PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 7, Pages 1893–1902
S 0002-9939(99)04662-6
Article electronically published on February 26, 1999

SPECIAL-VALUED SUBGROUPS OF LATTICE-ORDERED GROUPS

YUANQIAN CHEN AND PAUL CONRAD

(Communicated by Lance W. Small)

ABSTRACT. We prove that the intersection of all maximal special-valued subgroups of a lattice-ordered group G is the special-valued quasi-torsion radical of a lattice-ordered group G, which extends our earlier result that the intersection of all maximal finite-valued subgroups of a lattice-ordered group G is the finite-valued torsion radical of G. We also show that the class A_f of almost finite-valued lattice-ordered groups is a quasi-torsion class, and the A_f quasi-torsion radical of a group is equal to the intersection of the group with the lateral completion of the finite-valued torsion radical of the group.

INTRODUCTION

For the basic definitions and results in lattice-ordered group theory, see M. Anderson and T. Feil [2] and M. Darnel [8]. A *lattice-ordered group*, written *ℓ-group*, is a partially ordered group $(G, \leq)$ where the partial order is a lattice (meaning that each pair of elements a, b of G has a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$). An *ℓ-subgroup* A of an ℓ-group G is both a subgroup and a sublattice of G. A is a *convex ℓ-subgroup* of G if a, $b \in A$ and $a \leq g \leq b$ imply that $g \in A$. A normal convex ℓ-subgroup is an *ℓ-ideal*. A convex ℓ-subgroup which is maximal with respect to not containing some $g \in G$ is called *regular* and is a *value* of g. A regular subgroup A is an *essential value* if it contains all the values for some $g \in G$. Element g is *special* if it has a unique value and in this case the value is called a *special value*. Regular subgroups of G form a *root system* under inclusion, written $\Gamma(G)$. (That is, $\Gamma(G)$ is a partially ordered set for which $\{\alpha \in \Gamma(G) \mid \alpha \geq \gamma\}$ is totally ordered, for any $\gamma \in \Gamma(G)$.) A subset $\Delta \subseteq \Gamma(G)$ is *plenary* if $\cap\Delta = \{0\}$ and Δ is a dual ideal in $\Gamma(G)$; that is, if $\delta \in \Delta, \gamma \in \Gamma(G)$ and $\gamma > \delta$, then $\gamma \in \Delta$. If G is an abelian ℓ-group, then G is ℓ-isomorphic to an ℓ-subgroup of $V(\Gamma(G), R)$ such that if γ is a value of $g \in G$, then γ is a maximal component of g after the embedding, where $V(\Gamma(G), R)$ is the abelian ℓ-group of all real-valued functions v on $\Gamma(G)$ for which $v(\gamma) \in R$ and the support of each v satisfies the ascending chain condition. This is a consequence of the Conrad-Harvey-Holland embedding theorem for abelian lattice-ordered groups.

Received by the editors July 16, 1996 and, in revised form, September 2, 1997.
1991 *Mathematics Subject Classification.* Primary 06F15, 06F20; Secondary 20F60.
Key words and phrases. Torsion class and quasi-torsion class, finite-valued and special-valued subgroups of a lattice-ordered group.

$\Sigma(\Delta, R)$ is the ℓ-subgroup of $V(\Delta, R)$ containing all elements $v \in V$ with finite supports. $F(\Delta, R)$ is the ℓ-subgroup of $V(\Delta, R)$ containing all elements $v \in V$ whose supports are contained in a finite number of chains in Δ.

For any $g \in G$, $G(g) = \{h \in G \mid \mid h \mid \leq n \mid g \mid$, for some positive integer $n\}$, the *principal convex ℓ-subgroup of G generated by g*, is the least convex ℓ-subgroup of G that contains g.

An ℓ-group G is *finite-valued* if every element of G has only a finite number of values; this is equivalent to the statement that every element of G can be expressed as a finite sum of disjoint special elements. Each element of G is also called finite-valued. An ℓ-group G is *special-valued* if G has a plenary subset of special values; this is equivalent to the statement that each positive element of G can be expressed as the join of a set of pairwise disjoint positive special elements. A positive element g of G is special-valued if g can be expressed as the join of disjoint special elements.

A lattice homomorphism is *complete* if it preserves all (not necessarily finite) meets and joins. A convex ℓ-subgroup is *closed* if it is closed with respect to infinite meets and joins which exist in the ℓ-group. A convex ℓ-subgroup C of an ℓ-group G is closed if and only if the natural lattice homomorphism from G onto its lattice G/C of right cosets is complete. Extensions which preserve the lattice of closed convex ℓ-subgroups are called *a^*-extensions.*

An ℓ-group is *laterally complete (conditionally laterally complete)* if for any subset (bounded subset) $\{g_\alpha \mid \alpha \in A\}$ of disjoint positive elements, $\vee_A g_\alpha$ exists.

