PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 6, Pages 1827–1836
S 0002-9939(99)04809-1
Article electronically published on February 18, 1999

ACCELERATIONS OF RIEMANNIAN QUADRATICS

LYLE NOAKES

(Communicated by Christopher Croke)

ABSTRACT. A Riemannian corner-cutting algorithm generalizing a classical construction for quadratics was previously shown by the author to produce a C^1 curve p_∞ whose derivative is Lipschitz. The present paper takes the analysis of p_∞ a step further by proving that it possesses left and right accelerations everywhere. Two-sided accelerations are shown to exist on the complement of a countable dense subset D of the domain. The results are shown to be sharp in the following sense. For almost any scaled triple in Euclidean space there is a Riemannian perturbation of the Euclidean metric such that the two-sided accelerations of the resulting curve p_∞ exist *nowhere* in D.

1. Background in brief

A very detailed description of the construction of the Riemannian quadratics is given in [11], but the following summary is enough for the present paper to be read independently. Let $\langle\, ,\, \rangle$ be a Riemannian metric on an open subset V of $\mathbb{R}^n$, possibly realised as a coordinate chart of a more general Riemannian manifold. Let V be *geodesically convex* in the sense that any two points in V are joined by a geodesic segment, unique up to reparameterization and minimal. Let U be another open subset of $\mathbb{R}^n$ whose closure is compact and contained in V.

Let d be the metric on U defined by the Riemannian distance. Then $d(x_a, x_b)$ is bounded for all $x_a, x_b \in U$. For $a < b \in \mathbb{R}$ let $C[a, b]$ be the complete metric space of continuous curves $\omega\colon [a, b] \to U$ with respect to the uniform metric d_U where

$$d_U(\omega, \omega') = \operatorname*{Max}_{t\in[a,b]} d(\omega(t), \omega'(t)).$$

The Christoffel transformations

$$\Gamma_x\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

are also bounded for $x \in U$.

Because U is convex we can define for any $x_a, x_b \in U$ the *midpoint* $M(x_a, x_b) \in U$ to be $\omega((a + b)/2)$ where $\omega\colon [a, b] \to U$ is the minimal geodesic from x_a to x_b. A *Riemannian scaled triple* is a quadruple $Y = (y_0, y_1, y_2, h) \in U^3 \times \mathbb{R}_+$ where y_0, y_1, y_2 are the *vertexes* of Y and h is the *scale*. The fundamental polygon $p\colon [0, 2h] \to U$ of Y is the track sum of the geodesic segments joining y_0, y_1 and y_1, y_2, parameterized by $[0, h]$ and $[h, 2h]$ respectively. Using the midpoint map

Received by the editors December 7, 1996 and, in revised form, June 11, 1997.
1991 *Mathematics Subject Classification.* Primary 53B20, 53B99; Secondary 41A15, 41A29, 41A99.
Key words and phrases. Geodesic, parallel translation, corner-cutting.

$M\colon U \times U \to U$, a *splitting* of Y into its *left triple* Y^L and *right triple* Y^R are defined as follows.

Definition 1. Let $y_3 = M(y_0, y_1), y_4 = M(y_1, y_2)$ and $y_5 = M(y_3, y_4)$. Then Y^L is the scaled triple $(y_0, y_3, y_5, h/2)$ and $Y^R = (y_5, y_4, y_2, h/2)$.

Splitting can also be applied to Y^L and then Y^R, producing scaled triples

$$Y^{LL}, Y^{LR}, Y^{RL}, Y^{RR}$$

of scales $h/4$. Continuing, after m iterations we obtain, for every word w of length m in the symbols L, R, a scaled triple Y^w of scale $h/2^m$ called *descendants* of Y in generation m. Writing the Y^w in dictionary order, the track-sum p_m of their fundamental polygons turns out to have a smoother appearance than the fundamental polygon p. Indeed the main result of [11] is

Theorem 1. *The sequence $\{p_m\colon m \geq 1\}$ converges uniformly in $C[0, 2h]$ to a curve $p_\infty \in C[0, 2h]$ with the properties*
1. *p_∞ is differentiable on $(0, 2h)$,*
2. *p_∞ is right-differentiable at 0,*
3. *p_∞ is left-differentiable at $2h$,*
4. *$\dot{p}_\infty$ is Lipschitz.*

For the classical quadratic algorithm, $\langle\, ,\, \rangle$ is the Euclidean inner product and p_∞ is well-known to be a quadratic polynomial curve. However, other generalizations of the classical algorithm produce curves with pathological properties [12], [4], [5], [6], [7], [8], [1]. The present Riemannian generalization turns out to be both regular *and* pathological.

