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ON THE ASYMPTOTIC STABILITY IN FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Consider a system of functional differential equations $dx/dt = f(t, x_t)$ where f is the vector-valued functional. The classical asymptotic stability result for such a system calls for a positive definite functional $V(t, \varphi)$ and negative definite functional dV/dt. In applications one can construct a positive definite functional V, whose derivative is not negative definite but is less than or equal to zero. Exactly for such cases J. Hale created the effective asymptotic stability criterion if the functional f in functional differential equations is autonomous (f does not depend on t), and N. N. Krasovskii created such criterion for the case where the functional f is periodic in t. For the general case of the non-autonomous functional f V. M. Matrosov proved that this criterion is not right even for ordinary differential equations. The goal of this paper is to prove this criterion for the case when f is almost periodic in t. This case is a particular case of the class of non-autonomous functionals.

1. Introduction

Let $x = (x_1, \dots, x_n) \in R^n, t \in R, |x| = \sqrt{x_1^2 + \cdots + x_n^2}$. For a given $h > 0, C$ denotes the space of continuous functions mapping $[-h; 0]$ into R^n and for $\varphi \in C$, $\|\varphi\| = \sup_{-h \le \theta \le 0} |\varphi(\theta)|$. According to [4] we denote

$$C_H = \{\varphi \in C : \|\varphi\| \le H\}.$$

If x is a continuous function of u defined on $-h \le u < A, A > 0$, and if t is a fixed number satisfying $0 \le t < A$, then x_t denotes the restriction of x to the segment $[t-h; t]$ so that x_t is an element of C defined by $x_t(\theta) = x(t+\theta)$ for $-h \le \theta \le 0$.

Consider a system of functional differential equations

$$\frac{dx}{dt} = f(t, x_t) \tag{1}$$

and obtain conditions on a Lyapunov functional to insure that the zero solution is asymptotically stable.

In this system dx/dt denotes the right-hand derivative of x at t, t is time, and $f(t, \varphi) \in R^n$ is defined on $[0; \infty) \times C_H$; $f(t; 0) \equiv 0$.

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According to [4] we denote by $x(t_0,\varphi)$ a solution of (1) with initial condition $\varphi \in C_H$ where $x_{t_0}(t_0,\varphi)=\varphi$, and we denote by $x(t,t_0,\varphi)$ the value of $x(t_0,\varphi)$ at t and $x_t(t_0,\varphi)=x(t+\theta,t_0,\varphi)$, $-h\le\theta\le 0$.

It is assumed that the vector-valued functional $f(t,\varphi)$ is continuous on $[0;\infty)\times C_H$ so that a solution will exist for each continuous initial condition. For continuation of solutions, we suppose that f takes closed bounded sets of $[0;\infty)\times C_H$ into closed bounded sets of R^n.

Let $V(t,\varphi)$ be a continuous functional defined for $t\ge 0, \varphi\in C_H$. The upper right-hand derivative of V along solutions of (1) is defined to be [9]

$$\frac{dV(t,x_t(t_0,\varphi))}{dt}=\overline{\lim_{\Delta t\to+0}}\{V(t+\Delta t,x_{t+\Delta t}(t_0,\varphi))-V(t,x_t(t_0,\varphi))\}\frac{1}{\Delta t}.$$

If V satisfies a Lipschitz condition in the second argument, then this limit is uniquely determined.

The classical criterion of asymptotic stability of zero solution of equations (1), which was obtained by N. N. Krasovskii [10], assumes the existence of a positive definite functional V and a negative definite functional dV/dt. In applications one can constract a positive definite functional V, whose derivative is not negative definite but is less than or equal to zero. Exactly for such cases J. Hale [8] created the effective asymptotic stability criterion if the functional f in equations (1) is autonomous (f does not depend on t), and N. N. Krasovskii [10] created such criterion for the case where the operator f is periodic in t. For the general case of the non-autonomous operator f V. M. Matrosov [12] proved that this criterion is not right even for ordinary differential equations. The goal of this paper is to prove this criterion for the case when f is almost periodic in t. This case is a particular case of the class of non-autonomous operators.