A *torsion class* is a class of lattice-ordered groups that is closed under convex ℓ-subgroups, ℓ-homomorphic images, and joins of convex ℓ-subgroups. For an ℓ-group G and a torsion class T, $T(G)$ indicates the join of all the convex ℓ-subgroups of G that belong to T. $T(G)$ is then the largest convex ℓ-subgroup of G that belongs to T, called the *torsion radical* of G. A *quasi-torsion class* is a class of ℓ-groups which is closed under convex ℓ-subgroups, complete ℓ-homomorphic images, and joins of convex ℓ-subgroups. For an ℓ-group G and a quasi-torsion class Q, $Q(G)$ indicates the join of all the convex ℓ-subgroups of G that belong to Q. $Q(G)$ is then the largest convex ℓ-subgroup of G that belongs to Q, called the *quasi-torsion radical* of G. Finite-valued ℓ-groups form a torsion class F_v, and special-valued ℓ-groups form a quasi-torsion class S.

We have shown that the finite-valued torsion radical of an ℓ-group G is the intersection of all maximal finite-valued subgroups of G [6]. Let S be the quasi-torsion class of special-valued ℓ-groups. We will show that the quasi-torsion radical $S(G)$ is the intersection of all the maximal special-valued subgroups of G. We will show that the class A_f of almost finite-valued ℓ-groups is a quasi-torsion class, and its quasi-torsion radical of G is equal to the intersection of G with the lateral completion of the finite-valued torsion radical of G: $A_f(G) = F_v(G)^L \cap G$. Also for each ℓ-group G, the following are equivalent:

1. There exists a largest finite-valued subgroup of G.
2. The set Δ of special values of G is an ℓ-ideal of $\Gamma(G)$.
3. $S(G)$ is the largest special-valued subgroup of G and $S(G)$ is almost finite-valued.

1. Maximal special-valued subgroups

Definition 1.1. A *special-valued subgroup* of an ℓ-group G is an ℓ-subgroup U such that for each $0 < g \in U$, $g = \vee_\Lambda g_\lambda$, where the g_λ's are disjoint and special in G.

Thus if $g \in U$, then $|\, g \,| = \vee_\Lambda g_\lambda$, where $g_\lambda \in G$ are disjoint and special, but we don't require that $g_\lambda \in U$.

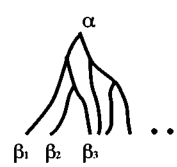
If $\cdots \subseteq C_\alpha \subseteq C_\beta \subseteq C_\gamma \subseteq \cdots$ is a chain of special-valued subgroups of G, then $\cup C_\lambda$ is a special-valued subgroup. Hence each special-valued subgroup is contained in a maximal special-valued subgroup.

Proposition 1.2. *If $0 < a = \vee_A a_\alpha$ and $0 < b = \vee_B b_\beta$ are special-valued elements in G, where A and B are sets of values, then $a + b$ is special-valued in G with the set of maximal elements in $A \cup B$ as special values.*

Proof. Consider

$$0 < a + b = \vee_A a_\alpha + b = \vee_A(a_\alpha + b) = \vee_A(a_\alpha + \vee_B b_\beta)$$
$$= \vee_A(\vee_B(a_\alpha + b_\beta)) = \vee_{A\cup B}(a_\alpha + b_\beta).$$

Suppose that $\alpha > \beta$, for some $\beta \in B$. Then α is greater than some subset of B and disjoint from the other elements of B.



$$a_\alpha \wedge b = a_\alpha \wedge (\vee_B b_\beta) = \vee_B(a_\alpha \wedge b_\beta)$$
$$= (a_\alpha \wedge b_{\beta_1}) \vee (a_\alpha \wedge b_{\beta_2}) \vee (a_\alpha \wedge b_{\beta_2}) \vee \ldots$$
$$= b_{\beta_1} \vee b_{\beta_2} \vee b_{\beta_3} \vee \ldots .$$

Thus $(a_\alpha + b_{\beta_1}) \vee (a_\alpha + b_{\beta_2}) \vee (a_\alpha + b_{\beta_3}) \vee \cdots = a_\alpha + (b_{\beta_1} \vee b_{\beta_2} \vee b_{\beta_3} \vee \ldots)$ is special with value α, and α is maximal in $A \cup B$.

If $\alpha = \beta$, for some $\beta \in B$, then $a_\alpha + b_\beta$ is special with value α, and α is maximal in $A \cup B$.

