

CONGRUENCE-PRESERVING EXTENSIONS  
OF FINITE LATTICES  
TO SECTIONALLY COMPLEMENTED LATTICES

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ABSTRACT. In 1962, the authors proved that every finite distributive lattice can be represented as the congruence lattice of a finite sectionally complemented lattice. In 1992, M. Tischendorf verified that every finite lattice has a congruence-preserving extension to an atomistic lattice.

In this paper, we bring these two results together. We prove that *every finite lattice has a congruence-preserving extension to a finite sectionally complemented lattice.*

1. INTRODUCTION

Early results on related structures (the automorphism group, the congruence lattice, and so on) of a lattice were characterization theorems, typified by the following result of R. P. Dilworth:

**Theorem 1.** *Let  $D$  be a finite distributive lattice. Then there exists a finite lattice  $L$  such that the congruence lattice of  $L$  is isomorphic to  $D$ .*

A stronger form of this result was published in G. Grätzer and E. T. Schmidt [6]:

**Theorem 2.** *Every finite distributive lattice  $D$  can be represented as the congruence lattice of a finite sectionally complemented lattice  $L$ .*

In the last decade, the emphasis has shifted from representation theorems to extension theorems, typified by the following important result of M. Tischendorf [12]:

**Theorem 3.** *Every finite lattice has a congruence-preserving embedding to a finite atomistic lattice.*

In this paper, in the spirit of Tischendorf's result, we shall prove the extension form of Theorem 2:

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**Theorem.** *Every finite lattice  $K$  has a congruence-preserving embedding into a finite sectionally complemented lattice  $L$ .*

This result does not hold for infinite lattices:  $F(\aleph_2)$  (the free lattice on  $\aleph_2$  generators) does not have a congruence-preserving embedding into a sectionally complemented lattice according to M. Ploščica, J. Tůma, and F. Wehrung [10]. The problem is open for lattices of size  $\aleph_0$  and  $\aleph_1$ .

See G. Grätzer and E. T. Schmidt [8] for a discussion of characterization theorems *vs.* extension theorems, and see G. Grätzer, H. Lakser, and E. T. Schmidt [2], [3], [4], and G. Grätzer and E. T. Schmidt [8] for further examples of extension theorems.

## 2. PRELIMINARIES

We use the basic concepts and notations as in [1].

A finite lattice  $L$  is *atomistic* if every element is a join of atoms;  $L$  is *sectionally complemented* if, for every  $a \leq b$  in  $L$ , there is an element  $c$  that is a complement of  $a$  in the interval  $[0, b]$ . Obviously, every finite sectionally complemented lattice is atomistic, but not conversely; the seven element lattice  $K$  of Figure 1 witnesses this.

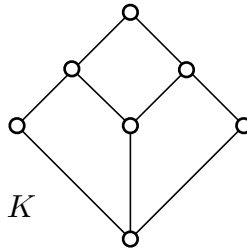


FIGURE 1

Let  $L$  be a lattice. If  $K$  is a sublattice of  $L$ , we call  $L$  an *extension* of  $K$ . If  $L$  is an extension of  $K$ ,  $\Theta$  is a congruence of  $K$ , and  $\Phi$  is a congruence of  $L$ , then  $\Phi$  is an *extension* of  $\Theta$  to  $L$  iff the restriction of  $\Phi$  to  $K$  equals  $\Theta$ . We say that  $K$  in  $L$  has the *Congruence Extension Property* iff every congruence of  $K$  has an extension to  $L$ .

We call  $L$  a *congruence-preserving extension* of  $K$  iff every congruence of  $K$  has *exactly one* extension to  $L$ . In this case, the congruence lattice of  $K$  is isomorphic to the congruence lattice of  $L$ ; in formula,  $\text{Con } K \cong \text{Con } L$ .

Let  $M$  be a finite poset such that  $\inf\{a, b\}$  exists in  $M$ , for all  $a, b \in M$ . We define in  $M$ :

$$\begin{aligned} a \wedge b &= \inf\{a, b\}, & \text{for all } a, b \in M; \\ a \vee b &= \sup\{a, b\}, & \text{whenever } \sup\{a, b\} \text{ exists.} \end{aligned}$$

This makes  $M$  into a finite *chopped lattice*. Note that if  $L$  is a finite lattice with unit,  $1$ , then  $L - \{1\}$  is a finite chopped lattice; and conversely, if  $M$  is a finite chopped lattice, then  $L = M \cup \{1\}$  is a finite lattice from which we obtain  $M$  by chopping off  $1$ .

