, e_4\}$  of pairwise disjoint and  $\mu$ -measurable subsets of  $\Omega$  such that  $\Omega_{x,y}^+ = e_1 \cup e_2$  and  $\Omega_{x,y}^- = e_3 \cup e_4$ , namely

$$E_{x,y} = \{e_j : \Omega_{x,y}^+ = e_1 \cup e_2, \Omega_{x,y}^- = e_3 \cup e_4; e_1 \cap e_2 = e_3 \cap e_4 = \emptyset\},$$

and denote by  $\mathcal{E}_{x,y}$  the set of all decompositions  $E_{x,y}$ . If  $\max(\|x\chi_{\Omega_{x,y}^+}\|, \|x\chi_{\Omega_{x,y}^-}\|) = \|x\|$ , then there exists some  $e_j \in E_{x,y}$  such that  $\|x\chi_{e_j}\| = \|x\|$ . Now we define the following functionals:

$$p_x(y) = \begin{cases} \inf_{E_{x,y} \in \mathcal{E}_{x,y}} \max_{\substack{e_j \in E_{x,y} \\ \|x\chi_{e_j}\| = \|x\|}} \|y\chi_{e_j}\| & \text{if } \max(\|x\chi_{\Omega_{x,y}^+}\|, \|x\chi_{\Omega_{x,y}^-}\|) = \|x\|, \\ 0 & \text{if } \max(\|x\chi_{\Omega_{x,y}^+}\|, \|x\chi_{\Omega_{x,y}^-}\|) < \|x\| \end{cases}$$

and

$$r_x(y) = \begin{cases} \inf_{E_{x,y} \in \mathcal{E}_{x,y}} \operatorname{ess\,sup}_{\substack{t \in e_j \in E_{x,y} \\ \|x\chi_{e_j}\| = \|x\|}} \|x\| \frac{|y(t)|}{|x(t)|} & \text{if } \max(\|x\chi_{\Omega_{x,y}^+}\|, \|x\chi_{\Omega_{x,y}^-}\|) = \|x\|, \\ 0 & \text{if } \max(\|x\chi_{\Omega_{x,y}^+}\|, \|x\chi_{\Omega_{x,y}^-}\|) < \|x\|, \end{cases}$$

where  $\operatorname{ess\,sup}_{t \in E} z(t) = \inf_{E_0 \subset E, \mu(E_0) = 0} \sup_{t \in E \setminus E_0} z(t)$ .

**Theorem 4.2.** Let  $X \in AMF$  and let  $x \in X \setminus \{0\}$ . Then we have

- (i)  $p_x(y) = r_x(y)$  for all  $y \in X$ ;
- (ii)  $p_x(\cdot)$  is the smallest support semi-norm at  $x$ .

*Proof.* (i) It suffices to consider the case  $\max(\|x\chi_{\Omega_{x,y}^+}\|, \|x\chi_{\Omega_{x,y}^-}\|) = \|x\|$ . Suppose that  $\|x\chi_{e_i}\| = \|x\|$  and  $\operatorname{ess\,sup}_{t \in e_i} \|x\| |y(t)|/|x(t)| = A$  for some  $e_i \in E_{x,y}$ . It is easy to see that there exists  $e_0 \subset e_i$  such that  $\mu(e_0) = 0$  and  $\sup_{t \in e_i \setminus e_0} \|x\| |y(t)|/|x(t)| = A$ .  $t \in e_i \setminus e_0$  implies that  $\|x\| |y(t)| \leq A|x(t)|$ . So we have  $\|x\| \|y\chi_{e_i}\| \leq A\|x\chi_{e_i}\| = A\|x\|$ , i.e.  $\|y\chi_{e_i}\| \leq A$ . Hence it follows that  $p_x(y) \leq r_x(y)$ .

Conversely, for any  $\varepsilon > 0$ , there exists  $E_{x,y} \in \mathcal{E}_{x,y}$  such that

$$\max_{\substack{e_j \in E_{x,y} \\ \|x\chi_{e_j}\| = \|x\|}} \|y\chi_{e_j}\| \leq p_x(y) + \varepsilon.$$

Put

$$e_k^1 = \begin{cases} \{t \in e_k : \|x\| |y(t)| \leq (1 + \varepsilon)\|y\chi_{e_k}\| |x(t)|\} & \text{if } \|x\chi_{e_k}\| = \|x\|, \\ \emptyset & \text{if } \|x\chi_{e_k}\| < \|x\|, \end{cases} \quad e_k^2 = e_k \setminus e_k^1.$$

We have  $\|x\chi_{e_k^2}\| < \|x\|$ . In fact, if  $\|x\chi_{e_k^2}\| = \|x\|$ , then

$$e_k^2 = \{t \in e_k : \|x\| |y(t)| > (1 + \varepsilon)\|y\chi_{e_k}\| |x(t)|\}.$$

This leads to

$$\|x\| \|y\chi_{e_k^2}\| \geq (1 + \varepsilon)\|y\chi_{e_k}\| \|x\chi_{e_k^2}\| = (1 + \varepsilon)\|y\chi_{e_k}\| \|x\|.$$