If α is not comparable with any β, then $a_\alpha + b_\beta = a_\alpha \vee b_\beta$ with values α and β, and both are maximal in $A \cup B$. □

Corollary 1.3. *If a and b are finite-valued, then so is $a + b$.*

Proposition 1.4. *If b is a special-valued element with $b > a > 0$, then $b + a$ is special-valued and has the same special values as b.*

Proof. Suppose that $b = \vee_\Lambda b_\lambda$ with b_λ disjoint and special. Then $a = b \wedge a = (\vee_\Lambda b_\lambda) \wedge a = \vee_\Lambda(b_\lambda \wedge a)$. We have

$$b + a = b + \vee_\Lambda(b_\lambda \wedge a)$$
$$= \vee_\Lambda(b + (b_\lambda + a))$$
$$= \vee_\Lambda(\vee_\Lambda b_\gamma + (b_\lambda \wedge a))$$
$$= \vee_\Lambda(\vee_\Lambda(b_\gamma + (b_\lambda \wedge a)))$$
$$= \vee_{\gamma\in\Lambda,\lambda\in\Lambda}(b_\gamma + (b_\lambda \wedge a)).$$

If $\gamma \neq \lambda$, then $b_\gamma + (b_\lambda \wedge a) = b_\gamma \vee (b_\lambda \wedge a) \leq (b_\gamma + (b_\gamma \wedge a)) \vee (b_\lambda + (b_\lambda \wedge a))$. Therefore $b + a = \vee_\Lambda(b_\lambda + (b_\lambda \wedge a))$.

Now we show that $b_\lambda + (b_\lambda \wedge a)$ is special with value λ. Since $b_\lambda \in G^\lambda \setminus G_\lambda$, where G_λ is the regular subgroup of G, and G^λ the cover of G_λ, we have $b_\lambda + (b_\lambda \wedge a) \in G^\lambda \setminus G_\lambda$. Now let α be a value of $b_\lambda + (b_\lambda \wedge a)$. Then $b_\lambda \notin G_\alpha$, so $G_\alpha \subseteq G_\lambda$. If $G_\alpha \subset G_\lambda$, then $b_\lambda \leq b_\lambda + (b_\lambda \wedge a) \in G^\alpha \subseteq G_\lambda$. This contradicts the fact that $b_\lambda \in G^\lambda \setminus G_\lambda$. Therefore $\alpha = \lambda$. □

Proposition 1.5. *An ℓ-subgroup C is the largest special-valued subgroup of G if and only if $C = S(G)$ which consists of all the special-valued elements of G.*

Proof. ($\Longleftarrow$) If C consists of all the special-valued elements of G, then it is the largest special-valued subgroup of G.

($\Longrightarrow$) If $0 < g$ is special-valued, then the ℓ-subgroup $\langle g \rangle$ of G generated by g is a special-valued subgroup of G. Therefore $\langle g \rangle \subseteq C$, and hence C consists of all the special-valued elements of G.

If b is special-valued and $b > a > 0$, then by the last proposition, b, $b + a \in C$, so $a \in C$. Thus C is convex and hence $C \subseteq S(G)$. Since $S(G)$ is a special-valued subgroup of G, $S(G) \subseteq C$. □

Corollary 1.6. *For an ℓ-group G, the following are equivalent:*

1. *There exists a largest special-valued subgroup of G.*
2. *$S(G)$ consists of all the special-valued elements of G.*
3. *$b \in G$ is special and $b > a > 0$ imply that a is special-valued.*
4. *$S(G)$ contains all the special elements of G.*

Proof. By the above proposition $1 \longleftrightarrow 2$, and clearly $2 \longrightarrow 3$.

$3 \longrightarrow 4$. b is special implies that $G(b) \subseteq S(G)$, so $S(G)$ contains all the special elements of G.

$4 \longrightarrow 2$. $0 < g$ is special-valued implies that $g = \vee_\Lambda g_\lambda$. Each $g_\lambda \in S(G)$, and $S(G)$ is closed. Hence, we have $g \in S(G)$. □

Proposition 1.7. *Let G be an ℓ-group. If a is a positive special-valued element of G, b is a negative element of $S(G)$, and $a+b$ is positive, then $a+b$ is special-valued in G.*

Proof. $0 < a = \vee_A a_\alpha$, where a_α are disjoint and special with value α, and $0 < -b = \vee_B(-b_\beta)$, where $-b_\beta$ are disjoint and special with value β.

Now $0 < a + b = \vee_A a_\alpha + b$, so $0 < -b < \vee_A a_\alpha$, and hence by Proposition 1.4, $(\vee_A a_\alpha - b)$ is special-valued with the same set of special values as $\vee_A a_\alpha$. Thus A is the set of special values for $\vee_A a_\alpha - b = \vee_A a_\alpha + \vee_B(-b_\beta)$. So each β is less than or equal to one and only one α and is incomparable with the other α's. Now consider

$$0 < a + b = \vee_A a_\alpha + b = \vee_A(a_\alpha + b) = \vee_A((a_\alpha + b) \vee 0).$$

Case I. $a_\alpha \wedge \mid b \mid = 0$. In this case α is incomparable with all β, thus $(a_\alpha + b) \vee 0 = a_\alpha$.