Let $D \subset [0, 2h]$ be the countable dense subset consisting of multiples of h by dyadic rationals. Except in the classical case it is rare for p_∞ to be twice differentiable at points in D. Interestingly, although Theorem 1 says p_∞ is C^1 everywhere, it is really only at points in D that this seems plausible. So we might expect higher derivatives of p_∞ to be better behaved on D than elsewhere: exactly the opposite is true. In fact the main result of the present paper is

Theorem 2. 1. *$\dot{p}_\infty$ is left-differentiable on $(0, 2h]$.*
2. *The left-acceleration $\ddot{p}_{\infty-}$ is left-continuous.*
3. *$\dot{p}_\infty$ is right-differentiable on $[0, 2h)$.*
4. *The right-acceleration $\ddot{p}_{\infty+}$ is right-continuous.*
5. *$\dot{p}_\infty$ is differentiable on the complement of D in $[0, 2h]$.*

The proof of Theorem 2 is carried out in two stages. First, candidates for the one-sided *covariant* derivatives of the velocity field $\dot{p}_\infty$ are constructed as limits of sequences of functions $\alpha_{m\pm}\colon [0, 2h] \to \mathbb{R}^n$. The $\alpha_{m\pm}$ are themselves constructed from accelerations of descendants of the scaled triple Y together with the geometric operation of *parallel translation*. Inheritance properties of accelerations of scaled triples lead to analytic results concerning the $\alpha_{m\pm}$ and their limits. The second step is to prove that the limits are in fact one-sided accelerations of p_∞. The main ingredients are an inheritance property for accelerations of scaled triples, and the well-known relationship between geodesics and parallel translation.

The question of whether $\dot{p}_\infty$ has a two-sided derivative at points in D may now be considered. For any Riemannian manifold there will always be special configurations of y_0, y_1, y_2 for which the answer is "yes".

Example 1. Suppose that y_1 lies on the minimal geodesic $\omega\colon [0,2h]$ from y_0 to y_2. Each p_m is obtained by preceding ω with a piecewise-linear function

$$q_m\colon [0,2h] \to [0,2h],$$

and the sequence $\{q_m : m \geq 1\}$ converges uniformly to a quadratic function of the form

$$q_\infty(t) = at + (1-a)t^2/(2h)$$

where $\omega(ah) = y_1$. Then $p_\infty = \omega \circ q_\infty$ and is therefore C^∞.

The answer is also "yes" when $U = \mathbb{R}^n$ with the Euclidean inner product, regardless of the scaled triple Y, because then p_∞ is a quadratic polynomial curve. However, a small Riemannian perturbation can completely change this, as the following result shows.

Theorem 3. *Let y_0, y_1, y_2 be non-colinear points in $\mathbb{R}^n$ with the Euclidean metric. Fix $h > 0$. Then there is a Riemannian metric $\langle\,,\,\rangle$, C^∞-close to the Euclidean inner product, with the following property. For the Riemannian scaled triple $Y = (y_0, y_1, y_2, h)$ of $(\mathbb{R}^n, \langle\,,\,\rangle)$, p_∞ has a two-sided derivative* nowhere *in D.*

The proof is by direct construction and is given in §4.

2. Accelerations of triples

The *mesh* $\mu(Y)$ of a scaled triple $Y = (y_0, y_1, y_2, h)$ is defined to be the larger of $d(y_0, y_1), d(y_1, y_2)$. Recall from [11] that $\mu(Y^w) \leq \mu(Y)/2^m$ where w is a word of length m in the symbols L, R. The *acceleration* $\alpha(Y)$ of a scaled triple Y is defined in [11] as

$$(\dot\omega_{12}(h) - \dot\omega_{01}(h))/(2h)$$

where, for $i = 0, 1$, $\omega_{ii+1}\colon [ih, (i+1)h] \to U$ is the minimal geodesic from y_i to y_{i+1}. Recall from [11] the following *inheritance property* of accelerations of scaled triples.

