SPECIAL-VALUED SUBGROUPS
OF LATTICE-ORDERED GROUPS

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(Communicated by Lance W. Small)

Abstract. We prove that the intersection of all maximal special-valued subgroups of a lattice-ordered group \( G \) is the special-valued quasi-torsion radical of a lattice-ordered group \( G \), which extends our earlier result that the intersection of all maximal finite-valued subgroups of a lattice-ordered group \( G \) is the finite-valued torsion radical of \( G \). We also show that the class \( A_f \) of almost finite-valued lattice-ordered groups is a quasi-torsion class, and the \( A_f \) quasi-torsion radical of a group is equal to the intersection of the group with the lateral completion of the finite-valued torsion radical of the group.

Introduction

For the basic definitions and results in lattice-ordered group theory, see M. Anderson and T. Feil [2] and M. Darnel [8]. A lattice-ordered group, written \( \ell \)-group, is a partially ordered group \((G, \leq)\) where the partial order is a lattice (meaning that each pair of elements \( a, b \) of \( G \) has a least upper bound \( a \lor b \) and a greatest lower bound \( a \land b \)). An \( \ell \)-subgroup \( A \) of an \( \ell \)-group \( G \) is both a subgroup and a sublattice of \( G \). \( A \) is a convex \( \ell \)-subgroup of \( G \) if \( a, b \in A \) and \( a \leq g \leq b \) imply that \( g \in A \). A normal convex \( \ell \)-subgroup is an \( \ell \)-ideal. A convex \( \ell \)-subgroup which is maximal with respect to not containing some \( g \in G \) is called regular and is a value of \( g \). A regular subgroup \( A \) is an essential value if it contains all the values for some \( g \in G \). Element \( g \) is special if it has a unique value and in this case the value is called a special value. Regular subgroups of \( G \) form a root system under inclusion, written \( \Gamma(G) \). (That is, \( \Gamma(G) \) is a partially ordered set for which \( \{ \alpha \in \Gamma(G) \mid \alpha \geq \gamma \} \) is totally ordered, for any \( \gamma \in \Gamma(G) \).) A subset \( \Delta \subseteq \Gamma(G) \) is plenary if \( \cap \Delta = \{0\} \) and \( \Delta \) is a dual ideal in \( \Gamma(G) \); that is, if \( \delta \in \Delta, \gamma \in \Gamma(G) \) and \( \gamma > \delta \), then \( \gamma \in \Delta \). If \( G \) is an abelian \( \ell \)-group, then \( G \) is \( \ell \)-isomorphic to an \( \ell \)-subgroup of \( V(\Gamma(G), R) \) such that if \( \gamma \) is a value of \( g \in G \), then \( \gamma \) is a maximal component of \( g \) after the embedding, where \( V(\Gamma(G), R) \) is the abelian \( \ell \)-group of all real-valued functions \( v \) on \( \Gamma(G) \) for which \( v(\gamma) \in R \) and the support of each \( v \) satisfies the ascending chain condition. This is a consequence of the Conrad-Harvey-Holland embedding theorem for abelian lattice-ordered groups.

Received by the editors July 16, 1996 and, in revised form, September 2, 1997.
1991 Mathematics Subject Classification. Primary 06F15, 06F20; Secondary 20F60.
Key words and phrases. Torsion class and quasi-torsion class, finite-valued and special-valued subgroups of a lattice-ordered group.

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\( \Sigma(\Delta, R) \) is the \( \ell \)-subgroup of \( V(\Delta, R) \) containing all elements \( v \in V \) with finite supports. \( F(\Delta, R) \) is the \( \ell \)-subgroup of \( V(\Delta, R) \) containing all elements \( v \in V \) whose supports are contained in a finite number of chains in \( \Delta \).

For any \( g \in G \), \( G(g) = \{ h \in G \mid |h| \leq n \mid g \} \), for some positive integer \( n \), the principal convex \( \ell \)-subgroup of \( G \) generated by \( g \), is the least convex \( \ell \)-subgroup of \( G \) that contains \( g \).

An \( \ell \)-group \( G \) is finite-valued if every element of \( G \) has only a finite number of values; this is equivalent to the statement that every element of \( G \) can be expressed as a finite sum of disjoint special elements. Each element of \( G \) is also called finite-valued. An \( \ell \)-group \( G \) is special-valued if \( G \) has a plenary subset of special values; this is equivalent to the statement that each positive element of \( G \) can be expressed as the join of a set of pairwise disjoint positive special elements. A positive element \( g \) of \( G \) is special-valued if \( g \) can be expressed as the join of disjoint special elements.

A lattice homomorphism is complete if it preserves all (not necessarily finite) meets and joins. A convex \( \ell \)-subgroup is closed if it is closed with respect to infinite meets and joins which exist in the \( \ell \)-group. A convex \( \ell \)-subgroup \( C \) of an \( \ell \)-group \( G \) is closed if and only if the natural lattice homomorphism from \( G \) onto its lattice \( G/C \) of right cosets is complete. Extensions which preserve the lattice of closed convex \( \ell \)-subgroups are called \( \alpha^* \)-extensions.