Case II. $\alpha = \beta$ for some β. We then have

$$0 < -b = \vee_B(-b_\beta) = -b_\beta + \vee_{B\setminus\{\beta\}}(-b_\gamma),$$

where $-b_\beta$ and $\vee_{B\setminus\{\beta\}}(-b_\gamma)$ are disjoint. Hence

$$(a_\alpha + b) \vee 0 = (a_\alpha + b_\beta - \vee_{B\setminus\{\beta\}}(-b_\gamma)) \vee 0 = (a_\alpha + b_\beta) \vee 0$$

is a positive element in $S(G)$ with all special values less than or equal to α, since $a_\alpha + b_\beta$ is disjoint from $\vee_{B\setminus\{\beta\}}(-b_\gamma)$.

Case III. $\alpha > \beta$ for some β. Let $B_\alpha = \{\beta \in B \mid \alpha > \beta\}$. Hence for $\beta \in B \backslash B_\alpha$, $\alpha \| \beta$.

We now have that $-b = \vee_B(-b_\beta)$ and that $a_\alpha \wedge (-b) = a_\alpha \wedge \vee_B(-b_\beta) = \vee_B(a_\alpha \wedge (-b_\beta)) = \vee_{B_\alpha}(-b_\beta)$, so $-b = \vee_{B_\alpha}(-b_\beta) + \vee_{B\backslash B_\alpha}(-b_\beta)$ and $a_\alpha + b = a_\alpha - \vee_{B_\alpha}(-b_\beta) - \vee_{B\backslash B_\alpha}(-b_\beta)$, where $a_\alpha - \vee_{B_\alpha}(-b_\beta)$ and $\vee_{B\backslash B_\alpha}(-b_\beta)$ are positive and disjoint. Therefore, $(a_\alpha + b) \vee 0 = a_\alpha - \vee_{B_\alpha}(-b_\beta)$ is special with value α.

Thus $(a_\alpha + b) \vee 0$ can be written as the join of disjoint special elements and has all its special values less than or equal to α. Hence $0 < a + b = \vee_A((a_\alpha + b) \vee 0)$ can be written as the join of disjoint special elements, hence is special valued. □

Proposition 1.8. *If U is a special-valued subgroup of G, then $U + S(G)$ is a special-valued subgroup of G.*

Proof. Since $S(G)$ is an ℓ-ideal, we have that $U + S(G)$ is an ℓ-subgroup of G. Consider $0 < g = a + b \in U + S(G)$, with $a \in U$ and $b \in S(G)$. We have $g + S(G) = a + S(G) = a \vee 0 + S(G)$, so without loss of generality, we may assume $a > 0$. $0 < a + b = a + b^+ - b^-$, where $a + b^+$ can be written as the join of disjoint special elements of G. So by Proposition 1.7, $a + b$ can be written as the join of disjoint special elements of G. □

We are now ready to describe the special-valued quasi-torsion radical for an ℓ-group by its maximal special-valued subgroups.

Theorem 1.9. *$S(G)$ is the intersection of all the maximal special-valued subgroups of G.*

Proof. By Proposition 1.8, $S(G)$ is contained in the intersection of all the maximal special-valued subgroups of G. Now we pick $0 < a \in G \setminus S(G)$. We need to show there exists a maximal special-valued subgroup that does not contain a. If a is not special-valued, then it does not belong to any special-valued subgroups of G. Now suppose that $a = a_1 \vee a_2 \vee a_3 \vee \cdots$, where $a_i \in G$ are disjoint and special. $G(a) \not\subseteq S(G)$, so there exists $b \in G$ such that $a > b > 0$ and b is not special-valued in G. By Proposition 1.4, $a + b$ is special-valued. Thus there exists a maximal special-valued subgroup of G that contains $a + b$ but not a. □

Proposition 1.10. *If $0 < a$ and $0 < b$ are special-valued elements in an ℓ-group G, then so are $a \wedge b$ and $a \vee b$.*

Proof. Consider $0 < \vee_A a_\alpha$ and $0 < \vee_B b_\beta$, where the a_α's are disjoint and a_α has special value α, and the b_β's are disjoint and b_β has special value β.

Let $C = \{\alpha \in A \mid \text{there exists } \beta \in B \text{ comparable with } \alpha\}$. Then $C = C_1 \cup C_2$, where $C_1 = \{\alpha \in C \mid \beta \leq \alpha \text{ for some } \beta \in B\}$ and $C_2 = \{\alpha \in C \mid \alpha < \beta \text{ for some } \beta \in B\}$.