Lemma 2.1.

$$\alpha(Y^L) = \alpha(Y) + O(1)\mu(Y) = \alpha(Y^R).$$

In a Riemannian setting the *parallel translation* of a vector based at x_a to a vector based at x_b depends on a choice of curve joining a, b. The curve needs to be C^1 and the velocity vector should satisfy a Lipschitz condition. Parallel translation is also defined along a continuous track-sum of such curves with finitely many summands. Parallel translation is transitive and preserves Riemannian inner products. More details can be found in introductory books on Riemannian geometry, for example [2]. For $m \geq 1$ accelerations of scaled triples can be used to define functions

$$\alpha_{m-}, \alpha_{m+}\colon [0,2h] \to \mathbb{R}^n.$$

Roughly speaking, $\alpha_{m-}(s)$ is the parallel translation along p_m to $p_m(s)$ of the acceleration of the nearest scaled triple to the *left* of s in generation m. Replacing "left" by "right" describes $\alpha_{m+}(s)$. The formal definitions are as follows.

Definition A. 1. For $t \in (2(i-1)h/2^m, 2ih/2^m]$ let w be the ith word of length m in the symbols L, R. Then $\alpha_{m-}(t)$ is the parallel translation of $\alpha(Y^w)$ along p_m from $p_m((2i-1)h/2^m)$ to $p_m(t)$.

2. For $t \in [2(i-1)h/2^m, 2ih/2^m)$ let w be the ith word of length m in the symbols L, R. Then $\alpha_{m+}(t)$ is the parallel translation of $\alpha(Y^w)$ along p_m from $p_m((2i-1)h/2^m)$ to $p_m(t)$.

3. Set $\alpha_{m-}(0) = \alpha_{m+}(0)$ and $\alpha_{m+}(2h) = \alpha_{m-}(2h)$.

From the definitions α_{m-} is left-continuous on $(0, 2h]$, and α_{m+} is right-continuous on $[0, 2h)$. Let $D_m = \{2ih/2^m : 1 \le i \le 2^m\}$. Then α_{m-} and α_{m+} agree on $[0, 2h] - D_m$ and so they are continuous except possibly at points in D_m. (Interestingly the left and right derivatives $\dot{p}_{m-}, \dot{p}_{m+}$ of the piecewise-geodesic p_m are *continuous* at points in D_m).

Lemma 2.2. *The sequences*

$$\{\alpha_{m-} : m \ge 1\} \quad \textit{and} \quad \{\alpha_{m+} : m \ge 1\}$$

converge uniformly with respect to the Euclidean norm on $\mathbb{R}^n$.

Proof. For $t \in [2(i-1)h/2^m, 2ih/2^m)$, $\alpha_{m+}(t) - \alpha_{m+1+}(t)$ is the difference after parallel translation along p_m between

$$\alpha(Y^w) \text{ and } \alpha((Y^w)^L), \quad \text{or} \quad \alpha(Y^w) \text{ and } \alpha((Y^w)^R)$$

where w is the ith word of length m in the symbols L, R.

In either case the norm of the difference is $O(1)\mu(Y^w) = O(1)\mu(Y)/2^m$ by Lemma 2.1 applied to the scaled triple Y^w. The same holds when $t = 2h$. So $\{\alpha_{m+} : m \ge 1\}$ is Cauchy and therefore convergent. A similar argument applies to the sequence of left-accelerations and the lemma is proved. □

Let $\alpha_{\infty-}, \alpha_{\infty+}$ be the limits of the respective sequences in Lemma 2.2. Because $\alpha_{m-}|(0, 2h]$ is left-continuous and $\alpha_{m+}|[0, 2h)$ is right-continuous for $m \ge 1$ we obtain