An \( \ell \)-group is laterally complete (conditionally laterally complete) if for any subset (bounded subset) \( \{g_\alpha \mid \alpha \in A\} \) of disjoint positive elements, \( \lor_{\alpha} g_\alpha \) exists.

A torsion class is a class of lattice-ordered groups that is closed under convex \( \ell \)-subgroups, \( \ell \)-homomorphic images, and joins of convex \( \ell \)-subgroups. For an \( \ell \)-group \( G \) and a torsion class \( T \), \( T(G) \) indicates the join of all the convex \( \ell \)-subgroups of \( G \) that belong to \( T \). \( T(G) \) is then the largest convex \( \ell \)-subgroup of \( G \) that belongs to \( T \), called the torsion radical of \( G \). A quasi-torsion class is a class of \( \ell \)-groups which is closed under convex \( \ell \)-subgroups, complete \( \ell \)-homomorphic images, and joins of convex \( \ell \)-subgroups. For an \( \ell \)-group \( G \) and a quasi-torsion class \( Q \), \( Q(G) \) indicates the join of all the convex \( \ell \)-subgroups of \( G \) that belong to \( Q \). \( Q(G) \) is then the largest convex \( \ell \)-subgroup of \( G \) that belongs to \( Q \), called the quasi-torsion radical of \( G \). Finite-valued \( \ell \)-groups form a torsion class \( F_v \), and special-valued \( \ell \)-groups form a quasi-torsion class \( S \).

We have shown that the finite-valued torsion radical of an \( \ell \)-group \( G \) is the intersection of all maximal finite-valued subgroups of \( G \) [6]. Let \( S \) be the quasi-torsion class of special-valued \( \ell \)-groups. We will show that the quasi-torsion radical \( S(G) \) is the intersection of all the maximal special-valued subgroups of \( G \). We will show that the class \( A_f \) of almost finite-valued \( \ell \)-groups is a quasi-torsion class, and its quasi-torsion radical of \( G \) is equal to the intersection of \( G \) with the lateral completion of the finite-valued torsion radical of \( G \) : \( A_f(G) = F_v(G)^L \cap G \). Also for each \( \ell \)-group \( G \), the following are equivalent:

1. There exists a largest finite-valued subgroup of \( G \).
2. The set \( \Delta \) of special values of \( G \) is an \( \ell \)-ideal of \( \Gamma(G) \).
3. \( S(G) \) is the largest special-valued subgroup of \( G \) and \( S(G) \) is almost finite-valued.

1. Maximal special-valued subgroups

Definition 1.1. A special-valued subgroup of an \( \ell \)-group \( G \) is an \( \ell \)-subgroup \( U \) such that for each \( 0 < g \in U \), \( g = \lor_{\alpha} g_\alpha \), where the \( g_\alpha \)'s are disjoint and special in \( G \).
Thus if $g \in U$, then $|g| = \vee_A g_\alpha$, where $g_\alpha \in G$ are disjoint and special, but we don’t require that $g_\alpha \in U$.

If $\cdots \subseteq C_\alpha \subseteq C_\beta \subseteq C_\gamma \subseteq \cdots$ is a chain of special-valued subgroups of $G$, then $\cup C_\lambda$ is a special-valued subgroup. Hence each special-valued subgroup is contained in a maximal special-valued subgroup.

**Proposition 1.2.** If $0 < a = \vee_A a_\alpha$ and $0 < b = \vee_B b_\beta$ are special-valued elements in $G$, where $A$ and $B$ are sets of values, then $a + b$ is special-valued in $G$ with the set of maximal elements in $A \cup B$ as special values.

**Proof.** Consider

$$0 < a + b = \vee_A a_\alpha + b = \vee_A (a_\alpha + b) = \vee_A (a_\alpha + \vee_B b_\beta) = \vee_A (\vee_B (a_\alpha + b_\beta)) = \vee_{A \cup B} (a_\alpha + b_\beta).$$

Suppose that $\alpha > \beta$, for some $\beta \in B$. Then $\alpha$ is greater than some subset of $B$ and disjoint from the other elements of $B$.

Thus $(a_\alpha + b_\beta) \vee (a_\alpha + b_\beta_3) \vee (a_\alpha + b_\beta_3) \vee \cdots = a_\alpha + (b_\beta_1 \vee b_\beta_2 \vee b_\beta_3 \vee \cdots)$ is special with value $\alpha$, and $\alpha$ is maximal in $A \cup B$.

If $\alpha = \beta$, for some $\beta \in B$, then $a_\alpha + b_\beta$ is special with value $\alpha$, and $\alpha$ is maximal in $A \cup B$.