We have

$$\begin{aligned} a \vee b &= (\vee_A a_\alpha) \vee (\vee_B b_\beta) = \vee_A (a_\alpha \vee (\vee_B b_\beta)) = \vee_A (\vee_B (a_\alpha \vee b_\beta)) \\ &= \vee_{A\cup B} (a_\alpha \vee b_\beta) = \vee_{A\backslash C} a_\alpha \vee_C (a_\alpha \vee b_\beta) \vee_{B\backslash C} b_\beta \\ &= \vee_{A\backslash C} a_\alpha \vee (a_\alpha \vee (\vee_{C_1} b_\beta)) \vee (b_\beta \vee (\vee_{C_2} a_\alpha)) \vee_{B\backslash C} b_\beta. \end{aligned}$$

Therefore, $a \vee b$ is special-valued with values maximal in $A \cup B$. Similarly,

$$a \wedge b = (\vee_A a_\alpha) \wedge (\vee_B b_\beta) = \vee_A (a_\alpha \wedge (\vee_B b_\beta)) = \vee_{A\cup B} (a_\alpha \wedge b_\beta) = \vee_C (a_\alpha \wedge b_\beta).$$

Therefore, $a \wedge b$ is special-valued with values minimal in C. □

Theorem 1.11. *A maximal special-valued subgroup U of an abelian ℓ-group G is weakly saturated; i.e., special components of elements of U belong to U.*

Proof. Suppose that $v \in U$. Then $v = g_1 \vee g_2 \vee g_3 \vee \dots$, where $g_i \in G$ are disjoint and special. Assume (by way of contradiction) that $g_1 \notin U$, and let $\langle U, g_1 \rangle$ be the ℓ-subgroup of G generated by U and g_1. Consider $0 < f \in \langle U, g_1 \rangle$. Then

$$f = f \vee 0 = \vee_I \wedge_J \left((u_{ij} \pm n_{ij} g_1) \vee 0\right)$$

where I and J are finite index sets, $u_{ij} \in U$ and n_{ij} are positive integers. We will first show that for fixed i and j, each $(u_{ij} \pm n_{ij} g_1) \vee 0$ is special-valued.

Consider $(u \pm n g_1) \vee 0$, $u \in U$ hence $\mid u \mid = h_1 \vee h_2 \vee h_3 \vee \dots$, where $h_i \in G$ are disjoint and special.

If the value of g_1 is different from all those of u, then clearly $(u \pm n g_1) \vee 0$, is special-valued.

If the value of g_1 is the same as a value of u, say, a value of h_1, but the value of g_1 is the same as that of $h_1 \pm n g_1$, then we have $(u \pm n g_1) \vee 0$ is special-valued.

Now consider the case when the value of g_1 is the same as that of h_1, and values of $h_1 \pm n g_1$ are less than that of g_1. We then have

$$(h_1 \pm n g_1) \vee 0 \leq (u \pm n v) \vee 0 = f_1 \vee f_2 \vee f_3 \vee \dots$$

where $0 < f_i \in G$ are disjoint and special, since $(u \pm n v) \vee 0 \in U$.

Let $F = \{f_j \mid f_j \leq g_1\}$, $\vee\{f_j \mid f_j \in F\}$ exists in G because $\vee\{f_j \mid f_j \in F\} = (f_1 \vee f_2 \vee f_3 \vee \dots) \wedge g_1$. We claim that

$$(h_1 \pm n g_1) \vee 0 = \vee\{f_j \mid f_j \in F\}.$$

Clearly $(h_1 \pm n g_1) \vee 0 \leq \vee\{f_j \mid f_j \in F\}$, since $(h_1 \pm n v) \vee 0 \leq f_1 \vee f_2 \vee f_3 \vee \dots$. On the other hand, each $f_j \leq (h_1 \pm n g_1) \vee 0$ implies that $\vee\{f_j \mid f_j \in F\} \leq (h_1 \pm n g_1) \vee 0$. We have shown that $(h_1 \pm n g_1) \vee 0$ is special-valued which implies that $(u \pm n g_1) \vee 0$ is special-valued. Thus by Proposition 1.10, $\vee_I \wedge_J ((u_{ij} \pm n_{ij} g_1) \vee 0)$ is special-valued in G. □

Corollary 1.12. *A maximal special-valued subgroup U of an abelian ℓ-group is special-valued as an ℓ-group.*

Proof. Let U be a maximal special-valued subgroup of G. For each $0 < u \in U$, $u = g_1 \vee g_2 \vee g_3 \vee \dots$, where $g_i \in G$ are disjoint and special. By Theorem 1.11, each $g_i \in U$. Also each g_i is special in U. Therefore U is special-valued as an ℓ-group. □

2. Almost finite-valued ℓ-groups

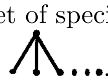
Definition 2.1. An ℓ-group G is *almost finite-valued* if it is special-valued and the principal convex ℓ-subgroup $G(g)$ is finite-valued for each finite-valued element $g \in G$.