An equivalence relation  $\Theta$  on the chopped lattice  $M$  is a *congruence relation* iff, for all  $a_0, a_1, b_0, b_1 \in M$ ,

$$\begin{aligned} a_0 &\equiv b_0(\Theta), \\ a_1 &\equiv b_1(\Theta), \end{aligned}$$

imply that

$$\begin{aligned} a_0 \wedge a_1 &\equiv b_0 \wedge b_1(\Theta), \\ a_0 \vee a_1 &\equiv b_0 \vee b_1(\Theta), \quad \text{whenever } a_0 \vee a_1 \text{ and } b_0 \vee b_1 \text{ both exist.} \end{aligned}$$

The set  $\text{Con } M$  of all congruence relations of  $M$  is a lattice.

An *ideal*  $I$  of a finite chopped lattice  $M$  is a non-empty subset  $I \subseteq M$  with the following two properties:

- (i)  $i \wedge a \in I$ , for  $i \in I$  and  $a \in M$ ;
- (ii)  $i \vee j \in I$ , for  $i, j \in I$ , provided that  $i \vee j$  exists in  $M$ .

The ideals of the finite chopped lattice  $M$  form the finite lattice  $\text{Id } M$ .

The following lemma was published in G. Grätzer [1].

**Lemma 4** (G. Grätzer and H. Lakser). *Let  $M$  be a finite chopped lattice. Then for every congruence relation  $\Theta$  of  $M$ , there exists exactly one congruence relation  $\bar{\Theta}$  of  $\text{Id } M$  such that, for  $a, b \in M$ ,*

$$[a] \equiv [b] (\bar{\Theta}) \quad \text{iff} \quad a \equiv b (\Theta).$$

*In particular,  $\text{Con } M \cong \text{Con}(\text{Id } M)$ .*

In the proof of the Theorem, we need a more detailed form of Theorem 2.

**Theorem 2'**. *For every finite distributive lattice  $D$ , there is a lattice  $L$  with the following properties:*

- (i)  $L$  is finite;
- (ii)  $L$  is sectionally complemented;
- (iii)  $\text{Con } L$  is isomorphic to  $D$ ;
- (iv) there is an independent set of atoms

$$Q = \{ q_\Phi \mid \Phi \in \text{J}(\text{Con } L) \}$$

*of  $L$  such that  $\Phi = \Theta(0, q_\Phi)$ , for  $\Phi \in \text{J}(\text{Con } L)$ , and there is a one-to-one correspondence between congruences and certain subsets of  $Q$ ; the subset that corresponds to the congruence  $\Theta$  of  $L$  is*

$$\{ q_\Phi \mid \Phi \in \text{J}(\text{Con } L) \text{ and } \Phi \leq \Theta \}.$$

(Recall that a set of atoms is *independent*, if they generate a Boolean sublattice;  $\text{J}(\text{Con } L)$  is the set of join-irreducible congruences of  $L$ .)

The two standard proofs of Theorem 2 (the original proof in [6] and the proof, due to the first author and H. Lakser, published in [1]) both exhibit the set  $Q$  satisfying (iv).

### 3. SECTIONALLY COMPLEMENTED LATTICES

In this section, we present a new construction of finite sectionally complemented lattices.

**Theorem 5.** *Let  $A$  and  $B$  be (disjoint) finite sectionally complemented lattices with zero elements  $0_A$  and  $0_B$ , respectively. Let  $p_A$  be an atom of  $A$  and let  $p_B$  be an atom of  $B$ .*

*Form the partial lattice  $M$  by identifying  $0_A$  with  $0_B$  and  $p_A$  with  $p_B$ . Then  $M$  is a finite chopped lattice and  $\text{Id } M$  is a finite sectionally complemented lattice.*

*Proof.* Let  $0 = 0_A = 0_B$  and  $p = p_A = p_B$ . Obviously,  $M$  is a finite chopped lattice.