If $\alpha$ is not comparable with any $\beta$, then $a_\alpha + b_\beta = a_\alpha \vee b_\beta$ with values $\alpha$ and $\beta$, and both are maximal in $A \cup B$. \hfill $\square$

**Corollary 1.3.** If $a$ and $b$ are finite-valued, then so is $a + b$.

**Proposition 1.4.** If $b$ is a special-valued element with $b > 0$, then $b + a$ is special-valued and has the same special values as $b$.

**Proof.** Suppose that $b = \vee_\Lambda b_\lambda$ with $b_\lambda$ disjoint and special. Then $a = b \wedge a = (\vee_\Lambda b_\lambda) \wedge a = \vee_\Lambda (b_\lambda \wedge a)$. We have

$$b + a = b + \vee_\Lambda (b_\lambda \wedge a) = \vee_\Lambda (b + (b_\lambda + a)) = \vee_\Lambda (\vee_\Lambda b_\gamma + (b_\lambda + a)) = \vee_\Lambda (\vee_\Lambda (b_\gamma + (b_\lambda + a))) = \vee_{\gamma, \lambda} (b_\gamma + (b_\lambda + a)).$$

If $\gamma \neq \lambda$, then $b_\gamma + (b_\lambda \wedge a) = b_\gamma \vee (b_\lambda \wedge a) \leq (b_\gamma + (b_\gamma \wedge a)) \vee (b_\lambda + (b_\lambda \wedge a))$. Therefore $b + a = \vee_\Lambda (b_\lambda + (b_\lambda \wedge a)).$
Now we show that $b_\lambda + (b_\lambda \land a)$ is special with value $\lambda$. Since $b_\lambda \in G^\lambda \setminus G_\lambda$, where $G_\lambda$ is the regular subgroup of $G$, and $G^\lambda$ the cover of $G_\lambda$, we have $b_\lambda + (b_\lambda \land a) \in G^\lambda \setminus G_\lambda$. Now let $\alpha$ be a value of $b_\lambda + (b_\lambda \land a)$. Then $b_\lambda \notin G_\alpha$, so $G_\alpha \subseteq G_\lambda$. If $G_\alpha \subset G_\lambda$, then $b_\lambda \leq b_\lambda + (b_\lambda \land a) \in G^\alpha \subseteq G_\lambda$. This contradicts the fact that $b_\lambda \in G^\lambda \setminus G_\lambda$. Therefore $\alpha = \lambda$. □

**Proposition 1.5.** An $\ell$-subgroup $C$ is the largest special-valued subgroup of $G$ if and only if $C = S(G)$ which consists of all the special-valued elements of $G$.

**Proof.** $(\Leftarrow)$ If $C$ consists of all the special-valued elements of $G$, then it is the largest special-valued subgroup of $G$.

$(\Rightarrow)$ If $0 < g$ is special-valued, then the $\ell$-subgroup $\langle g \rangle$ of $G$ generated by $g$ is a special-valued subgroup of $G$. Therefore $\langle g \rangle \subseteq C$, and hence $C$ consists of all the special-valued elements of $G$.

If $b$ is special-valued and $b > a > 0$, then by the last proposition, $b, b + a \in C$, so $a \in C$. Thus $C$ is convex and hence $C \subseteq S(G)$. Since $S(G)$ is a special-valued subgroup of $G$, $S(G) \subseteq C$. □

**Corollary 1.6.** For an $\ell$-group $G$, the following are equivalent:

1. There exists a largest special-valued subgroup of $G$.
2. $S(G)$ consists of all the special-valued elements of $G$.
3. $b \in G$ is special and $b > a > 0$ imply that $a$ is special-valued.
4. $S(G)$ contains all the special elements of $G$.

**Proof.** By the above proposition $1 \iff 2$, and clearly $2 \implies 3$.

3 $\implies$ 4. $b$ is special implies that $G(b) \subseteq S(G)$, so $S(G)$ contains all the special elements of $G$.

4 $\implies$ 2. $0 < g$ is special-valued implies that $g = \lor_A g_\lambda$. Each $g_\lambda \in S(G)$, and $S(G)$ is closed. Hence, we have $g \in S(G)$.

**Proposition 1.7.** Let $G$ be an $\ell$-group. If $a$ is a positive special-valued element of $G$, $b$ is a negative element of $S(G)$, and $a + b$ is positive, then $a + b$ is special-valued in $G$.

**Proof.** $0 < a = \lor_A a_\alpha$, where $a_\alpha$ are disjoint and special with value $\alpha$, and $0 < -b = \lor_B(-b_\beta)$, where $-b_\beta$ are disjoint and special with value $\beta$.