We use A_f to indicate the class of almost finite-valued ℓ-groups. We first list some basic properties of A_f and brief proofs.

1. $F_v \subseteq A_f \subseteq S$, where F_v is the torsion class of finite-valued ℓ-groups, and S is the quasi-torsion class of special-valued ℓ-groups.
2. A_f is closed with respect to products.

Proof. The quasi-torsion class S of special-valued ℓ-groups is closed with respect to products, and if $0 < g$ is special in a product, then it is special in one of the factors, so $G(g)$ is finite-valued. □

3. A_f is not closed with respect to ℓ-subgroups.
   For example, for each $x \in [0,1]$, let R_x be the real numbers. Then $\prod_{x\in[0,1]} R_x \in A_f$, but $C[0,1] \notin S$.
4. $G \in A_f$ does not imply that $G^L \in A_f$.
   For example, let $\Lambda =$ . Then $\Sigma(\Lambda, R) \in A_f$, but $\Sigma(\Lambda, R)^L = V(\Lambda, R) \notin A_f$.
5. If $G \in A_f$, then $F_v(G)$ is the ℓ-ideal generated by all the $G(g)$ with g special. So $F_v(G)$ is the largest finite-valued subgroup of G.
6. A laterally complete special-valued ℓ-group is almost finite-valued if and only if its set of special values Δ contains no copy of

   $\Lambda =$ .

   Proof. Suppose Λ is contained in Δ. Then let g be a special element with value δ which is the maximal element of Δ. Since G is laterally complete, $G(g)$ contains an element with an infinite number of values. On the other hand, let g be a special element with value δ, and Δ contains no copy of Λ. Then the ideal $\overline{\delta} = \{\alpha \in \Delta \mid \alpha \leq \delta\}$ contains only a finite number of roots, so $G(g)$ is finite-valued. □

7. For a completely distributive ℓ-group G, the following are equivalent:
   (a) G^L is almost finite-valued.
   (b) The plenary set Δ of all essential values of $\Gamma(G)$ contains no copy of the set Λ of (6).

   Proof. By [3], $\Delta \cong$ the set of all the special elements of $\Gamma(G^L)$. □

8. If G is a special-valued vector lattice, then without loss of generality, we assume that $\Sigma(\Delta, R) \subseteq G \subseteq V(\Delta, R)$. $V(\Delta, R) \in A_f$ if and only if Δ contains no copy of Λ as in (6). Thus each abelian a^*-extension of G is almost finite-valued if and only if Δ contains no copy of Λ.

Theorem 2.2. *For an ℓ-group G, the following are equivalent*:

1. *There exists a largest finite-valued subgroup of G.*
2. *$F_v(G)$ consists of all the finite-valued elements of G.*
3. *$0 < a < b$ and b is special imply that a is finite-valued.*
4. *b is special implies that $G(b)$ is finite-valued.*
5. *$F_v(G)$ contains all the special elements of G.*
6. *The set Δ of special values of G is an ideal of $\Gamma(G)$.*
7. *$S(G)$ is the largest special-valued subgroup of G, and $S(G)$ is almost finite-valued.*

Proof. We proved in [6] that 1, 2, and 3 are equivalent, and clearly $3 \longleftrightarrow 4$ and $2 \longleftrightarrow 5$.

Now suppose $0 < g \in G$ is special with value G_α. Then $G_\alpha \longrightarrow G_\alpha \cap G(g)$ is a one-to-one order-preserving map of the regular subgroups of G that do not contain $G(g)$ onto the regular subgroups of $G(g)$.

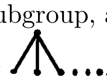
$6 \longrightarrow 4$. Δ is an ℓ-ideal of $\Gamma(G)$ implies that all the regular subgroups of $G(g)$ are special if g is special. So $G(g)$ is finite-valued.

$4 \longrightarrow 7$. $G(g)$ is finite-valued implies that $G(g) \subseteq S(G)$. So $S(G)$ contains all the special elements.

$7 \longrightarrow 4$. g is special implies $g \in S(G)$. Thus $G(g) \subseteq S(G)$, hence $G(g)$ is finite-valued.

$4 \longrightarrow 6$ is clear. □

This theorem shows that if G has a largest finite-valued subgroup, then G has a largest special-valued subgroup, although the converse is not true.

For example, let $\Lambda =$ .

$G = V(\Lambda, R) \in S$, but G has no largest finite-valued subgroup.

Actually, there exists a largest special-valued subgroup of G if and only if for each special $b \in G$, $G(b)$ is special-valued. In turn, this is equivalent to the assertion that for each special value $\delta \in \Delta$, the ideal $\overline{\delta} = \{\gamma \in \Gamma(G) \mid \gamma \leq \delta\}$ contains a dual ideal of special elements with zero intersection.