An ideal  $I$  of  $M$  can be represented by a pair of elements  $\langle i_A, i_B \rangle$  such that  $i_A \in A$ ,  $i_B \in B$ ,  $I \cap A = (i_A]$ ,  $I \cap B = (i_B]$ , and  $i_A \wedge p = i_B \wedge p$ .

To show that  $\text{Id } M$  is sectionally complemented, let  $I \subseteq J$  be two ideals of  $M$ , represented by  $\langle i_A, i_B \rangle$  and  $\langle j_A, j_B \rangle$ , respectively. Let  $s_A$  be the sectional complement of  $i_A$  in  $j_A$  and let  $s_B$  be the sectional complement of  $i_B$  in  $j_B$ . If

$$p \wedge s_A = p \wedge s_B,$$

then  $\langle s_A, s_B \rangle$  is an ideal  $S$  that is a sectional complement of  $I$  in  $[(0), J]$ . Otherwise, without loss of generality, we can assume that

$$\begin{aligned} p \wedge s_A &= 0, \\ p \wedge s_B &= p. \end{aligned}$$

Let  $s'_B$  be a sectional complement of  $p$  in  $[0, s_B]$ . Then  $\langle s_A, s'_B \rangle$  satisfies

$$p \wedge s_A = p \wedge s'_B$$

(= 0), and so the pair represents an ideal  $S$  of  $M$ . Obviously,  $I \wedge S = (0)$ .

Since  $p \wedge s_B = p$ , it follows that  $p \leq s_B \leq j_B$ . Since  $J$  is an ideal and  $j_B \wedge p = p$ , it follows that  $j_A \wedge p = p$ , that is,

$$p \leq j_A.$$

Obviously,  $I \vee S \subseteq J$ . So to show that  $I \vee S = J$ , it is sufficient to verify that  $j_A, j_B \in I \vee S$ . Evidently,  $j_A = i_A \vee s_A \in I \vee S$ . Note that  $p \leq j_A = i_A \vee s_A \in I \vee S$ . Thus,  $p, s'_B, i_B \in I \vee S$ , and therefore

$$p \vee s'_B \vee i_B = (p \vee s'_B) \vee i_B = s_B \vee i_B = j_B \in I \vee S,$$

which was to be proved. □

An easy consequence of this theorem is the following:

**Corollary 6.** *Let  $A, B$ , and  $M$  be as in Theorem 5. In addition, let us assume that  $B$  is a simple lattice. Then  $\text{Con } M \cong \text{Con } A$  and  $\text{Id } M$  is a sectionally complemented congruence-preserving extension of  $A$ .*

*Proof.* It is clear that a congruence  $\Theta$  of  $A$  has exactly one extension to a congruence  $\Theta_M$  of  $M$ ; indeed, if  $p_A \equiv 0_A(\Theta)$ , then let  $B$  be collapsed under  $\Theta_M$ ; otherwise, let  $\Theta_M$  be  $\omega_B$  on  $B$ . So this statement follows from Theorem 5 and Lemma 4. □

**Lemma 7.** *For every finite lattice  $L$ , there is a finite, simple, sectionally complemented extension  $S(L)$ .*

*Proof.* If  $|L| \leq 2$ , then we can take  $S(L) = L$ . So we can assume that  $|L| > 2$ .

Let 0 and 1 be the zero and unit element of  $L$ , respectively. We can assume that 1 is join-reducible; otherwise, adjoin a common complement  $u$ , that is, an element  $u$  satisfying

$$\begin{aligned} u \wedge x &= 0, \\ u \vee x &= 1, \end{aligned}$$

for all  $x \in L - \{0, 1\}$ . So there are elements  $x_1, x_2 \in L - \{1\}$  satisfying  $x_1 \vee x_2 = 1$ , and 1 is join-reducible.