2. Definitions and preliminary results

Definition 1 ([1], [2], [3], [5], [6], [11], [14], [15])**.** A continuous function $F(t):R\to R^n$ is called almost periodic if for every $\varepsilon>0$ there exists $l=l(\varepsilon)>0$ such that any segment $[\alpha;\alpha+l],\alpha\in R$, contains at least one number τ such that $|F(t+\tau)-F(t)|<\varepsilon$ for every $t\in R$. A number τ is called an ε-almost period of F.

Let us introduce the following definition which is analogous to [11], [14].

Definition 2. A continuous functional $F(t,\varphi):R\times C_r\to R^n$ $(0<r<\infty)$ is called uniformly almost periodic in t if for every $\varepsilon>0$ there exists $l=l(\varepsilon,r)>0$ such that any segment $[\alpha;\alpha+l],\alpha\in R$, contains at least one number τ such that $|F(t+\tau,\varphi)-F(t,\varphi)|<\varepsilon$ for every $t\in R,\varphi\in C_r$.

Remark. A continuous function $F(t)$, which satisfies Definition 1 is called uniformly almost periodic in papers [1], [2], [3], [11], so Definitions 1 and 2 are somewhat different from corresponding Definitions in [1], [2], [3], [11].

Lemma 1 ([11])**.** *Let $F_1(t),...,F_N(t):R\to R^n$ be almost periodic functions. Then for every $\varepsilon>0$ there exists $l=l(\varepsilon)>0$ such that any segment $[\alpha;\alpha+l],\alpha\in R$, contains a number τ such that*

$$|F_i(t+\tau)-F_i(t)|<\varepsilon,\qquad i=1,2,...,N;\ t\in R.$$

We denote

$$C_{H(L)} = \{\varphi \in C_H : |\varphi(x_1) - \varphi(x_2)| \le L|x_1 - x_2| \quad \text{for each} \quad x_1, x_2 \in [-h; 0]\} \subset C_H.$$

Lemma 2. *If the functional $F(t,\varphi) : R \times C_{H(L)} \to R^n$ is Lipschitzian in φ and almost periodic in t for every fixed $\varphi \in C_{H(L)}$, then it is uniformly almost periodic in t.*

Proof. Since the functional $F(t,\varphi)$ satisfies Lipschitz conditions in φ, then

$$|F(t,\varphi) - F(t,\psi)| \le L_1\|\varphi - \psi\| \tag{2}$$

where L_1 is the Lipschitz constant.

Let $\varepsilon > 0$ be any real number. $C_{H(L)}$ is the set of uniformly bounded equicontinuous functions, therefore $C_{H(L)}$ is a compact set. Hence there is a finite set of functions $\varphi_1, ..., \varphi_N$ such that $\varphi_j \in C_{H(L)}$ $(j = 1, ..., N)$ and for each $\varphi \in C_{H(L)}$ there exists a number i $(1 \le i \le N)$ such that

$$\|\varphi - \varphi_i\| < \frac{\varepsilon}{3L_1}. \tag{3}$$

From Lemma 1 it follows that there exists $l > 0$ such that in any segment $[\alpha; \alpha + l]$ there exists a number τ , such that

$$|F(t,\varphi_i) - F(t+\tau,\varphi_i)| < \frac{\varepsilon}{3} \tag{4}$$

for each $t \in R, i = 1, ..., N$.

We will now show that for every $\varphi \in C_{H(L)}$, each number τ, which satisfies inequality (4), is an ε-almost period of the functional $F(t,\varphi)$. Let φ_k be the same element of the set $\varphi_1, ..., \varphi_N$ for which $\|\varphi - \varphi_k\| < \varepsilon/(3L_1)$. Then from (2)-(4) we obtain

$$|F(t+\tau,\varphi) - F(t,\varphi)| \le |F(t+\tau,\varphi) - F(t+\tau,\varphi_k)|$$

$$+|F(t+\tau,\varphi_k) - F(t,\varphi_k)| + |F(t,\varphi_k) - F(t,\varphi)|$$

$$< \frac{\varepsilon}{3} + 2L_1 \cdot \frac{\varepsilon}{3L_1} = \varepsilon. \tag{5}$$

The inequality (5) proves Lemma 2. □

3. Main results

In this section we consider the system of functional differential equations (1) under the assumptions above. We also assume that the functional $f(t,\varphi)$ is Lipschitzian in φ and almost periodic in t for every fixed $\varphi \in C_H$.