We now give the proof that A_f is a quasi-torsion class.

Theorem 2.3. *The almost finite-valued ℓ-groups form a quasi-torsion class A_f.*

Proof. G is almost finite-valued if and only if the set Δ of special values is a plenary subset and an ideal of $\Gamma(G)$. A special-valued ℓ-group G is almost finite-valued if and only if Δ is an ideal of $\Gamma(G)$.

1. A_f is closed with respect to convex ℓ-subgroups.

Suppose C is a convex ℓ-subgroup of an almost finite-valued ℓ-group G. Then $C \in S$, since S is a quasi-torsion class. The set C_Δ of all special values of C is an ideal and a plenary subset of G_Δ, the set of all special values of G, and G_Δ is an ideal in $\Gamma(G)$. Therefore C_Δ is an ideal in $\Gamma(C)$ and a plenary subset of $\Gamma(C)$.

2. A_f is closed with respect to complete ℓ-homomorphic images.

Let C be a closed ℓ-ideal of $G \in A_f$. Then $G/C \in S$. Suppose $C + g$ is special in G/C, $g = g_1 \vee g_2 \vee g_3 \vee \dots$ in G, with not all g_i belonging to C. If there is more than one component of g not in C, say, g_1 is one of them, then $C + g_1$ is a proper component of $C + g$, so $C + g$ is not special. So without loss of generality, g is special in G. The convex ℓ-subgroup of G/C generated by $C + g$ is isomorphic to $\dfrac{G(g)}{C \cap G(g)}$, and it is finite-valued, since F_v is a torsion class.

3. A_f is closed with respect to joins of convex ℓ-subgroups.

$S(G)$ is the largest convex ℓ-subgroup that belongs to S. If the largest convex ℓ-subgroup of G that is almost finite-valued exists, then it is an ℓ-ideal of $S(G)$. So without loss of generality, we assume that G is special-valued. A convex ℓ-subgroup C of G is almost finite-valued if and only if C_Δ is an ideal of $\Gamma(C)$, but $\Gamma(C)$ is an ideal of $\Gamma(G)$. So C is almost finite-valued if and only if C_Δ is an ideal in $\Gamma(G)$.

If $C_1 \subseteq C_2 \subseteq C_3 \subseteq \dots$ form a chain of convex ℓ-subgroups of G that belong to A_f, then their sets of special values $C_{1\Delta} \subseteq C_{2\Delta} \subseteq C_{3\Delta} \subseteq \dots$ form a chain of ideals in $\Gamma(G)$, so $\cup C_{i\Delta}$ is an ideal in $\Gamma(G)$, hence $\cup C_i$ is almost finite-valued.

If B and C are almost finite-valued, and $0 < g \in B + C$ is special with value δ, then $g \in B$ or $g \in C$. In fact, g is equal to the sum of positive elements from $B \cup C$, and one of them must have value δ. Without loss of generality, there exists $0 < b \in B$ with a value δ, hence $nb > g$ for some $n \in N$, so $g \in B$. Thus $B(g)$ is finite-valued, which implies $(B+C)(g)$ is finite-valued. Therefore, $B+C$ is almost finite-valued. □

The torsion radical $F_v(G)$ is a characteristic ℓ-ideal of G, and $F_v(G) \subseteq G \subseteq G^L$, where G^L is the lateral completion of G. Actually, $F_v(G)^L \subseteq G^L$. This follows from the following proposition.

Proposition 2.4. *$C \in \mathcal{C}(G)$, the set of all convex ℓ-subgroups of G, implies that $C^L \subseteq G^L$.*

Proof. $C \subseteq C''$, the polar of C in G^L, and since C'' is closed, it is laterally complete. Now G is dense in G^L, so $0 < x \in C'' \subseteq G^L$ implies $x > y > 0$, for some $y \in G \cap C''$. In particular, y is not disjoint from C^+, so $y \wedge c > 0$, for some $c \in C$. Therefore $x > y \wedge c \in C$, so C is dense in the laterally complete ℓ-group C'', and hence $C^L \subseteq C'' \subseteq G^L$. □

The following theorem demonstrates the relation between the torsion radicals $A_f(G)$ and $F_v(G)$.

Theorem 2.5. *$A_f(G) = F_v(G)^L \cap G$.*

Proof. $F_v(G)^L$ is special-valued and has the same set Δ of special values as $F_v(G)$ by the result in [3]. Consider $0 < g \in F_v(G)^L \cap G$, $g = h_1 \vee h_2 \vee h_3 \vee \cdots$, where h_i are disjoint and special in $F_v(G)^L$. Now there exists a special element $0 < g_1 \in F_v(G)$ with the same value as h_1. Thus $ng_1 > h_1$ for some $n \in N$, so $h_1 = ng_1 \wedge g \in F_v(G)$; thus $G(h_1)$ is finite-valued.