Now $0 < a + b = \lor_A a_\alpha + b$, so $0 < -b < \lor_A a_\alpha$, and hence by Proposition 1.4, $(\lor_A a_\alpha - b)$ is special-valued with the same set of special values as $\lor_A a_\alpha$. Thus $A$ is the set of special values for $\lor_A a_\alpha - b = \lor_A a_\alpha + \lor_B(-b_\beta)$. So each $\beta$ is less than or equal to one and only one $\alpha$ and is incomparable with the other $\alpha$’s. Now consider

$$0 < a + b = \lor_A a_\alpha + b = \lor_A (a_\alpha + b) = \lor_A ((a_\alpha + b) \lor 0).$$

Case I. $a_\alpha \land |b| = 0$. In this case $a$ is incomparable with all $\beta$, thus $(a_\alpha + b) \lor 0 = a_\alpha$.

Case II. $\alpha = \beta$ for some $\beta$. We then have

$$0 < -b = \lor_B(-b_\beta) = -b_\beta + \lor_{B \setminus \{\beta\}}(-b_\gamma),$$

where $-b_\beta$ and $\lor_{B \setminus \{\beta\}}(-b_\gamma)$ are disjoint. Hence

$$(a_\alpha + b) \lor 0 = (a_\alpha + b_\beta - \lor_{B \setminus \{\beta\}}(-b_\gamma)) \lor 0 = (a_\alpha + b_\beta) \lor 0$$

is a positive element in $S(G)$ with all special values less than or equal to $\alpha$, since $a_\alpha + b_\beta$ is disjoint from $\lor_{B \setminus \{\beta\}}(-b_\gamma)$. 

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Case III. \( \alpha > \beta \) for some \( \beta \). Let \( B_{\alpha} = \{ \beta \in B \mid \alpha > \beta \} \). Hence for \( \beta \in B \setminus B_{\alpha} \), \( \alpha \parallel \beta \).

We now have that \( -b = \vee_B(-b_3) \) and that \( a_\alpha \wedge (-b) = a_\alpha \wedge \vee_B(-b_3) = \vee_B(a_\alpha \wedge (-b_3)) = \vee_B(-b_3) \), so \( -b = \vee_{B_{\alpha}}(-b_3) + \vee_{B \setminus B_{\alpha}}(-b_3) \) and \( a_\alpha + b = a_\alpha - \vee_{B_{\alpha}}(-b_3) - \vee_{B \setminus B_{\alpha}}(-b_3) \), where \( a_\alpha - \vee_{B_{\alpha}}(-b_3) \) and \( \vee_{B \setminus B_{\alpha}}(-b_3) \) are positive and disjoint. Therefore, \( (a_\alpha + b) \vee 0 = a_\alpha - \vee_{B_{\alpha}}(-b_3) \) is special with value \( \alpha \).

Thus \( (a_\alpha + b) \vee 0 \) can be written as the join of disjoint special elements and has all its special values less than or equal to \( \alpha \). Hence \( 0 < a + b = \vee_A((a_\alpha + b) \vee 0) \) can be written as the join of disjoint special elements, hence is special valued.

**Proposition 1.8.** If \( U \) is a special-valued subgroup of \( G \), then \( U + S(G) \) is a special-valued subgroup of \( G \).

**Proof.** Since \( S(G) \) is an \( \ell \)-ideal, we have that \( U + S(G) \) is an \( \ell \)-subgroup of \( G \). Consider \( 0 < g = a + b \in U + S(G) \), with \( a \in U \) and \( b \in S(G) \). We have \( g + S(G) = a + S(G) = a \vee 0 + S(G) \), so without loss of generality, we may assume \( a > 0 \). Then \( 0 < a + b = a + b^+ - b^- \), where \( a + b^+ \) can be written as the join of disjoint special elements of \( G \). So by Proposition 1.7, \( a + b \) can be written as the join of disjoint special elements of \( G \).

We are now ready to describe the special-valued quasi-torsion radical for an \( \ell \)-group by its maximal special-valued subgroups.

**Theorem 1.9.** \( S(G) \) is the intersection of all the maximal special-valued subgroups of \( G \).

**Proof.** By Proposition 1.8, \( S(G) \) is contained in the intersection of all the maximal special-valued subgroups of \( G \). Now we pick \( 0 < a \in G \setminus S(G) \). We need to show there exists a maximal special-valued subgroup that does not contain \( a \). If \( a \) is not special-valued, then it does not belong to any special-valued subgroups of \( G \). Now suppose that \( a = a_1 \vee a_2 \vee a_3 \vee \cdots \), where \( a_i \in G \) are disjoint and special. \( G(a) \not\subseteq S(G) \), so there exists \( b \in G \) such that \( a > b > 0 \) and \( b \) is not special-valued in \( G \). By Proposition 1.4, \( a + b \) is special-valued. Thus there exists a maximal special-valued subgroup of \( G \) that contains \( a + b \) but not \( a \).