A contradiction. Now we construct a new decomposition  $E'_{x,y} = \{\sigma_i : 1 \leq i \leq 4\}$  by letting

$$\sigma_1 = \bigcup_{k=1,2} e_k^1, \quad \sigma_2 = \Omega_{x,y}^+ \setminus \sigma_1, \quad \sigma_3 = \bigcup_{k=3,4} e_k^1, \quad \sigma_4 = \Omega_{x,y}^- \setminus \sigma_3.$$

It is easy to see that  $E'_{x,y} \in \mathcal{E}_{x,y}$ . From the construction of  $\sigma_i$  we see that  $\max_{i=1,3} \|x\chi_{\sigma_i}\| = \|x\|$  and  $\max_{i=2,4} \|x\chi_{\sigma_i}\| < \|x\|$ . Thus

$$\begin{aligned} \sup_{\substack{t \in \sigma_i \in E'_{x,y} \\ \|x\chi_{\sigma_i}\| = \|x\|}} \|x\| \frac{|y(t)|}{|x(t)|} &= \sup_{t \in \sigma_1 \cup \sigma_3} \|x\| \frac{|y(t)|}{|x(t)|} \\ &\leq (1 + \varepsilon) \max_{\substack{e_i \in E_{x,y} \\ \|x\chi_{e_i}\| = \|x\|}} \|y\chi_{e_i}\| \\ &\leq (1 + \varepsilon)(p_x(y) + \varepsilon). \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, this means that  $r_x(y) \leq p_x(y)$ . The proof of (i) is complete.

(ii) First, we shall prove that  $|\langle f, y \rangle| \leq p_x(y)$  for any  $f \in S_x^*$  and  $y \in X$ . Assume for the contrary that there exist  $f \in S_x^*$  and  $y \in X$  such that  $|\langle f, y \rangle| > p_x(y)$ . In view of Corollaries 2.3 and 2.4, we have  $\max(\|x\chi_{\Omega_{x,y}^+}\|, \|x\chi_{\Omega_{x,y}^-}\|) = \|x\|$ . It follows from the definition of  $p_x(\cdot)$  that there exists a decomposition

$$E_{x,y} = \{e_i : \Omega_{x,y}^+ = e_1 \cup e_2, \Omega_{x,y}^- = e_3 \cup e_4; e_1 \cap e_2 = e_3 \cap e_4 = \emptyset\}$$

such that

$$|\langle f, y \rangle| > \max_{\substack{e_j \in E_{x,y} \\ \|x\chi_{e_j}\| = \|x\|}} \|y\chi_{e_j}\|.$$

Suppose that  $e_k \in E_{x,y}$  satisfying  $\|x\chi_{e_k}\| = \|x\|$  and

$$\|y\chi_{e_k}\| = \max_{\substack{e_j \in E_{x,y} \\ \|x\chi_{e_j}\| = \|x\|}} \|y\chi_{e_j}\|.$$

We have  $|\langle f, y \rangle| > \|y\chi_{e_k}\|$ . Let

$$I_1 = \{i : \|x\chi_{e_i}\| = \|x\|\}, \quad I_2 = \{i : \|x\chi_{e_i}\| < \|x\|\}.$$

In virtue of Corollaries 2.3 and 2.4 we have

$$\langle f, y\chi_{\Omega \setminus (\Omega_{x,y}^+ \cup \Omega_{x,y}^-)} \rangle = 0 \quad \text{and} \quad \langle f, y\chi_{\bigcup_{i \in I_2} e_i} \rangle = 0.$$

Therefore

$$\begin{aligned} |\langle f, y \rangle| &= |\langle f, y\chi_{\Omega_{x,y}^+} \rangle + \langle f, y\chi_{\Omega_{x,y}^-} \rangle| \\ &= |\langle f, y\chi_{\Omega_{x,y}^+ \cap (\bigcup_{i \in I_1} e_i)} \rangle + \langle f, y\chi_{\Omega_{x,y}^- \cap (\bigcup_{i \in I_1} e_i)} \rangle| \\ &\leq \|y\chi_{\bigcup_{i \in I_1} e_i}\| = \|y\chi_{e_k}\|. \end{aligned}$$

This is a contradiction

Finally, we prove that  $p_x(\cdot)$  is the smallest support semi-norm at  $x$ , i.e.  $p_x(y) = \sup\{|\langle f, y \rangle| : f \in S_x^*\}$  for any  $y \in X$ . If this equality does not hold, then there exist

$\delta > 0$  and  $y \in X$  such that for any  $f \in S_x^*$ ,  $|\langle f, y \rangle| \leq p_x(y) - \delta$ . It is easy to see that  $p_x(y) > 0$ . It follows that  $\max(\|x\chi_{\Omega_{x,y}^+}\|, \|x\chi_{\Omega_{x,y}^-}\|) = \|x\|$ . Let  $E_{x,y} \in \mathcal{E}_{x,y}$  and

$$E_{x,y} = \{e_i: \Omega_{x,y}^+ = e_i \cup e_2, \Omega_{x,y}^- = e_3 \cup e_4; e_1 \cap e_2 = e_3 \cap e_4 = \emptyset\}.$$