Lemma 2.3. 1. $\alpha_{\infty-}|(0, 2h]$ *is left-continuous.*

2. $\alpha_{\infty+}|[(0, 2h)$ *is right-continuous.*

Lemma 2.4. *If* $s \notin D_m$, *then*

$$\alpha_{\infty+}(s) = \alpha_{\infty-}(s) + O(1)\mu(Y)/2^m.$$

Proof. Because $s \notin D_m$, $s \in (2(i-1)h/2^m, 2ih/2^m)$ for some $1 \le i \le 2^m$. For $r \ge 1$, $\alpha_{m+r-}(s) = \alpha(Z_-) + O(1)\mu(Y)/2^m$ and $\alpha_{m+r+}(s) = \alpha(Z_+) + O(1)\mu(Y)/2^m$ where Z_+, Z_- are descendants in generation r of Y^w (possibly the same descendant). Here w is the ith word in the symbols L, R. By Lemma 2.1

$$\alpha(Z_+) = \alpha(Y^w) + O(1)\mu(Y)/2^m = \alpha(Z_-)$$

and this proves the lemma. □

Let $D_\infty = \bigcup_{m \ge 1} D_m$. By Lemma 2.4, $\alpha_{\infty-}$ and $\alpha_{\infty+}$ agree on the complement of D_∞.

Lemma 2.5. $\alpha_{\infty-}$ *and* $\alpha_{\infty+}$ *are continuous at any point in the complement of* D_∞.

Proof. Because $s \notin D_\infty$, given any $m \ge 1$ and any t sufficiently near s, we have $t \notin D_m$ and therefore $\alpha_{m-}(t) = \alpha_{m+}(t)$. As in the proof of Lemma 2.4,

$$\alpha_{m-}(t) - \alpha_{\infty-}(t) = O(1)\mu(Y)/2^m = \alpha_{m+}(t) - \alpha_{\infty+}(t)$$

and therefore

$$\alpha_{\infty-}(t) - \alpha_{\infty+}(t) = O(1)\mu(Y)/2^m.$$

Similarly, $\alpha_{\infty-}(s) = \alpha_{\infty+}(s)$. Because $\alpha_{\infty-}, \alpha_{\infty+}$ are left and right-continuous this proves the lemma. □

Next we establish relationships between the functions

$$\alpha_{\infty-}, \alpha_{\infty+} : [0, 2h] \to \mathbb{R}^n$$

and the left and right *covariant accelerations* of the curve p_∞ resulting from the Riemannian quadratic construction.

3. Proof of Theorem 2

Let $q\colon [a,b] \to U$ be C^1 where $\dot{q}$ is Lipschitz, and let $X_t \in \mathbb{R}^n$ be a vector based at $q(t)$. The parallel translation of X_t along q from $q(t)$ to $q(s)$ is denoted by $X_{t\to s}$. In particular, $\dot{q}$ may be regarded as a vector field defined along the curve q, so that $\dot{q}(t)$ is based at $q(t)$. We have two kinds of cases in mind. Firstly, q might be p_∞ and then $\dot{q}$ is Lipschitz according to Theorem 1. Secondly, q might be a geodesic segment. Parallel translation along a track-sum of geodesics is defined by parallely translating along successive summands.

Definition C. 1. The *left covariant acceleration* $\nabla_{\dot{q}(s)-}\dot{q}$ of q at time $s \in (0, 2h]$ is defined to be

$$\lim_{t\to s-} (\dot{q}(t)_{t\to s} - \dot{q}(s))/(t - s)$$

whenever the limit exists.

2. The *right covariant acceleration* $\nabla_{\dot{q}(s)+}\dot{q}$ of q at time $s \in [0, 2h)$ is

$$\lim_{t\to s+} (\dot{q}(t)_{t\to s} - \dot{q}(s))/(t - s)$$

when the limit exists.

A necessary and sufficient condition for $\dot{q}$ to be left- (respectively right-) differentiable at s is that the left (respectively right) covariant acceleration of q should exist at time s. A necessary and sufficient condition for the two-sided acceleration $\ddot{q}(s)$ to exist is that

$$\nabla_{\dot{q}(s)-}\dot{q} = \nabla_{\dot{q}(s)+}\dot{q}.$$

The reason for considering covariant accelerations instead of $\ddot{q}$ is that geometric constructions are most easily investigated using covariant objects. When $q = p_\infty$ it is by no means clear whether either of the covariant accelerations exist. A large part of the answer provided by Theorem 2 comes from the following result.

