

THE CREMMER-GERVAIS SOLUTION OF THE YANG-BAXTER EQUATION

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ABSTRACT. A direct proof is given of the fact that the Cremmer-Gervais R -matrix satisfies the (Quantum) Yang-Baxter equation

1. INTRODUCTION

Let V be a vector space of rank n over a field F . Let $c \in \text{End } V \otimes V$ be a linear operator. Define $c_{12}, c_{23} \in \text{End } V \otimes V \otimes V$ by $c_{12} = c \otimes Id$, $c_{23} = Id \otimes c$. Then c is said to satisfy the Yang-Baxter equation (YBE) if

$$c_{12}c_{23}c_{12} = c_{23}c_{12}c_{23}.$$

An extremely interesting solution of this equation was found by Cremmer and Gervais in their paper [1]. In its slightly more general two parameter form, it is (up to a scalar)

$$c(e_i \otimes e_j) = \begin{cases} qe_j \otimes e_i & \text{if } i = j, \\ qp^{i-j}e_j \otimes e_i + \sum_{i \leq k < j} (q - q^{-1})p^{i-k}e_k \otimes e_{i+j-k} & \text{if } i < j, \\ q^{-1}p^{i-j}e_j \otimes e_i + \sum_{j < k < i} (q^{-1} - q)p^{i-k}e_k \otimes e_{i+j-k} & \text{if } i > j, \end{cases}$$

where $\{e_1, \dots, e_n\}$ is a basis for V , and q and p are non-zero elements of F . Taking $q = p^{n/2}$ yields the original operator given by Cremmer and Gervais. The derivation of this solution used some fairly technical calculations involving chiral vertex operators and is a little inaccessible to the non-specialist. Here we give an elementary proof of this result along the same lines as the proof in [3] for the standard solutions of the Yang-Baxter equation.

2. LINEAR COMBINATIONS OF SOLUTIONS OF THE YBE

Suppose f and g are solutions of the YBE and let $\alpha, \beta \in F$. Expanding the equation

$$(\alpha f + \beta g)_{12}(\alpha f + \beta g)_{23}(\alpha f + \beta g)_{12} = (\alpha f + \beta g)_{23}(\alpha f + \beta g)_{12}(\alpha f + \beta g)_{23},$$

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we see that $c = \alpha f + \beta g$ will be a solution of the YBE for all $\alpha, \beta \in A$ if the following two conditions are satisfied:

$$f_{12}g_{23}g_{12} + g_{12}f_{23}g_{12} + g_{12}g_{23}f_{12} = f_{23}g_{12}g_{23} + g_{23}f_{12}g_{23} + g_{23}g_{12}f_{23},$$

$$g_{12}f_{23}f_{12} + f_{12}g_{23}f_{12} + f_{12}f_{23}g_{12} = g_{23}f_{12}f_{23} + f_{23}g_{12}f_{23} + f_{23}f_{12}g_{23}.$$

In the case where f is the permutation operator $P(e_i \otimes e_j) = e_j \otimes e_i$, the second condition is true for any g (since $g_{12}P_{23}P_{12} = P_{23}P_{12}g_{23}$ and similar equalities hold for the other terms). Thus we obtain the following simple condition which we shall refer to as the *compatibility condition*.

Lemma 2.1. *Suppose that $g \in \text{End } V \otimes V$ is a solution of the YBE. Then $c = \alpha P + \beta g$ will be a solution of the YBE for all $\alpha, \beta \in F$ if*

$$(2.1) \quad g_{12}g_{23}P_{12} + g_{12}P_{23}g_{12} + P_{12}g_{23}g_{12} = g_{23}g_{12}P_{23} + g_{23}P_{12}g_{23} + P_{23}g_{12}g_{23}.$$

We shall apply this result to the case where

$$(2.2) \quad g(e_i \otimes e_j) = \sum_k \eta(i, j, k)e_k \otimes e_{i+j-k}$$

and

$$(2.3) \quad \eta(i, j, k) = \begin{cases} 1 & \text{if } i \leq k < j, \\ -1 & \text{if } j \leq k < i, \\ 0 & \text{otherwise.} \end{cases}$$

Taking $\alpha = q$ and $\beta = (q - q^{-1})$ yields

$$c(e_i \otimes e_j) = \begin{cases} qe_j \otimes e_i & \text{if } i = j, \\ qe_j \otimes e_i + \sum_{i \leq k < j} (q - q^{-1})e_k \otimes e_{i+j-k} & \text{if } i < j, \\ q^{-1}e_j \otimes e_i + \sum_{j < k < i} (q^{-1} - q)e_k \otimes e_{i+j-k} & \text{if } i > j, \end{cases}$$

which is the Cremmer-Gervais operator described in the introduction in the case where $p = 1$. Once we have shown that this operator satisfies the Yang-Baxter equation, it follows from some well-known “twisting” results [2] that the more general operator is also a solution.