**Proposition 1.10.** If \( 0 < a \) and \( 0 < b \) are special-valued elements in an \( \ell \)-group \( G \), then so are \( a \wedge b \) and \( a \vee b \).

**Proof.** Consider \( 0 < \vee_A a_\alpha \) and \( 0 < \vee_B b_\beta \), where the \( a_\alpha \)'s are disjoint and \( a_\alpha \) has special value \( \alpha \), and the \( b_\beta \)'s are disjoint and \( b_\beta \) has special value \( \beta \).

Let \( C = \{ \alpha \in A \mid \text{there exists } \beta \in B \text{ comparable with } \alpha \} \). Then \( C = C_1 \cup C_2 \), where \( C_1 = \{ \alpha \in C \mid \beta \leq \alpha \text{ for some } \beta \in B \} \) and \( C_2 = \{ \alpha \in C \mid \alpha < \beta \text{ for some } \beta \in B \} \).

We have
\[
 a \vee b = (\vee_A a_\alpha) \vee (\vee_B b_\beta) = \vee_A(a_\alpha \wedge (\vee_B b_\beta)) = \vee_A(\vee_B(a_\alpha \vee b_\beta)) \\
 = \vee_{A \cup B}(a_\alpha \vee b_\beta) = \vee_A \vee_C (a_\alpha \vee b_\beta) \vee_{B \setminus C} b_\beta \\
 = \vee_A \vee_C (a_\alpha \vee (\vee_{C_1} b_\beta)) \vee (b_\beta \vee (\vee_{C_2} a_\alpha)) \vee_{B \setminus C} b_\beta.
\]

Therefore, \( a \vee b \) is special-valued with values maximal in \( A \cup B \). Similarly,
\[
 a \wedge b = (\vee_A a_\alpha) \wedge (\vee_B b_\beta) = \vee_A(a_\alpha \wedge (\vee_B b_\beta)) = \vee_{A \cup B}(a_\alpha \wedge b_\beta) = \vee_C(a_\alpha \wedge b_\beta).
\]

Therefore, \( a \wedge b \) is special-valued with values minimal in \( C \).
Theorem 1.11. A maximal special-valued subgroup $U$ of an abelian $\ell$-group $G$ is weakly saturated; i.e., special components of elements of $U$ belong to $U$.

Proof. Suppose that $v \in U$. Then $v = g_1 \lor g_2 \lor g_3 \lor \ldots$, where $g_i \in G$ are disjoint and special. Assume (by way of contradiction) that $g_1 \notin U$, and let $\langle U, g_1 \rangle$ be the $\ell$-subgroup of $G$ generated by $U$ and $g_1$. Consider $0 < f \in \langle U, g_1 \rangle$. Then

$$f = f \lor 0 = \lor_I \land_J ((u_{ij} \pm n_{ij}g_1) \lor 0)$$

where $I$ and $J$ are finite index sets, $u_{ij} \in U$ and $n_{ij}$ are positive integers. We will first show that for fixed $i$ and $j$, each $(u_{ij} \pm n_{ij}g_1) \lor 0$ is special-valued.

Consider $(u \pm ng_1) \lor 0$, $u \in U$ hence $|u| = h_1 \lor h_2 \lor h_3 \lor \ldots$, where $h_i \in G$ are disjoint and special. If the value of $g_1$ is different from all those of $u$, then clearly $(u \pm ng_1) \lor 0$, is special-valued.

If the value of $g_1$ is the same as a value of $u$, say, a value of $h_1$, but the value of $g_1$ is the same as that of $h_1 \pm ng_1$, then we have $(u \pm ng_1) \lor 0$ is special-valued.

Now consider the case when the value of $g_1$ is the same as that of $h_1$, and values of $h_1 \pm ng_1$ are less than that of $g_1$. We then have

$$(h_1 \pm ng_1) \lor 0 \leq (u \pm nv) \lor 0 = f_1 \lor f_2 \lor f_3 \lor \ldots$$

where $0 < f_i \in G$ are disjoint and special, since $(u \pm nv) \lor 0 \in U$.

Let $F = \{f_j \mid f_j \leq g_1\}$, $\lor\{f_j \mid f_j \in F\}$ exists in $G$ because $\lor\{f_j \mid f_j \in F\} = (f_1 \lor f_2 \lor f_3 \lor \ldots) \land g_1$. We claim that

$$(h_1 \pm ng_1) \lor 0 = \lor\{f_j \mid f_j \in F\}.$$ 

Clearly $(h_1 \pm ng_1) \lor 0 \leq \lor\{f_j \mid f_j \in F\}$, since $(h_1 \pm nv) \lor 0 \leq f_1 \lor f_2 \lor f_3 \lor \ldots$. On the other hand, each $f_j \leq (h_1 \pm ng_1) \lor 0$ implies that $\lor\{f_j \mid f_j \in F\} \leq (h_1 \pm ng_1) \lor 0$. We have shown that $(h_1 \pm ng_1) \lor 0$ is special-valued which implies that $(u \pm ng_1) \lor 0$ is special-valued. Thus by Proposition 1.10, $\lor_I \land_J ((u_{ij} \pm n_{ij}g_1) \lor 0)$ is special-valued in $G$. \hfill \Box

Corollary 1.12. A maximal special-valued subgroup $U$ of an abelian $\ell$-group is special-valued as an $\ell$-group.