Lemma 3.1. 1. *For* $s \in (0, 2h]$

$$\nabla_{\dot{p}_\infty(s)-}\dot{p}_\infty = \alpha_{\infty-}(s).$$

2. *For* $s \in [0, 2h)$

$$\nabla_{\dot{p}_\infty(s)+}\dot{p}_\infty = \alpha_{\infty+}(s).$$

Proof. Given $s \in [0, 2h)$ and $l \geq 1$ we have $s \in [2(k-1)h/2^l, 2kh/2^l)$ for some k. Let $t > s$ lie in the same subinterval. Given $m > l$ let

$$s \in (2(i-1)h/2^m, 2ih/2^m] \quad \text{and} \quad t \in (2(j-1)h/2^m, 2jh/2^m).$$

Then $i \leq j$. Because s, t are separated by at least $j - i$ subintervals of length $2h/2^m$

$$\text{(SEP)} \qquad 2(j-i)h/2^m \leq t - s.$$

Let X_- and X_+ be the piecewise-continuous left and right velocity vector fields $\dot{p}_{m-}$ and $\dot{p}_{m+}$, defined along the piecewise geodesic curve p_m. When u is an integer each restriction of p_m to a subinterval of $[0, 2h]$ of the form $[(2u-1)h/2^m, (2u+1)h/2^m]$ is a geodesic. Because velocities of geodesics are translated parallely,

$$X_{-(2u+1)h/2^m \to (2u-1)h/2^m} = X_{+(2u-1)h/2^m}.$$

Therefore $X_{+t \to s} - \dot{p}_{m+}(s)$ can be written in the following form:

$$\begin{gathered}
(X_{+t \to (2j-1)h/2^m} - X_{-(2j-1)h/2^m)})_{(2j-1)h/2^m \to s} + \\
(X_{+(2j-3)h/2^m} - X_{-(2j-3)h/2^m})_{(2j-3)h/2^m \to s} + \\
\cdots \\
(X_{+(2u+1)h/2^m} - X_{-(2u+1)h/2^m})_{(2u+1)h/2^m \to s} + \\
(X_{+(2u-1)h/2^m} - X_{-(2u-1)h/2^m})_{(2u-1)h/2^m \to s} + \\
\cdots \\
(X_{+(2i+1)h/2^m} - X_{-(2i+1)h/2^m})_{(2i+1)h/2^m \to s} + \\
(X_{+(2i-1)h/2^m \to s} - \dot{p}_{m+}(s)).
\end{gathered}$$

Here every term is a parallel translation to $p_m(s)$ of a difference of translated velocities parameterized within a subinterval of width $2h/2^m$.

All but the first and last terms are scalar multiples by $2h/2^m$ of parallel translations to $p_m(s)$ of accelerations of the descendants

$$Y^1, Y^2, \ldots, Y^{2^m}$$

of the scaled triple Y. The sum of these intermediate terms is

$$2h/2^m \sum_{i<u<j} \alpha(Y^u)_{(2u-1)h/2^m \to s}.$$

The first and last terms are small multiples of parallel translations $\alpha(Y^j)$ and $\alpha(Y^i)$, depending on the precise locations of s and t. In any case

$$X_{+t \to s} - \dot{p}_{m+}(s) - 2h/2^m \sum_{i<u<j} \alpha(Y^u)_{(2u-1)h/2^m \to s}$$

is bounded in norm by $O(1)\mu(Y)/2^m$ according to Lemma 2.1.