Let

$$N(L) = L - \{x \mid x = 0 \text{ or } x \text{ is an atom}\}.$$

For every  $a \in N(L)$ , we adjoin an atom  $p_a < a$  so that if  $a \neq b$ , then  $p_a \neq p_b$ . We make

$$S(L) = L \cup \{p_a \mid a \in N(L)\}$$

(a disjoint union) into a poset by the following rules:

( $\alpha$ )  $L$  is a subposet of  $S(L)$ ;

( $\beta$ ) if  $a \in N(L)$  and  $x \in L$ , then

$$x < p_a \quad \text{iff } x = 0,$$

$$p_a < x \quad \text{iff } a \leq x;$$

( $\gamma$ ) if  $a$  and  $b \in N(L)$ , then

$$p_a \leq p_b \quad \text{iff } a = b.$$

Then  $S(L)$  is a lattice. The meet and the join in  $S(L)$  are described by the following rules:

(i)  $L$  is a sublattice of  $S(L)$ ;

(ii) if  $a \in N(L)$  and  $x \in L$ , then

$$p_a \wedge x = \begin{cases} 0, & \text{if } a \not\leq x, \\ p_a, & \text{if } a \leq x; \end{cases}$$

(iii) if  $a \in N(L)$  and  $x \in L$ , then

$$p_a \vee x = \begin{cases} a \vee x, & \text{if } x \neq 0, \\ p_a, & \text{if } x = 0; \end{cases}$$

(iv) if  $a$  and  $b \in N(L)$ ,  $a \neq b$ , then

$$p_a \wedge p_b = 0,$$

$$p_a \vee p_b = a \vee b.$$

Obviously,  $S(L)$  is a finite lattice; 0 and 1 are the zero and unit elements of  $S(L)$ , respectively.  $S(L)$  an extension of  $L$ . To show that  $S(L)$  is sectionally complemented, take  $0 < u < v$  in  $S(L)$ . We consider two cases:

*Case 1.*  $u \in L$ . Then  $v \in N(L)$  and  $p_v$  is a sectional complement of  $u$  in  $[0, v]$  by (ii) and (iii).

*Case 2.*  $u \notin L$ . Then  $u = p_a$ , for some  $a \in N(L)$ , and  $v \in L$  satisfies  $a \leq v$ . If  $a = v$ , then any  $x$  satisfying  $0 < x < a$  is a sectional complement of  $u$  in  $[0, v]$ ; and there is such an  $x$  because  $a \in N(L)$ . If  $a < v$ , then  $p_v$  is a sectional complement of  $u$  in  $[0, v]$  by (iii) and (iv).

Finally,  $S(L)$  is simple. Indeed, let  $\Theta > \omega$  be a congruence of  $L$ . We verify that there is an  $a > 0$ ,  $a \in S(L)$ , such that  $a \equiv 0(\Theta)$ . Indeed, since  $\Theta > \omega$ , there are  $u, v \in L$  such that  $u < v$  and  $u \equiv v(\Theta)$ . If  $u = 0$ , then  $a = v$  satisfies the requirements. If  $0 < u$ , then  $v \in N(L)$ , so there is an element  $p_v \in S(L)$ . We have two cases to consider:

*Case 1.*  $u \in L$ . In this case,  $u \equiv v(\Theta)$  implies that  $p_v \equiv 0(\Theta)$ , so we can take  $a = p_v$ .

*Case 2.*  $u \in S(L) - L$ . In this case,  $u = p_x$ , for some  $x \in N(L)$ . Obviously,  $v \in L$  and  $x \leq v$ . Since  $x \in N(L)$ , there is an  $a \in L$  satisfying  $0 < a < x$ . Now  $u \equiv v(\Theta)$  implies that  $p_x \equiv x(\Theta)$  and so  $a \equiv 0(\Theta)$ .

Using the congruence  $a \equiv 0(\Theta)$ , we conclude that  $p_1 \equiv 1(\Theta)$ . Meeting with  $x_1$ , we obtain that  $x_1 \equiv 0(\Theta)$ , and similarly  $x_2 \equiv 0(\Theta)$ . Joining these two congruences, we obtain that  $0 \equiv 1(\Theta)$ , that is,  $\Theta = \iota$ . □

We should point out that since a finite partition lattice is simple and sectionally complemented by O. Ore [9], Lemma 7 follows from the following very deep result of P. Pudlák and J. Tůma [11]: *Every finite lattice can be embedded into a finite partition lattice.*

For the lattice  $K$  of Figure 1, we can choose for  $S(K)$  the lattice depicted in Figure 2; note that this is much smaller than the lattice constructed in the proof of Lemma 7.