Proof. Let $U$ be a maximal special-valued subgroup of $G$. For each $0 < u \in U$, $u = g_1 \lor g_2 \lor g_3 \lor \ldots$, where $g_i \in G$ are disjoint and special. By Theorem 1.11, each $g_i \in U$. Also each $g_i$ is special in $U$. Therefore $U$ is special-valued as an $\ell$-group. \hfill \Box

2. Almost finite-valued $\ell$-groups

Definition 2.1. An $\ell$-group $G$ is almost finite-valued if it is special-valued and the principal convex $\ell$-subgroup $G(g)$ is finite-valued for each finite-valued element $g \in G$.

We use $A_f$ to indicate the class of almost finite-valued $\ell$-groups. We first list some basic properties of $A_f$ and brief proofs.

1. $F_v \subseteq A_f \subseteq S$, where $F_v$ is the torsion class of finite-valued $\ell$-groups, and $S$ is the quasi-torsion class of special-valued $\ell$-groups.

2. $A_f$ is closed with respect to products.
Proof. The quasi-torsion class $S$ of special-valued $\ell$-groups is closed with respect to products, and if $0 < g$ is special in a product, then it is special in one of the factors, so $G(g)$ is finite-valued.

3. $A_f$ is not closed with respect to $\ell$-subgroups.
   For example, for each $x \in [0, 1]$, let $R_x$ be the real numbers. Then
   \[
   \prod_{x \in [0, 1]} R_x \in A_f,
   \]
   but $C[0, 1] \notin S$.

4. $G \in A_f$ does not imply that $G_L \in A_f$.
   For example, let $\Lambda = \{0\}$. Then $\Sigma(\Lambda, R) \in A_f$, but $\Sigma(\Lambda, R)_L = V(\Lambda, R) \notin A_f$.

5. If $G \in A_f$, then $F_v(G)$ is the $\ell$-ideal generated by all the $G(g)$ with $g$ special.
   So $F_v(G)$ is the largest finite-valued subgroup of $G$.

6. A laterally complete special-valued $\ell$-group is almost finite-valued if and only if its set of special values $\Delta$ contains no copy of $\Lambda$.
   Proof. Suppose $\Lambda$ is contained in $\Delta$. Then let $g$ be a special element with value $\delta$ which is the maximal element of $\Delta$. Since $G$ is laterally complete, $G(g)$ contains an element with an infinite number of values. On the other hand, let $g$ be a special element with value $\delta$, and $\Delta$ contains no copy of $\Lambda$. Then the ideal $\delta = \{\alpha \in \Delta \mid \alpha \leq \delta\}$ contains only a finite number of roots, so $G(g)$ is finite-valued.

7. For a completely distributive $\ell$-group $G$, the following are equivalent:
   (a) $G_L$ is almost finite-valued.
   (b) The plenary set $\Delta$ of all essential values of $\Gamma(G)$ contains no copy of the set $\Lambda$ of (6).

   Proof. By [3], $\Delta \cong$ the set of all the special elements of $\Gamma(G_L)$.

8. If $G$ is a special-valued vector lattice, then without loss of generality, we assume that $\Sigma(\Delta, R) \subseteq G \subseteq V(\Delta, R)$. $V(\Delta, R) \in A_f$ if and only if $\Delta$ contains no copy of $\Lambda$ as in (6). Thus each abelian $a^*$-extension of $G$ is almost finite-valued if and only if $\Delta$ contains no copy of $\Lambda$.

Theorem 2.2. For an $\ell$-group $G$, the following are equivalent:
1. There exists a largest finite-valued subgroup of $G$.
2. $F_v(G)$ consists of all the finite-valued elements of $G$.
3. $0 < a < b$ and $b$ is special imply that $a$ is finite-valued.
4. $b$ is special implies that $G(b)$ is finite-valued.
5. $F_v(G)$ contains all the special elements of $G$.
6. The set $\Delta$ of special values of $G$ is an ideal of $\Gamma(G)$.
7. $S(G)$ is the largest special-valued subgroup of $G$, and $S(G)$ is almost finite-valued.

Proof. We proved in [6] that 1, 2, and 3 are equivalent, and clearly 3 $\iff$ 4 and 2 $\iff$ 5.