Recall that t was chosen to lie in the subinterval $[2(k-1)h/2^l, 2kh/2^l)$ containing s where l was given. Let Z be the kth descendant of Y in generation l. Then the scaled triples

$$Y^i, Y^{i+1}, \ldots, Y^j$$

are descendants of Z in generation $m - l$. So be Lemma 2.1 the accelerations

$$\alpha(Y^j), \alpha(Y^{j-1}), \ldots, \alpha(Y^{i+1}), \alpha(Y^i)$$

differ in norm by $O(1)\mu(Z) = O(1)\mu(Y)/2^l$. Therefore

$$\begin{aligned}
&\|X_{+t\to s} - \dot{p}_{m+}(s) - (t-s)\alpha(Y^i)\| \\
&\quad \le \|X_{+t\to s} - \dot{p}_{m+}(s) - (j-i)(2h/2^m)\alpha(Y^i)\| + O(1)\mu(Y)/2^m \\
&\quad \le (j-i+1)(2h/2^m)O(1)\mu(Y)/2^l + O(1)\mu(Y)/2^m \\
&\quad = (t-s)O(1)\mu(Y)/2^l + O(1)\mu(Y)/2^m
\end{aligned}$$

by (SEP). As $m \to \infty$ the inequality becomes

$$\|\dot{p}_\infty(t)_{t\to s} - \dot{p}_\infty(s) - (t-s)\alpha_{\infty+}(s)\| = (t-s)O(1)\mu(Y)/2^l.$$

To complete the proof of part 2 of the lemma let $l \to \infty$. Part 1 follows from part 2 applied to the scaled triple (y_2, y_1, y_0, h). □

So the left and right covariant derivatives exist on $(0, 2h]$ and on $[0, 2h)$, respectively. The left covariant derivative is left-continuous and the right covariant derivative is right-continuous by Lemma 2.3. They also agree on the complement of D_∞ by Lemma 2.4. Theorem 2 is proved.

4. Proof of Theorem 3

Let Y be a scaled triple. We first deform the Euclidean metric so as to change the descendants Y^L, Y^R of Y in generation 1 while retaining the Euclidean metric on the convex hull of the vertices of Y^L, Y^R. Only the convex hull is relevant to further corner-cutting and so this kind of perturbation can be carried out with Y^L, Y^R in place of Y, as in Lemma 4.2. The perturbation in Lemma 4.1 is chosen to create differences between the accelerations of Y, Y^L, Y^R.

Lemma 4.1. *Let $Y = (y_0, y_1, y_2, h)$ be a Riemannian scaled triple for the Euclidean metric on $U \subseteq \mathbb{R}^n$ where y_0, y_1, y_2 are not colinear. Then there is a perturbation of the Euclidean metric to a Riemannian metric $\langle\, ,\, \rangle$ with the following properties*:

(a) *$\langle\, ,\, \rangle$ is C^∞-close to Euclidean,*

(b) *$\langle\, ,\, \rangle$ is Euclidean on the convex hull of y_0, y_3, y_4, y_2 where*

$$y_3 = M(y_0, y_1), \qquad y_4 = M(y_1, y_2)$$

and midpoints are calculated using the Riemannian metric $\langle\, ,\, \rangle$,

(c) *neither y_0, y_3, y_5 nor y_5, y_4, y_2 are colinear,*

(d) *$\alpha(Y)$, $\alpha(Y^L)$, $\alpha(Y^R)$ are distinct, where Y^L, Y^R and their accelerations are calculated using $\langle\, ,\, \rangle$.*

Proof. Let $c = (3y_1 + y_2)/4$. Because y_0, y_1, y_2 are not colinear there is an open ball $B(c, r)$ which does not intersect the segment y_0y_1. Without loss $0 < r \le \|y_1 - y_2\|/8$. Let r be so small that $B(x, r)$ does not intersect the segment whose endpoints are $(y_0 + y_1)/2$ and $(y_1 + y_2)/2$.

Modify the Euclidean metric on $\mathbb{R}^n$ by inserting a ridge within $B(x, r/2)$ whose axis A is orthogonal to the segment y_1y_2. Flatten the ends of A within $B(x, r) - B(x, r/2)$ so that the resulting Riemannian metric is Euclidean outside $B(x, r)$. Then

$$y_4 = ay_1 + (1-a)y_2 \quad \text{where } a \in (1/2, 1)$$

and a is close to $1/2$ when the ridge has small height ρ. Choose $\rho > 0$ so small that y_4 is very close to $(y_1 + y_2)/2$, namely so close that the segment y_3y_4 does not intersect $B(x,r)$. Here $y_3 = M(y_0, y_1) = (y_0 + y_1)/2$ as with the Euclidean metric. Then (a), (b) are satisfied.