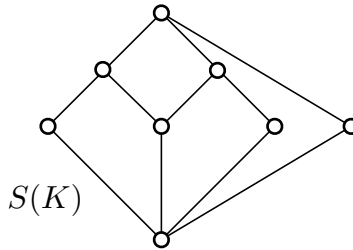


FIGURE 2

#### 4. RECTANGULAR EXTENSIONS

The *rectangular extension*  $\mathbb{R}(K)$  of a finite lattice  $K$  is defined as the direct product of *all* subdirect factors of  $K$ , that is,

$$\mathbb{R}(K) = \prod (K/\Phi \mid \Phi \in M(\text{Con } K)),$$

where  $M(\text{Con } K)$  is the set of all meet-irreducible congruences of  $K$ .

$K$  has a natural (diagonal) embedding into  $\mathbb{R}(K)$  by

$$\psi: a \mapsto a^{\mathbb{R}} = \langle [a]\Phi \mid \Phi \in M(\text{Con } K) \rangle.$$

Let  $K\psi = K^{\mathbb{R}}$  and for a congruence  $\Theta$  of  $K$ , let  $\Theta^{\mathbb{R}}$  denote the corresponding congruence of  $K^{\mathbb{R}}$ , that is,  $\Theta^{\mathbb{R}} = \Theta\psi$ .

**Theorem 8.** *Let  $K$  be a finite lattice. Then  $K^{\mathbb{R}}$  has the Congruence Extension Property in  $\mathbb{R}(K)$ .*

*Proof.* Let  $\Theta \in \text{Con } K$  and  $\Phi \in M(\text{Con } K)$ ; define the congruence  $\Theta_{\Phi}$  of  $K/\Phi$  as follows:

$$\Theta_{\Phi} = \begin{cases} \omega, & \text{if } \Theta \leq \Phi, \\ \iota, & \text{if } \Theta \not\leq \Phi; \end{cases}$$

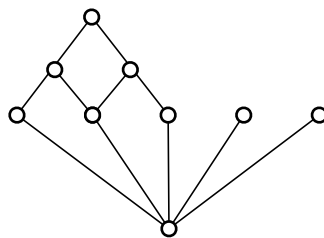


FIGURE 3

and define

$$\bar{\Theta}^{\mathbb{R}} = \prod (\Theta_{\Phi} \mid \Phi \in M(\text{Con } K)).$$

Obviously,  $\bar{\Theta}^{\mathbb{R}}$  is a congruence of  $\mathbb{R}(K)$ . It remains to show the Congruence Extension Property, that is, that

$$u^{\mathbb{R}} \equiv v^{\mathbb{R}} (\Theta^{\mathbb{R}}) \text{ in } K^{\mathbb{R}} \quad \text{iff} \quad u^{\mathbb{R}} \equiv v^{\mathbb{R}} (\bar{\Theta}^{\mathbb{R}}) \text{ in } \mathbb{R}(K), \text{ for all } u, v \in K.$$

Indeed,  $u^{\mathbb{R}} \equiv v^{\mathbb{R}} (\Theta^{\mathbb{R}})$  in  $K^{\mathbb{R}}$  is equivalent to  $uv(\Theta)$  in  $K$ , by the definition of  $u^{\mathbb{R}}$ ,  $v^{\mathbb{R}}$ ,  $K^{\mathbb{R}}$ , and  $\Theta^{\mathbb{R}}$ . The last congruence is equivalent to  $uv(\Phi)$  (that is,  $[u]\Phi = [v]\Phi$ ), for all  $\Phi \in M(\text{Con } K)$  satisfying  $\Theta \leq \Phi$ , which, in turn, can be written as

$$[u]\Phi \equiv [v]\Phi (\Theta_{\Phi}), \text{ for all } \Phi \in M(\text{Con } K) \text{ satisfying } \Theta \leq \Phi$$