Take an arbitrary  $\varepsilon > 0$ . Let

$$e_{j\varepsilon} = \{t \in e_j : (1 - \varepsilon)p_x(y)|x(t)| \leq \|x\| |y(t)|\} \quad \text{if } \|x\chi_{e_j}\| = \|x\|.$$

It follows that there exists at least one  $e_{j\varepsilon}$  such that  $\|x\chi_{e_{j\varepsilon}}\| = \|x\|$ . Indeed, in the opposite case it would deduce that

$$\max_{\substack{\sigma_j \in E'_{x,y} \\ \|x\chi_{\sigma_j}\| = \|x\|}} \|y\chi_{\sigma_j}\| < (1 - \varepsilon)p_x(y),$$

where  $E'_{x,y} = \{\sigma_j : 1 \leq j \leq 4\}$ ,

$$\sigma_1 = \bigcup_{j \in \{1,2\}} \{e_{j\varepsilon} : \|x\chi_{e_j}\| = \|x\|\}, \quad \sigma_2 = \Omega_{x,y}^+ \setminus \sigma_1,$$

$$\sigma_3 = \bigcup_{j \in \{3,4\}} \{e_{j\varepsilon} : \|x\chi_{e_j}\| = \|x\|\}, \quad \sigma_4 = \Omega_{x,y}^- \setminus \sigma_3,$$

and  $\sigma_1$  and  $\sigma_3$  take empty sets if  $\|x\chi_{e_j}\| < \|x\|$  for  $j = 1, 2$  and for  $j = 3, 4$  respectively. This is a contradiction. Let  $f_\varepsilon$  be a support functional at  $x\chi_{e_{j\varepsilon}}$ . In view of Corollary 2.3 one can easily verify that  $f_\varepsilon$  is also a support functional at  $x$ . So we have  $\langle f_\varepsilon, y\chi_{\Omega \setminus e_{j\varepsilon}} \rangle = 0$ . Moreover,  $t \in e_{j\varepsilon}$  implies that  $y(t)$  has the same symbol as  $x(t)$ . Therefore,

$$\begin{aligned} |\langle f_\varepsilon, y \rangle| &= |\langle f_\varepsilon, y\chi_{e_{j\varepsilon}} \rangle| \\ &\geq |\langle f_\varepsilon, x\chi_{e_{j\varepsilon}} \rangle| (1 - \varepsilon)p_x(y) / \|x\| \\ &= (1 - \varepsilon)p_x(y). \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, this means that  $\sup_{f \in S_x^*} |\langle f, y \rangle| \geq p_x(y)$ . This contradicts to the inequality  $\sup_{f \in S_x^*} |\langle f, y \rangle| \leq p_x(y) - \delta$ . This ends the proof.

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#### REFERENCES

1. S. Chen, H. Hudzik and A. Kaminska, *Support functionals and smooth points in Orlicz function spaces equipped with the Orlicz norm*, Math. Japonica **39** (1994), 271–279. MR **95f**:46039
2. S. Chen and H. Sun, *Weak convergence and weak compactness in abstract  $M$  spaces*, Proc. Amer. Math. Soc. **123** (1995), 1441–1447. MR **96b**:46020
3. H. Hudzik and Y. N. Ye, *Support functionals and smoothness of Musielak-Orlicz spaces endowed with the Luxemburg norm*, Comment. Math. Univ. Carolinae **31** (1990), 661–684. MR **92g**:46008;CMP 94:17
4. E. de Jonge, *Representation of linear functionals on a class of normed Köthe spaces*, J. Funct. Anal. **23** (1976), 119–134. MR **54**:11045
5. S. Kakutani, *Concrete representations of abstract  $M$  spaces*, Ann. of Math. **42** (1941), 994–1024. MR **3**:205g

6. L. V. Kantorovich and G. B. Akilov, *Functional Analysis*, Izdat. Nauka, Moscow, 1977. MR **58**:23465
7. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer-Verlag, 1979. MR **81c**:46001
8. W. A. Luxemburg and A. C. Zaanen, *Riesz Spaces I*, North Holland Math. Library, 1971. MR **58**:23483
9. Baoxiang Wang, *Support functional and its applications of Orlicz spaces*, Comment. Math. **33** (1993), 185–192. MR **94m**:46052
10. Baoxiang Wang and Meilan Xu, *On some applications of support functionals of Orlicz spaces*, Comment. Math. **35** (1995), 235–244. MR **97a**:46038
11. B. X. Wang and T. F. Wang, *The characteristic of weak convergence in a class of Banach sequence lattices*, Collect. Math. **48** (1997), 799–807. CMP 98:08
12. Xintai Yu, *Geometry of Banach Spaces*, Shanghai, 1986.
13. A. C. Zaanen, *Riesz Spaces II*, North Holland Math. Library, 1983. MR **86b**:46001

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