Since y_0, y_1, y_2 are not colinear neither are

$$y_0(y_0 + y_1)/2, \qquad (y_0 + 2y_1 + y_2)/4$$

nor

$$(y_0 + 2y_1 + y_2)/4, \qquad (y_1 = y_2)/2, y_2.$$

Now $y_3 = (y_0 + y_1)/2$ and

$$y_4 \approx (y_1 + y_2)/2, \qquad y_5 \approx (y_0 + 2y_1 + y_2)/4.$$

If $\rho > 0$ is small enough these approximations ensure (c). To prove (d) note that

$$y_5 = M(y_3, y_4) = (y_0 + (1 + 2a)y_1 + 2(1 - a)y_2)/4$$

and then

$$\alpha(Y^R) - \alpha(Y^L) = 2(1 - 2a)(y_1 - y_2)/h^2.$$

Similarly, $\alpha(Y^L) \neq \alpha(Y) \neq \alpha(Y^R)$. □

Next Lemma 4.1 is used to generate a sequence of perturbations of the Euclidean metric. Perturbations in generation $m+1$ are negligible in comparison with those in generation m, so that differences in accelerations in generation m are not wiped out by subsequent perturbations. Then the main differences between accelerations of triples in generation $m+1$ are attributable to differences in accelerations of parents in generation m.

Lemma 4.2. *Let $Y = (y_0, y_1, y_2, h)$ be a Riemannian scaled triple for the Euclidean metric on $\mathbb{R}^n$ where y_0, y_1, y_2 are not colinear. Then there is a sequence $\{\beta_m > 0 : m \geq 1\} \subset \mathbb{R}$ and, for each $m \geq 1$, a perturbation of the Euclidean metric to a Riemannian metric $\langle\, ,\, \rangle_m$ on $\mathbb{R}^n$ with the following properties, where*

$$Y^1, Y^2, \ldots, Y^{2m}$$

are the descendants of the scaled triple Y of $(\mathbb{R}^n, \langle\, ,\, \rangle)$ in generation m.

(a) *$\langle\, ,\, \rangle_m$ is C^∞-close to Euclidean.*

(b) *$\langle\, ,\, \rangle_m$ is Euclidean on the convex hull of the vertices of any Y^i where $i = 1, 2, \ldots, 2^m$.*

(c) *The vertices of Y^i are not colinear for any $i = 1, 2, \ldots, 2^m$.*

(d) *Let $\alpha(Y^j)_l$ denote the parallel translation*

$$\alpha(Y^j)_{(2j+1)h/2^m \to (2j+3)h/2^m}$$

of $\alpha(Y^j)$ along p_m all calculated with respect to the Riemannian metric $\langle\, ,\, \rangle_l$ where $l \geq m$. Then

$$\|\alpha(Y^j)_l - \alpha(Y^{j+1})\| > \beta_m$$

for $j = 1, 2, \ldots, 2^m - 1$ and all $l \geq m$.

(e) *If Y^i, Y^j have a common ancestor in generation $r < m$, then the norms of differences, after parallel translation along p_m, of $\alpha(Y^i), \alpha(Y^j)$ are smaller than $\beta_r/4$ for the Riemannian metric $\langle\, ,\, \rangle_l$ and any $l \geq m$.*

(f) *The sequence $\{\langle\, ,\, \rangle_m : m \geq 1\}$ converges as a sequence of C^∞ Riemannian metrics to a C^∞ Riemannian metric $\langle\, ,\, \rangle_\infty$.*

Proof. In Lemma 4.1 write $\beta_1 = \|\alpha(Y^l) - \alpha(Y^R)\|/2$. Set $\langle\,,\rangle_1 = \langle\,,\rangle$. Then since $\langle\,,\rangle_1$ is Euclidean on the convex hull of the vertices of Y^l, Y^R condition (e) holds when $m = 1$. The other conclusions depend on the β_m where $m > 1$ and we define these inductively as follows.