Lemma 3. *Consider the solution $x(t_0,\varphi_0)$ of the system (1). We suppose that $x_t(t_0,\varphi_0)$ belongs to C_r $(0 < r < H)$ for $t \ge 0$. Let $\{\varepsilon_k\}$ be a monotonically approaching zero sequence of positive numbers and $\{\tau_k\}$ a sequence of ε_k-almost periods of $f(t,\varphi)$ (for every ε_k there corresponds an ε_k-almost period τ_k). Then the limit relation*

$$\lim_{k\to\infty} \|x_{t^*}(t_0,\varphi_k) - x_{t^*+\tau_k}(t_0,\varphi_0)\| = 0 \tag{6}$$

holds, where $\varphi_k = x_{t_0+\tau_k}(t_0,\varphi_0)$ and t^* is a fixed moment of time which is more than t_0 $(t^* > t_0)$.

Proof. Consider the solutions of the system (1)

$$x(t_0, \varphi_k) \tag{7}$$

and

$$x(t_0 + \tau_k, \varphi_k). \tag{8}$$

For the time $\Delta t = t^* - t_0$ the function φ_k moves to the function $x_{t^*}(t_0, \varphi_k)$ along the trajectory (7) and φ_k moves to the function

$$x_{t^*+\tau_k}(t_0 + \tau_k, \varphi_k) = x_{t^*+\tau_k}(t_0, \varphi_0)$$

along the solution (8). The restriction of the solution x of the system (1), $x_t(t_0 + \tau_k, \varphi_k)$, with initial boundary value problem $\varphi_k = x_{t_0+\tau_k}$ may be interpreted as one of the system

$$\frac{dx}{dt} = f(t + \tau_k, x_t) \tag{9}$$

with initial function φ_k and initial moment of time t_0. If t is large enough, then $x_t \in C_{H(L)}$. But according to Lemma 2 the right-hand side of the system (1) is uniformly almost periodic in t on the set $R \times C_{H(L)}$, therefore the right-hand sides of the systems (1), (9) differ from each other no matter how small, if k is a large enough natural number. Hence the limit relation (6) follows. □

Theorem 1. *Let functional differential equations (1) satisfy the above conditions. There exists a continuous functional $V(t, \varphi) : R \times C_H \to R$ which is locally Lipschitz in φ such that the following conditions are fulfilled on the set $R \times C_H$:*

- (i) *$a(|\varphi(0)|) \le V(t, \varphi) \le b(\|\varphi\|)$, where $a, b \in K$; K is a class of Hahn's functions* [7], [13]*;*
- (ii) *$V(t, \varphi)$ is almost periodic in t for each fixed $\varphi \in C_H$;*
- (iii) *$dV/dt \le 0$, $dV/dt \not\equiv 0$ on each solution of the system (1).*

Then the solution

$$x = 0 \tag{10}$$

of functional differential equations (1) is asymptotically stable.

Proof. From conditions (i), (iii) it follows that the solution (10) is uniformly stable [8], [9]. Let $\varepsilon \in (0; H)$ be any positive number. Denote by $t_0 \in R$ the initial moment of time. By the stability of the zero solution there exists $\delta > 0$ such that if $\varphi \in C_\delta$, then $x_t(t_0, \varphi) \in C_\varepsilon$ for every $t \ge t_0$. Choose such a $\delta > 0$ and show that any solution $x(t_0, \varphi)$ with $\varphi \in C_\delta$ tends to zero as $t \to \infty$. Suppose that this is not true, i.e. there exist $\eta > 0$ and $\varphi_0 \in C_\delta$ such that $|x(t, t_0, \varphi_0)| > \eta > 0$ as $t \ge t_0$.

The function $V(t) = V(t, x_t(t_0, \varphi_0))$ is monotonically non-increasing because $dV/dt \le 0$. Hence there exists the limit

$$\lim_{t\to\infty} V(t) = \lim_{t\to\infty} V(t, x_t(t_0, \varphi_0)) = V_0 \ge a(\eta) > 0$$

and it is easy to see that $V(t, x_t(t_0, \varphi_0)) \ge V_0$ for $t \in [t_0; \infty)$.