(since for  $\Theta \leq \Phi$ , by definition,  $\Theta_{\Phi} = \omega$ ). Since  $\Theta_{\Phi} = \iota$ , for  $\Theta \not\leq \Phi$ , the congruence  $[u]\Phi \equiv [v]\Phi (\Theta_{\Phi})$  always holds if  $\Theta \not\leq \Phi$ . Therefore, the last displayed condition is equivalent to  $[u]\Phi \equiv [v]\Phi (\Theta_{\Phi})$ , for all  $\Phi \in M(\text{Con } K)$ , which is the same as  $u^{\mathbb{R}} \equiv v^{\mathbb{R}} (\bar{\Theta}^{\mathbb{R}})$ , as claimed.  $\square$

For the lattice  $K$  of Figure 1,  $\mathbb{R}(K)$  is  $K \times (\mathfrak{C}_2)^2$  (where  $\mathfrak{C}_2$  denotes the two-element lattice), which is representable as the ideal lattice of the chopped lattice of Figure 3. Indeed,  $K$  has three meet-irreducible congruences,  $\omega$ ,  $\Phi_1$ , and  $\Phi_2$ , so the three direct factors of  $\mathbb{R}(K)$  are  $K \cong K/\omega$ ,  $K/\Phi_1$ , and  $K/\Phi_2$ ; and the latter two are isomorphic to  $\mathfrak{C}_2$ .

For each  $\Phi \in M(\text{Con } K)$ , we select  $S(K/\Phi)$ , a finite, simple, sectionally complemented extension of  $K/\Phi$ , as provided by Lemma 7; let  $p_{\Phi}$  be an atom of  $S(K/\Phi)$ . Then we form the extension

$$\widehat{\mathbb{R}}(K) = \prod (S(K/\Phi) \mid \Phi \in M(\text{Con } K))$$

of  $\mathbb{R}(K)$ . We shall also denote by  $p_{\Phi}$  the corresponding atom of  $\widehat{\mathbb{R}}(K)$ , that is, the element whose  $\Phi$  component is  $p_{\Phi}$  and all the other components are zero.

For the lattice  $K$  of Figure 1, we can choose  $S(\mathfrak{C}_2) = \mathfrak{C}_2$  and then  $\widehat{\mathbb{R}}(K)$  can be represented as the ideal lattice of the chopped lattice of Figure 4. The three atoms singled out in the previous paragraph are black-filled in the diagram. (The first of the black-filled atoms could be any one of the first four atoms.)

**Lemma 9.** *Let  $K$  be a finite lattice. Then the extension  $\widehat{\mathbb{R}}(K)$  of  $K \cong K^{\mathbb{R}}$  has the following properties:*

- (i)  $\widehat{\mathbb{R}}(K)$  is sectionally complemented;
- (ii)  $\widehat{\mathbb{R}}(K)$  is finite;

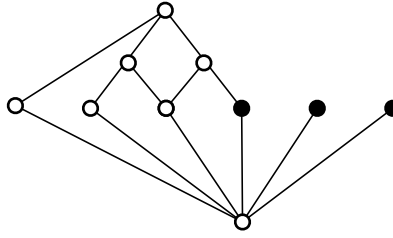


FIGURE 4

(iii) let

$$P = \{ p_\Phi \mid \Phi \in M(\text{Con } K) \};$$

then there is a one-to-one correspondence between subsets of  $P$  and congruences  $\Theta$  of  $\widehat{\mathbb{R}}(K)$ ; the subset of  $P$  corresponding to the congruence  $\Theta$  of  $\widehat{\mathbb{R}}(K)$  is

$$\{ p_\Phi \mid p_\Phi \equiv 0(\Theta) \};$$

hence, the congruence lattice of  $\widehat{\mathbb{R}}(K)$  is a finite Boolean lattice;

(iv) every congruence  $\Theta$  of  $K$  has an extension  $\widehat{\Theta}$  to a congruence of  $\widehat{\mathbb{R}}(K)$  corresponding to the subset

$$\{ p_\Phi \mid \Phi \in M(\text{Con } K) \text{ and } \Theta \not\leq \Phi \}$$

of  $P$ .

*Proof.* Properties (i) and (ii) follow from Lemma 7. Property (iii) is trivial since  $\widehat{\mathbb{R}}(K)$  is a finite direct product of simple lattices.