3. THE COMPATIBILITY CONDITION

In this section we check the compatibility condition (2.1) for the operator g given above.

Lemma 3.1. *Let $g \in \text{End } V \otimes V$ be an operator of the form*

$$g(e_i \otimes e_j) = \sum_k \eta(i, j, k)e_k \otimes e_{i+j-k},$$

where $\eta(i, j, k) = 0$ if k is not between i and j . Then the condition of Lemma 2.1 is satisfied if and only if

$$(3.1) \quad \begin{aligned} &\eta(i, k, a + b - j)\eta(j, a + b - j, a) + \eta(i, j, b + a - k)\eta(b + a - k, k, a) \\ &+ \eta(i, j, b)\eta(i + j - b, k, a) = \eta(i, k, a)\eta(i + k - a, j, b) \\ &+ \eta(j, k, a)\eta(i, j + k - a, b) + \eta(j, k, j + k - b)\eta(i, j + k - b, a) \end{aligned}$$

for all $i, j, k, a, b \in \{1, 2, \dots, n\}$.

Proof. Let $d_l = g_{12}g_{23}P_{12} + g_{12}P_{23}g_{12} + P_{12}g_{23}g_{12}$ and $d_r = g_{23}g_{12}P_{23} + g_{23}P_{12}g_{23} + P_{23}g_{12}g_{23}$. Denote $e_i \otimes e_j \otimes e_k$ by $[ijk]$. Then

$$\begin{aligned} d_l[ijk] &= \sum_{s,t} \eta(i, k, t)\eta(j, t, s)[s, j + t - s, i + k - t] \\ &\quad + \sum_{s,t} \eta(i, j, s)\eta(s, k, t)[t, s + k - t, i + j - s] \\ &\quad + \sum_{s,t} \eta(i, j, s)\eta(i + j - s, k, t)[t, s, i + j + k - s - t] \end{aligned}$$

and

$$\begin{aligned} d_r[ijk] &= \sum_{s,t} \eta(i, k, s)\eta(k + i - s, j, t)[s, t, i + j + k - s - t] \\ &\quad + \sum_{s,t} \eta(j, k, s)\eta(i, j + k - s, t)[s, t, i + j + k - t - s] \\ &\quad + \sum_{s,t} \eta(j, k, s)\eta(i, t, s)[s, j + k - t, i + t - s]. \end{aligned}$$

Comparing the coefficients of $[a, b, i + j + k - a - b]$ then yields the result. □

Now set

$$u(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases}$$

and

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Notice that $\eta(i, j, k) = u(k - i) - u(k - j)$.

Lemma 3.2. *For any integers a, b, i, j, k ,*

$$\begin{aligned} &u(a + b - i - j)(u(a - j) + u(b - i) - u(b - j) - u(j - b)) + u(k - b)u(a + b - i - k) \\ &= u(a - i)(u(k - b) - u(j - b) - u(b - j) + u(b + a - i - k)) + u(b - i)u(a - j). \end{aligned}$$

Proof. First note that

$$u(x) + u(-x) = 1 + \delta(x)$$

and

$$u(x + y)(u(x) + u(y)) = u(x)u(y) + u(x + y).$$

From this it follows that

$$u(a + b - i - j)((u(a - j) + u(b - i)) = u(a - j)u(b - i) + u(a + b - i - j),$$

$$u(a + b - i - j)(-u(b - j) - u(j - b)) = -u(a + b - i - j) - \delta(b - j)u(a - i)$$

and

$$\begin{aligned}
 u(k-b)u(a+b-i-k) &= (1-u(b-k) + \delta(b-k))u(a+b-i-k) \\
 &= u(a+b-i-k)(1-u(b-k)) + \delta(b-k)u(a-i) \\
 &= u(a+b-i-k)u(a-i) - u(a-i)u(b-k) \\
 &\quad + \delta(b-k)u(a-i) \\
 &= u(a+b-i-k)u(a-i) + u(a-i)(u(k-b) - 1).
 \end{aligned}$$

Combining these equations yields the desired result. \square

Theorem 3.3. For η as defined in (2.3), the operator $g(e_i \otimes e_j) = \sum_k \eta(i, j, k)e_k \otimes e_{i+j-k}$ satisfies the compatibility condition (2.1).