Suppose that Lemma 4.2 holds for $m < k$ and let

$$Z^1, Z^2, \ldots, Z^{2^{k-1}}$$

be the descendants of Y in generation $k-1$. Apply Lemma 4.1 to each Z^j, choosing perturbations so small that for any $j = 1, 2, \ldots, 2^{k-1}$ the difference in norms of accelerations of any pair from

$$Z^j, (Z^j)^L, (Z^j)R$$

is less than

$$\left(\min_{m=1,2,\ldots,k-1} \beta_m\right)/2^{k+2}.$$

Take care also that the perturbations are so small that they do not undo the previous inequalities (d) for $m \leq k-1$. Now let $W^1, W^2, \ldots, W^{2k}$ be the immediate descendants of the Z^j and set

$$\beta_k = \min_{i=1,2,\ldots,2^k-1} \|\alpha(W^i) - \alpha(W^{i+1})\|/2.$$

To ensure the convergence in (f) make each perturbation so much smaller than the last that $\{\langle\,,\rangle_m : m \geq 1\}$ is Cauchy. As in [3], Theorem 1.1.11, the C^∞ Riemannian metrics comprise a complete metric space, which proves (f). □

To prove Theorem 3 consider the perturbation $\langle\,,\rangle_\infty$ of the Euclidean metric and the associated Y^w for words w in L, R. If $s = 2ih/2^m \in D_m$, then

$$\|\alpha_{m-}(s) - \alpha_{m+}(s)\| > \beta_m$$

by Lemma 4.2(d). By Lemma 4.2(e)

$$\|\alpha_{m\pm}(s) - \alpha_{\infty\pm}(s)\| \leq \beta_m/4$$

and therefore

$$\|\alpha_{\infty-}(s) - \alpha_{\infty+}(s)\| \geq \beta_m/2.$$

Theorem 3 now follows from Lemma 3.1.

Acknowledgments

The author thanks the referee for a thoughtful reading and constructive suggestions.

References

1. A. S. Cavaretta, W. Dahmen and C. A. Micchelli, *Stationary subdivision*, Mem. Amer. Math. Soc. **93** (1991). MR **92h:**65017
2. S. Gallot, D. Hulin and J. Lafontaine, *Riemannian Geometry*, Springer-Verlag, 1990. MR **91j:**53001
3. R. Narasimhan, *Analysis on real and complex manifolds*, Advanced Studies in Pure Math. 1, Elsevier, 1973. MR **49:**11576
4. G. de Rham, *Sur quelques fonctions différentiables dont toutes les valeurs sont des valeurs critiques*, Celebrazioni archimedee de secolo XX, Siracusa, Vol. II, 61–65, 11-16 aprile 1961. MR **27:**1551

5. ______, *Un peu de mathématiques á propos d'une courbe plane*, Revue de Mathématiques Élémentaires **2** (1947), 73–76, 89–97. MR **9:**246g
6. ______, *Sur certaines équations fonctionelles*, Ouvrage publié á l'occasion de son centenaire par l'École polytechnique de l'Université de Lausanne, 95–97, 1953. MR **19:**842d
7. ______, *Sur une courbe plane*, J. de Math. Pures Appl. **35** (1956), 25–42. MR **19:**842e
8. ______, *Sur les courbes limites de polygones obtenus par trisection*, L'Enseignement Mathématique **5** (1959), 29–43. MR **21:**7508
9. L. Noakes, *Asymptotically smooth splines*, Advances in Computational Math. **4** (1994), 131–137. MR **96h:**65016
10. ______, *A global algorithm for geodesics*, to appear in Bull. Austral. Math. Soc. Series A.
11. ______, *Riemannian quadratics*, submitted to Proc. Third Chamonix Conference on Curves and Surfaces, A. Le Méhauté, C. Rabut and L. L. Schumaker (eds.) (1997), 319–328.
12. H. von Koch, *Sur une courbe continue sans tangente obtenue par une construction géométrique élémentaire*, Arkiv. Mat. Astronomik Fysik **1** (1904), 681–702.

Department of Mathematics, The University of Western Australia, Nedlands, Western Australia 6907, Australia

E-mail address: lyle@maths.uwa.edu.au