Now suppose $0 < g \in G$ is special with value $G_\alpha$. Then $G_\alpha \rightarrow G_\alpha \cap G(g)$ is a one-to-one order-preserving map of the regular subgroups of $G$ that do not contain $G(g)$ onto the regular subgroups of $G(g)$.
6 \rightarrow 4. \Delta is an \ell\text{-ideal of } \Gamma(G) implies that all the regular subgroups of \Gamma(g) are special if \gamma is special. So \Gamma(g) is finite-valued.

4 \rightarrow 7. \Gamma(g) is finite-valued implies that \Gamma(g) \subseteq S(G). So S(G) contains all the special elements.

7 \rightarrow 4. \gamma is special implies \gamma \in S(G). Thus \Gamma(g) \subseteq S(G), hence \Gamma(g) is finite-valued.

4 \rightarrow 6 is clear.

This theorem shows that if \Gamma has a largest finite-valued subgroup, then \Gamma has a largest special-valued subgroup, although the converse is not true.

For example, let \Lambda = \bigwedge \ldots\).

\Gamma = V(\Lambda, R) \in S, but \Gamma has no largest finite-valued subgroup.

Actually, there exists a largest special-valued subgroup of \Gamma if and only if for each special \gamma \in \Gamma, G(\gamma) is special-valued. In turn, this is equivalent to the assertion that for each special value \delta \in \Delta, the ideal \delta = \{\gamma \in \Gamma(G) | \gamma \leq \delta\} contains a dual ideal of special elements with zero intersection.

We now give the proof that \Lambda is a quasi-torsion class.

**Theorem 2.3.** The almost finite-valued \ell\text{-groups form a quasi-torsion class } \Lambda.

*Proof.\* \Gamma is almost finite-valued if and only if the set \Delta of special values is a plenary subset and an ideal of \Gamma(G). A special-valued \ell\text{-group } \Gamma is almost finite-valued if and only if \Delta is an ideal of \Gamma(G).

1. \Lambda is closed with respect to convex \ell\text{-subgroups.}

Suppose \Gamma is a convex \ell\text{-subgroup of an almost finite-valued \ell\text{-group } \Gamma. Then } \Gamma \subseteq S, since S is a quasi-torsion class. The set \Gamma of all special values of \Gamma is an ideal and a plenary subset of \Gamma, the set of all special values of \Gamma, and \Gamma is an ideal in \Gamma(G). Therefore \Gamma is an ideal in \Gamma(C) and a plenary subset of \Gamma(C).

2. \Lambda is closed with respect to complete \ell\text{-homomorphic images.}

Let \Gamma be a closed \ell\text{-ideal of } \Gamma \subseteq \Lambda. Then \Gamma/C \subseteq S. Suppose \gamma + g is special in \Gamma/C, g = g_1 \vee g_2 \vee g_3 \vee \ldots in G, with not all \gamma_i belonging to C. If there is more than one component of \gamma not in C, say, g_1 is one of them, then \gamma + g_1 is a proper component of \gamma + g, so \gamma + g is not special. So without loss of generality, \gamma is special in G. The convex \ell\text{-subgroup of } G/C generated by \gamma + g is isomorphic to \frac{\Gamma(g)}{C \cap \Gamma(g)} and it is finite-valued, since F_\ell is a torsion class.

3. \Lambda is closed with respect to joins of convex \ell\text{-subgroups.}

S(\Gamma) is the largest convex \ell\text{-subgroup that belongs to } S. If the largest convex \ell\text{-subgroup } \Gamma is almost finite-valued exists, then it is an \ell\text{-ideal of } S(\Gamma). So without loss of generality, we assume that \Gamma is special-valued. A convex \ell\text{-subgroup } C of \Gamma is almost finite-valued if and only if \Gamma is an ideal of \Gamma. So \Gamma is almost finite-valued if and only if \Gamma is an ideal in \Gamma(G).

If \Gamma \subseteq \Gamma \subseteq \Gamma \subseteq \ldots form a chain of convex \ell\text{-subgroups of } \Gamma that belong to \Lambda, then their sets of special values C_1 \subseteq C_2 \subseteq C_3 \subseteq \ldots form a chain of ideals in \Gamma(G), so \Gamma \subseteq C_\Lambda is an ideal in \Gamma(G), hence \Gamma is almost finite-valued.

If B and C are almost finite-valued, and 0 < g \in B + C is special with value \delta, then \gamma \in B or \gamma \in C. In fact, g is equal to the sum of positive elements from B, and one of them must have value \delta. Without loss of generality, there exists 0 < b \in B with a value \delta, hence nb > g for some n \in N, so g \in B. Thus B(g) is
finite-valued, which implies \((B + C)(g)\) is finite-valued. Therefore, \(B + C\) is almost finite-valued.

The torsion radical \(F_v(G)\) is a characteristic \(\ell\)-ideal of \(G\), and \(F_v(G) \subseteq G \subseteq G^L\), where \(G^L\) is the lateral completion of \(G\). Actually, \(F_v(G)^L \subseteq G^L\). This follows from the following proposition.