Proof. Expanding the left-hand side of (3.1) using $\eta(i, j, k) = u(k-i) - u(k-j)$ yields

$$\begin{aligned}
 &u(a+b-j-k)[u(a-k) - u(k-b) + u(j-b) - u(a-j)] \\
 &+ u(a+b-i-j)[-u(j-b) + u(a-j) + u(b-i) - u(b-j)] \\
 &+ u(a+b-i-k)[u(k-b) - u(a-k)] \\
 &+ u(a-k)[u(b-j) - u(b-i)].
 \end{aligned}$$

Similarly, the right-hand side becomes

$$\begin{aligned}
 &u(a+b-j-k)[u(a-k) - u(k-b) + u(j-b) - u(a-j)] \\
 &+ u(a-k)[u(b-j) - u(b-i) - u(a+b-i-k)] \\
 &+ u(a-i)[u(k-b) - u(b-j) - u(j-b) + u(a+b-i-k)] \\
 &+ u(b-i)u(a-j).
 \end{aligned}$$

The equality of these two expressions follows from the identity in Lemma 3.2. \square

4. THE YANG-BAXTER EQUATION

In this section we verify that the operator $g(e_i \otimes e_j) = \sum_k \eta(i, j, k)e_k \otimes e_{i+j-k}$ given above satisfies the Yang-Baxter equation. We begin by converting the problem into an identity for η .

Lemma 4.1. Let $g \in \text{End } V \otimes V$ be an operator of the form

$$g(e_i \otimes e_j) = \sum_k \eta(i, j, k)e_k \otimes e_{i+j-k},$$

where $\eta(i, j, k) = 0$ if k is not between i and j . Then g satisfies the Yang-Baxter equation if and only if

$$\begin{aligned}
 (4.1) \quad &\sum_a \eta(j, k, a)\eta(i, a, c)\eta(i+a-c, j+k-a, h) \\
 &= \sum_s \eta(i, j, s)\eta(i+j-s, k, h+c-s)\eta(s, h+c-s, c)
 \end{aligned}$$

for all $i, j, k, c, h \in \{1, 2, \dots, n\}$.

Proof. The left-hand side of (4.1) is the coefficient of $e_c \otimes e_h \otimes e_{i+j+k-c-h}$ in the expansion of $g_{23}g_{12}g_{23}(e_i \otimes e_j \otimes e_k)$. Similarly, the right-hand side is the coefficient of $e_c \otimes e_h \otimes e_{i+j+k-c-h}$ in $g_{12}g_{23}g_{12}(e_i \otimes e_j \otimes e_k)$. \square

The following identities are used in the proof of the next three results.

Lemma 4.2. For integers a, b, c, d, e ,

1. $\eta(a + d, b + d, c + d) = \eta(a, b, c)$
2. $\eta(a, b, c) = -\eta(b, a, c)$
3. $\eta(a, b, c) = \eta(-b, -a, -c - 1) = \eta(a, b, a + b - c - 1)$
4. $\eta(a, a + 1, c) = \delta(a - c)$
5. $\sum_a \eta(b, c, a) = c - b$
6. $\eta(a, b, d) + \eta(b, c, d) = \eta(a, c, d)$
7. $\eta(a, b + 1, c)\eta(c, a, b) = 0$
8. $\eta(a, b, c)\eta(c, b, d) = \eta(a, b, d)\eta(a, d + 1, c)$
9. $\eta(a, b, c)\eta(d, c, e) = \eta(a, b, c)\eta(d, a, e) + \eta(a, b, e)\eta(e + 1, b, c)$.