Consider some monotonically approaching zero sequence $\{\varepsilon_k\}$ of positive numbers, where ε_1 is sufficiently small. By Lemma 2 for every ε_i there exists a sequence of ε_i-almost periods $\tau_{i,1}, \tau_{i,2}, ..., \tau_{i,n}, ... \to \infty$ for functionals $f(t, \varphi)$ and $V(t, \varphi)$, such that inequalities

$$|V(t + \tau_{i,n}, \varphi) - V(t, \varphi)| < \varepsilon_i,$$

$$|f(t + \tau_{i,n}, \varphi) - f(t, \varphi)| < \varepsilon_i$$

hold for each $t \in R, \varphi \in C_\varepsilon$. Without loss of generality one can suppose $\tau_{i,n} < \tau_{i+1,n}$ for every i, n. Designate $\tau_k = \tau_{k,k}$.

Consider the sequence of functions $\varphi_k = x_{t_0+\tau_k}(t_0, \varphi_0) \quad (k = 1, 2, ...)$. It is a bounded sequence of equicontinuous functions because $\varphi_k \in C_\varepsilon$, therefore there is a limit function φ^* of this sequence. Without loss of generality one can assume the sequence φ_k itself converges to φ^*. Because of continuity and almost periodicity of the functional $V(t, \varphi)$ we obtain

$$V(t_0, \varphi^*) = \lim_{n\to\infty} V(t_0, \varphi_n)$$

$$= \lim_{k\to\infty} \lim_{n\to\infty} V(t_0 + \tau_k, \varphi_n)$$

$$= \lim_{n\to\infty} V(t_0 + \tau_n, \varphi_n)$$

$$= \lim_{n\to\infty} V(t_0 + \tau_n, x_{t_0+\tau_n}(t_0, \varphi_0)) = V_0.$$

Now consider the solution $x(t_0, \varphi^*)$. From the condition (iii) of the theorem, the existence of such a moment of time t^* $(t^* > t_0)$ follows, when the inequality

$$V(t^*, x_{t^*}(t_0, \varphi^*)) = V_1 < V_0$$

holds.

Solutions of functional differential equations (1) are continuous in initial data, so one can write

$$\lim_{k\to\infty} \|x_{t^*}(t_0, \varphi_k) - x_{t^*}(t_0, \varphi^*)\| = 0$$

because

$$\lim_{k\to\infty} \|\varphi_k - \varphi^*\| = 0.$$

Hence it follows that

$$\lim_{k\to\infty} V(t^*, x_{t^*}(t_0, \varphi_k)) = V_1 \tag{11}$$

Using the uniform almost periodicity property of $f(t, \varphi)$ and the limit relation (6), we obtain the inequality

$$\|x_{t^*}(t_0, \varphi_k) - x_{t^*+\tau_k}(t_0, \varphi_0)\| \le \gamma_k \tag{12}$$

where $\gamma_k \to 0$ as $k \to \infty$. Because of the uniform almost periodicity property of $V(t, \varphi)$ we have

$$|V(t^*, \varphi) - V(t^* + \tau_k, \varphi)| < \varepsilon_k \tag{13}$$

for every $\varphi \in C_H$, and from conditions (11), (12) it follows that

$$|V(t^*, x_{t^*+\tau_k}(t_0, \varphi_0)) - V_1| < \eta_k, \tag{14}$$

where $\eta_k \to 0$ as $k \to \infty$.

From (13) we obtain

$$|V(t^*, x_{t^*+\tau_k}(t_0,\varphi_0)) - V(t^*+\tau_k, x_{t^*+\tau_k}(t_0,\varphi_0))| < \varepsilon_k. \tag{15}$$

From (14), (15) we have

$$|V(t^*+\tau_k, x_{t^*+\tau_k}(t_0,\varphi_0)) - V_1| < \eta_k + \varepsilon_k, \tag{16}$$

where $\eta_k + \varepsilon_k \to 0$ as $k \to \infty$.

On the other hand

$$\lim_{k\to\infty} V(t^*+\tau_k, x_{t^*+\tau_k}(t_0,\varphi_0)) = V_0. \tag{17}$$

The relations (16), (17) are in contradiction to the inequality $V_1 < V_0$. This contradiction proves Theorem 1. □

Theorem 2. *Let the right-hand side of functional differential equations (1) be such that there exists a continuous functional $V(t,\varphi) : R \times C_H \to R$ which is locally Lipschitz in φ such that the following conditions are fulfilled on the set $R \times C_H$:*

(i) $|V(t,\varphi)| \le b(\|\varphi\|),\ b \in K$;
(ii) $V(t,\varphi)$ *is almost periodic in t for each fixed $\varphi \in C_H$;*
(iii) *for every $t \in R$ and for every $\delta > 0$ there exists $\varphi \in C_\delta$, such that $V(t,\varphi) > 0$;*
(iv) $dV/dt \ge 0;\ dV/dt \not\equiv 0$ *on each solution of the system (1).*

Then the solution (10) of functional differential equations (1) is unstable.