**Proposition 2.4.** \(C \in C(G)\), the set of all convex \(\ell\)-subgroups of \(G\), implies that \(C^L \subseteq G^L\).

**Proof.** \(C \subseteq C''\), the polar of \(C\) in \(G^L\), and since \(C''\) is closed, it is laterally complete. Now \(G\) is dense in \(G^L\), so \(0 < x \in C'' \subseteq G^L\) implies \(x > y > 0\), for some \(y \in G \cap C''\). In particular, \(y\) is not disjoint from \(C^+\), so \(y \wedge c > 0\), for some \(c \in C\). Therefore \(x > y \wedge c \in C\), so \(C\) is dense in the laterally complete \(\ell\)-group \(C''\), and hence \(C^L \subseteq C'' \subseteq G^L\).

The following theorem demonstrates the relation between the torsion radicals \(A_f(G)\) and \(F_v(G)\).

**Theorem 2.5.** \(A_f(G) = F_v(G)^L \cap G\).

**Proof.** \(F_v(G)^L\) is special-valued and has the same set \(\Delta\) of special values as \(F_v(G)\) by the result in [3]. Consider \(0 < g \in F_v(G)^L \cap G\), \(g = h_1 \vee h_2 \vee h_3 \vee \cdots\), where \(h_i\) are disjoint and special in \(F_v(G)^L\). Now there exists a special element \(0 < g_1 \in F_v(G)\) with the same value as \(h_1\). Thus \(ng_1 > h_1\) for some \(n \in N\), so \(h_1 = ng_1 \wedge g \in F_v(G)\); thus \(G(h_1)\) is finite-valued.

Suppose \(0 < u < g\) with \(u \in G\). Then \(u = u \wedge g = u \wedge (\vee h_i) = \vee (u \wedge h_i)\). Each \(u \wedge h_i \in G(h_i)\) is finite-valued. Therefore \(u\) is special-valued.

If \(0 < a \in G(g)\) is special, then \(a < ng\), so \(a < nh_i\), for some \(n \in N\). Thus \(G(a)\) is finite-valued. Therefore, \(G(g) \subseteq A_f(G)\), so \(F_v(G)^L \cap G \subseteq A_f(G)\).

Conversely, consider \(0 < g \in A_f(G)\). Then \(g = g_1 \vee g_2 \vee g_3 \vee \cdots\), where \(g_i\) are disjoint and special, and each \(G(g_i)\) is finite-valued. Therefore each \(g_i \in F_v(G)\), so \(g \in F_v(G)^L \cap G\).

Thus if \(G\) is laterally complete, then \(A_f(G) = F_v(G)^L\). So \(G \in A_f\) if and only if \(G = F_v(G)^L\). In fact, the proof of the theorem shows that \(F_v(G)(1) \cap G = A_f(G)\), where \(F_v(G)(1)\) is the \(\ell\)-group generated by the joins of disjoint elements from \(F_v(G)\) [9]. Therefore \(F_v(G)(1) \cap G = F_v(G)^L \cap G\). We now discuss almost finite-valued \(\ell\)-groups that are laterally complete.

**Proposition 2.6.** Let \(G\) be a special-valued laterally complete \(\ell\)-group. If \(g \in G\) is infinite valued, then \(F_v(G) + g\) is not finite-valued in \(G/F_v(G)\).

**Proof.** If \(F_v(G) + g\) is finite-valued, then without loss of generality, it is special in \(G/F_v(G)\). But \(g\) can be written as \(a + b\) with \(a \wedge b = 0\), where \(a\) and \(b\) are both infinite-valued. Thus \(F_v(G) + a\) and \(F_v(G) + b\) are non-zero components of \(F_v(G) + g\). This contradicts the fact that \(F_v(G) + g\) is special in \(G/F_v(G)\).

**Proposition 2.7.** Let \(G\) be a special-valued laterally complete \(\ell\)-group. \(G \in A_f\) if and only if \(G/F_v(G)\) contains no special elements.

**Proof.** \((\implies)\) If \(g \in G \backslash F_v(G)\), then \(g\) is infinite-valued and hence \(F_v(G) + g\) is not special.
If $0 < g \in G$ is special, then $g \in F_v(G)$. For otherwise $F_v(G) + g$ is not special in $G/F_v(G)$. Therefore $F_v(G)$ contains all the special elements of $G$, and hence $G \in A_f$.

The next theorem follows from the above theorem and propositions.

**Theorem 2.8.** For a special-valued laterally complete $\ell$-group $G$, the following are equivalent:

1. $G \in A_f$.
2. $G = F_v(G)^L$.
3. $F_v(G)$ contains all the special elements of $G$.
4. The set of special values of $G$ contains no copy of $\bigwedge \ldots$.

In particular, this theorem holds for the $\ell$-group $V(\Delta, R)$, where $\Delta$ is any root system.

**References**


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