Proof. The proofs are either trivial or routine calculations. □

Lemma 4.3. For any integers, t, s, b, d, h ,

$$\begin{aligned} \sum_a \eta(t, s, a)\eta(b + a, d - a, h) &= (s - t)\eta(b + t, d - t, h) \\ &\quad + (d - h - s)\eta(d - s, d - t, h) + (h - b - s + 1)\eta(b + t, b + s, h). \end{aligned}$$

Proof. Using the identities of Lemma 4.2,

$$\begin{aligned} &\sum_a \eta(t, s, a)\eta(b + a, d - a, h) \\ &= -\sum_a \eta(t, s, a)\eta(d - b - a, a, h - b) \\ &= -\sum (\eta(t, s, a)\eta(d - b - a, t, h - b) + \eta(t, s, h - b)\eta(h - b + 1, s, a)) \\ &= (h - b - s + 1)\eta(b + t, b + s, h) - \sum_a \eta(t, s, a)\eta(d - b - t, a, d - h - 1). \end{aligned}$$

Now

$$\begin{aligned} &-\sum_a \eta(t, s, a)\eta(d - b - t, a, d - h - 1) \\ &= -\sum_a \eta(t, s, a)\eta(d - b - t, t, d - h - 1) + \eta(t, s, d - h - 1)\eta(d - h, s, a) \\ &= (s - t)\eta(t + b - d, -t, h - d) + (d - s - h)\eta(-s, -t, h - d) \\ &= (s - t)\eta(b + t, d - t, h) + (d - h - s)\eta(d - s, d - t, h). \end{aligned}$$

Combining these two equations yields the assertion. □

Lemma 4.4. For any integers i, j, k, c, h ,

$$\begin{aligned} &\sum_a \eta(j, k, a)\eta(i, a, c)\eta(i + a - c, j + k - a, h) \\ &= \eta(j, k, c)((k - c - 1)\eta(i - c + k, j + k - c, h) + (j - h)\eta(j, j + k - c, h) \\ &\quad + (h - i)\eta(i, i + k - c, h)) \\ &\quad + \eta(i, j, c)((c - i + 1)\eta(i + j - c, i + k - c, h) + (h - j)\eta(i + j - c, j, h) \\ &\quad + (k - h)\eta(i + k - c, k, h)). \end{aligned}$$

Proof. By part 7 of Lemma 4.2

$$\begin{aligned} & \sum_a \eta(j, k, a) \eta(i, a, c) \eta(i + a - c, j + k - a, h) \\ &= \sum_a (\eta(j, k, a) \eta(i, j, c) + \eta(j, k, c) \eta(c + 1, k, a)) \eta(i + a - c, j + k - a, h). \end{aligned}$$

Using Lemma 4.3 we obtain that

$$\begin{aligned} & \sum_a \eta(j, k, a) \eta(i, j, c) \eta(i + a - c, j + k - a, h) \\ &= \eta(i, j, c) ((k - j) \eta(i + j - c, k, h) + (j - h) \eta(j, k, h) \\ & \quad + (h + c - i - k + 1) \eta(i + j - c, i + k - c, h)) \end{aligned}$$

and that

$$\begin{aligned} & \sum_a \eta(j, k, c) \eta(c + 1, k, a) \eta(i + a - c, j + k - a, h) \\ &= \eta(j, k, c) ((k - c - 1) \eta(i + 1, j + k - c - 1, h) + (j - h) \eta(j, j + k - c - 1, h) \\ & \quad + (h + c - i - k + 1) \eta(i + 1, i + k - c, h)) \\ &= \eta(j, k, c) ((k - c - 1) \eta(i, j + k - c, h) + (j - h) \eta(j, j + k - c, h) \\ & \quad + (h + c - i - k + 1) \eta(i, i + k - c, h)). \end{aligned}$$

Using these formulas and repeated application of the identity

$$\eta(a, b, h) + \eta(b, c, h) = \eta(a, c, h)$$

yields the result. \square

Theorem 4.5. For η as defined in (2.3), the operator $g(e_i \otimes e_j) = \sum_k \eta(i, j, k) e_k \otimes e_{i+j-k}$ satisfies the Yang-Baxter equation.