Suppose $0 < u < g$ with $u \in G$. Then $u = u \wedge g = u \wedge (\vee h_i) = \vee(u \wedge h_i)$. Each $u \wedge h_i \in G(h_i)$ is finite-valued. Therefore u is special-valued.

If $0 < a \in G(g)$ is special, then $a < ng$, so $a < nh_i$, for some $n \in N$. Thus $G(a)$ is finite-valued. Therefore, $G(g) \subseteq A_f(G)$, so $F_v(G)^L \cap G \subseteq A_f(G)$.

Conversely, consider $0 < g \in A_f(G)$. Then $g = g_1 \vee g_2 \vee g_3 \vee \cdots$, where g_i are disjoint and special, and each $G(g_i)$ is finite-valued. Therefore each $g_i \in F_v(G)$, so $g \in F_v(G)^L \cap G$. □

Thus if G is laterally complete, then $A_f(G) = F_v(G)^L$. So $G \in A_f$ if and only if $G = F_v(G)^L$. In fact, the proof of the theorem shows that $F_v(G)(1) \cap G = A_f(G)$, where $F_v(G)(1)$ is the ℓ-group generated by the joins of disjoint elements from $F_v(G)$ [9]. Therefore $F(G)(1) \cap G = F_v(G)^L \cap G$. We now discuss almost finite-valued ℓ-groups that are laterally complete.

Proposition 2.6. *Let G be a special-valued laterally complete ℓ-group. If $g \in G$ is infinite valued, then $F_v(G) + g$ is not finite-valued in $G/F_v(G)$.*

Proof. If $F_v(G) + g$ is finite-valued, then without loss of generality, it is special in $G/F_v(G)$. But g can be written as $a + b$ with $a \wedge b = 0$, where a and b are both infinite-valued. Thus $F_v(G) + a$ and $F_v(G) + b$ are non-zero components of $F_v(G) + g$. This contradicts the fact that $F_v(G) + g$ is special in $G/F_v(G)$. □

Proposition 2.7. *Let G be a special-valued laterally complete ℓ-group. $G \in A_f$ if and only if $G/F_v(G)$ contains no special elements.*

Proof. ($\Longrightarrow$) If $g \in G \setminus F_v(G)$, then g is infinite-valued and hence $F_v(G) + g$ is not special.

($\Longleftarrow$) If $0 < g \in G$ is special, then $g \in F_v(G)$. For otherwise $F_v(G) + g$ is not special in $G/F_v(G)$. Therefore $F_v(G)$ contains all the special elements of G, and hence $G \in A_f$. □

The next theorem follows from the above theorem and propositions.

Theorem 2.8. *For a special-valued laterally complete ℓ-group G, the following are equivalent*:

1. $G \in A_f$.
2. $G = F_v(G)^L$.
3. $F_v(G)$ *contains all the special elements of* G.
4. *The set of special values of* G *contains no copy of* .

In particular, this theorem holds for the ℓ-group $V(\Delta, R)$, where Δ is any root system.

References

1. M. Anderson, P. Bixler, and P. Conrad, *Vector lattices with no proper a-subspaces.* Archiv Math. (basel), 41 (1983), 427-433. MR **85h:**06038
2. M. Anderson, and T. Feil, Lattice-Ordered Groups: An Introduction. D. Reidel, 1987. MR **90b:**06001
3. P. Bixler, and M. Darnel, *Special-valued ℓ-groups.* Algebra Universalis, 22 (1986), 172-191. MR **88h:**06021
4. R. Byrd, *Lattice-ordered groups. Ph.D Dissertation.* Tulane University, 1966.
5. Y. Chen, *Torsion classes of lattice-ordered groups and special-valued subgroups.* Ph.D dissertation, University of Kansas, 1992.
6. Y. Chen, P. Conrad, and M. Darnel, *Finite-valued subgroups of lattice-ordered groups.* Czech Math Journal, 46(121) 1996. CMP 97:01
7. P. Conrad, J. Harvey, and W.C. Holland, *The Hahn embedding theorem for lattice-ordered groups.* Trans. Amer. Math. Soc., 108 (1963), 143-149. MR **27:**1519
8. M. Darnel, Theory of Lattice-Ordered Groups. Marcel Dekker, Inc, 1994. MR **95k:**06032
9. J. Martinez, *Torsion theory for lattice-ordered groups.* Czech. Math. J., 25 (1975), 284-299. MR **52:**10536

Department of Mathematical Sciences, Central Connecticut State University, New Britain, Connecticut 06050

E-mail address: chen@ccsua.ctstateu.edu

Department of Mathematics, University of Kansas, Lawrence, Kansas 66045