Proof. Let $\varepsilon \in (0; H)$. We shall take arbitrary $t_0 \in R$ and arbitrary small $\delta > 0$. Let us choose $\varphi_0 \in C_\delta$ such that $V(t_0,\varphi_0) > 0$. We can do it by condition (iii) of the theorem. By the condition (i) there exists $\eta > 0$ such that $|V(t,\varphi)| < V(t_0,\varphi_0)$ for every $\varphi \in C_\eta$. The function $V(t) = V(t, x_t(t_0,\varphi_0))$ is nondecreasing, i.e. $V(t, x_t(t_0,\varphi_0)) \ge V(t_0,\varphi_0)$ for $t \ge t_0$. It means that $\|x_t(t_0,\varphi_0)\| \ge \eta$ for each $t \ge t_0$. We shall show that there exists a moment of time t_1 $(t_1 > t_0)$, such that $\|x_{t_1}(t_0,\varphi_0)\| > \varepsilon$. Suppose that this is not true, i.e. inequalities

$$\eta \le \|x_t(t_0,\varphi_0)\| \le \varepsilon \tag{18}$$

hold for each $t > t_0$.

Using inequalities (18) and the condition (iv) of the theorem, we obtain a contradiction by means of the same way as in the proof of Theorem 1. We omit the literal repetition of these reasonings. The contradiction proves that the semitrajectory $x(t_0,\varphi_0)$ leaves C_ε. The proof is complete. □

Example. Consider the non-linear equation

$$\begin{aligned} \frac{dx}{dt} &= -2x^3(t) + 4x^2(t)x(t-h) \\ &\quad + (-12 + 3\sin^2(\sqrt{2}t) + 3\sin^2 t)x(t)x^2(t-h) + 4x^3(t-h) \end{aligned} \tag{19}$$

and the functional

$$V(t,x_t)=\frac{1}{2}x^2(t)+\int_{t-h}^{t}x^4(\theta)d\theta.$$

Its time derivative dV/dt along the solutions of (19) is

$$\begin{aligned}\frac{dV}{dt}&=\frac{1}{2}\cdot 2x(t)\left[-2x^3(t)+4x^2(t)x(t-h)\right.\\&\qquad\left.+(-12+3\sin^2(\sqrt{2}t)+3\sin^2 t)x(t)x^2(t-h)+4x^3(t-h)\right]\\&\quad+x^4(t)-x^4(t-h)\\&=-2x^4(t)+4x^3(t)x(t-h)+(-12+3\sin^2(\sqrt{2}t)+3\sin^2 t)x^2(t)x^2(t-h)\\&\quad+4x(t)x^3(t-h)+x^4(t)-x^4(t-h)\\&=-x^4(t)+4x^3(t)x(t-h)-6x^2(t)x^2(t-h)+4x(t)x^3(t-h)-x^4(t-h)\\&\quad+(-6+3\sin^2(\sqrt{2}t)+3\sin^2 t)x^2(t)x^2(t-h)\\&=-\left[x(t)-x(t-h)\right]^4-3(2-\sin^2(\sqrt{2}t)-\sin^2 t)x^2(t)x^2(t-h).\end{aligned}$$

For any $\varepsilon>0$ small enough there exists a sequence $t_1,t_2,\dots,t_n,\dots\to+\infty$ such that

$$0<2-\sin^2(\sqrt{2}t_i)-\sin^2 t_i<\varepsilon\quad (i=1,2,\dots).$$

The right-hand side of the equation (19) is not periodic in t. Hence we cannot apply Krasovskii's corresponding theorems on the asymptotic stability, but we can use Theorem 1 because the right-hand side of (19) is almost periodic in t and $dV/dt<0$ for each $t>0$. Therefore the zero solution of (19) is asymptotically stable.

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