Proof. Let

$$\zeta(i, j, k, c, h) = \sum_a \eta(j, k, a) \eta(i, a, c) \eta(i + a - c, j + k - a, h)$$

(the left-hand side of equation (4.1)). It is easily verified that the right-hand side of equation (4.1) is then $\zeta(i + j - k, i, j, h + c - k, i + j - h)$. Now

$$\begin{aligned} & \zeta(i + j - k, i, j, h + c - k, i + j - h) \\ &= (j + k - h - c - 1) \eta(j, k, c) \eta(i - c + k, j + k - c, h) \\ & \quad + (h - j) \eta(h, j + k - c, k) \eta(i - c + k, j + k - c, h) \\ & \quad + (k - h) \eta(h, j + k - c, k) \eta(i - c + k, j + k - c, h) \\ & \quad + (h + c - i - j + 1) \eta(i, j, c) \eta(i + j - c, i + k - c, h) \\ & \quad + (j - h) \eta(i + j - c, h, j) \eta(i + j - c, i + k - c, h) \\ & \quad + (h - i) \eta(i + j - c, h, i) \eta(i + j - c, i + k - c, h). \end{aligned}$$

We may then rearrange these terms one at a time using Lemma 4.2:

$$\begin{aligned} & (j + k - h - c - 1) \eta(j, k, c) \eta(i - c + k, j + k - c, h) \\ &= (k - c - 1) \eta(j, k, c) \eta(i - c + k, j + k - c, h) \\ & \quad + (j - h) (j + k - h - c - 1) \eta(j, k, c) \eta(i - c + k, j + k - c, h) \end{aligned}$$

and

$$\begin{aligned} &(h - j)\eta(h, j + k - c, k)\eta(i - c + k, j + k - c, h) \\ &= (h - j)\eta(i + k - c, j + k - c, j)\eta(i + k - c, j + 1, h) \\ &= (h - j)\eta(i + k - j, k, c)\eta(i + k - c, j, h). \end{aligned}$$

Similarly

$$\begin{aligned} (k - h)\eta(h, j + k - c, k)\eta(i - c + k, j + k - c, h) &= (k - h)\eta(i, j, c)\eta(i + k - c, k, h), \\ (h + c - i - j + 1)\eta(i, j, c)\eta(i + j - c, i + k - c, h) \\ &= (c - i + 1)\eta(i, j, c)\eta(i + j - c, i + k - c, h) \\ &\quad + (h - j)\eta(i, j, c)\eta(i + j - c, i + k - c, h), \\ (j - h)\eta(i + j - c, h, j)\eta(i + j - c, i + k - c, h) \\ &= (j - h)\eta(i + k - j, i, c)\eta(i + k - c, j, h) \end{aligned}$$

and

$$(h - i)\eta(i + j - c, h, i)\eta(i + j - c, i + k - c, h) = (h - i)\eta(k, j, c)\eta(i + k - c, i, h).$$

Adding these terms and rearranging easily yields $\zeta(i, j, k, c, h)$ as required. \square

Finally we make some observations about invertibility and the Hecke condition. Recall that R is said to be *Hecke* if it satisfies the condition

$$(R - q)(R + q^{-1}) = 0$$

for some q .

- Lemma 4.6.** 1. $g^2 = g$.
 2. $gP = -g$.
 3. $Pg = g + P - I$.

Proof. The first part follows from the identity

$$\sum_k \eta(i, j, k)\eta(k, i + j - k, l) = \eta(i, j, l),$$

which is a consequence of Lemma 4.3. The second and third parts follow from the identities $\eta(j, i, k) = -\eta(i, j, k)$ and $\eta(i, j, i + j - k) = \eta(i, j, k) + \delta(k - j) - \delta(k - i)$, respectively. \square

Proposition 4.7. *Let α and β be non-zero elements of F . The operator $R = \alpha P + \beta g$ is invertible if and only if $\alpha \neq \beta$. It is Hecke if and only if $\beta = \alpha - \alpha^{-1}$.*

Proof. Using Lemma 4.6 we find that

$$R^2 = \beta R + \alpha(\alpha - \beta)I,$$

and the proposition then follows immediately. \square

Theorem 4.8. *Let F be a field and let V be a vector space with basis $\{e_1, \dots, e_n\}$. Let $c \in \text{End } V \otimes V$ be the linear operator*

$$c(e_i \otimes e_j) = qp^{i-j}e_j \otimes e_i + \sum_k (q - q^{-1})p^{i-k}\eta(i, j, k)e_k \otimes e_{i+j-k}.$$

Then c is an invertible solution of the Yang-Baxter equation.

Proof. For $p = 1$ the result follows from Lemma 2.1, Theorem 3.3, Proposition 4.7 and Theorem 4.5. For more general p we apply [2, Theorem 3.3]. \square

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