To verify property (iv), observe that a congruence  $\Theta$  of  $K$  extends to  $\mathbb{R}(K)$  to a congruence that is a direct product of congruences of the form  $\omega$  and  $\iota$  of the factors; the corresponding direct product of congruences of the form  $\omega$  and  $\iota$  (on the factors  $S(K/\Phi)$ ) extends  $\Theta$  to a congruence  $\widehat{\Theta}$  of  $\widehat{\mathbb{R}}(K)$ . The last statement is a rewrite of the definition of  $\Theta^{\mathbb{R}}$  in the proof of Theorem 8.  $\square$

### 5. PROOF OF THE THEOREM

Let  $K$  be a finite lattice. In this section, we construct a finite, sectionally complemented, congruence-preserving extension  $L$  of  $K$ , as required by the Theorem.

Using Lemma 7, for every  $\Phi \in M(\text{Con } K)$ , we select a finite, sectionally complemented, simple extension  $K_\Phi = S(K/\Phi)$  of  $K/\Phi$ , with an atom,  $p_\Phi$ , and zero,  $0_\Phi$ .

Let  $D = \text{Con } K$ , and let  $L_0$  be a lattice whose existence was stated in Theorem 2' for the finite distributive lattice  $D$ . Since

$$J(\text{Con } L_0) \cong J(D) \cong J(\text{Con } K),$$

the independent set of atoms  $Q = \{ q_\Phi \mid \Phi \in J(\text{Con } L_0) \}$  of  $L_0$  (described in (iv) of Theorem 2') can also be denoted as follows:

$$Q = \{ q_\Phi \mid \Phi \in J(\text{Con } K) \},$$

and for  $\Phi, \Phi' \in J(\text{Con } K)$ ,  $\Theta(0, q_\Phi) \leq \Theta(0, q_{\Phi'})$  iff  $\Phi \leq \Phi'$  in  $J(\text{Con } K)$ .

For a congruence  $\Theta$  of  $K$ , let  $\Theta_0$  denote the congruence of  $L_0$  determined by the set  $\{q_\Phi \mid \Phi \leq \Theta\}$ , that is,

$$\Theta_0 = \bigvee (\Theta(q_\Phi, 0_{L_0}) \mid \Phi \leq \Theta);$$

obviously,  $\Theta \mapsto \Theta_0$  sets up an isomorphism  $\text{Con } K \rightarrow \text{Con } L_0$ .

Let  $\Phi \in \text{M}(\text{Con } K)$ ; then let  $\Phi^\dagger$  be the smallest congruence of  $K$  not contained in  $\Phi$ . Note that  $\Phi^\dagger \in \text{J}(\text{Con } K)$ . The map  $\Phi \mapsto \Phi^\dagger$  is a natural order isomorphism between  $\text{M}(\text{Con } K)$  and  $\text{J}(\text{Con } K)$ .

The following observation is trivial but crucial:

**Lemma 10.** *The map  $p_\Phi \mapsto q_{\Phi^\dagger}$  sets up a bijection between the set*

$$\{p_\Phi \mid \Phi \in \text{M}(\text{Con } K) \text{ and } \Theta \not\leq \Phi\}$$

*that describes the congruence  $\widehat{\Theta}$  of  $\widehat{\mathbb{R}}(K)$  in Lemma 9(iv) and the set*

$$\{q_\Phi \mid \Phi \in \text{J}(\text{Con } K) \text{ and } \Phi \leq \Theta\}$$

*that describes the congruence  $\Theta_0$  in  $L_0$ .*

*In addition,*

$$\{q_\Phi \mid \Phi \in \text{J}(\text{Con } K) \text{ and } \Phi \leq \Theta\} = \{q_{\Phi^\dagger} \mid \Phi \in \text{M}(\text{Con } K) \text{ and } \Theta \not\leq \Phi\}.$$

Now we inductively construct the lattice  $L$  of the Theorem. Let

$$\Phi_1, \Phi_2, \dots, \Phi_n$$

list the meet-irreducible congruences of  $K$ . We apply Theorem 5 with  $A = L_0$ ,  $B = S(K/\Phi_1)$ , and the atoms  $q_{\Phi_1^\dagger}$  of  $L_0$  and  $p_{\Phi_1}$  of  $S(K/\Phi_1)$ , to obtain the chopped lattice  $M_1$  and  $L_1 = \text{Id } M_1$ . From Corollary 6, it follows that  $L_1$  is a congruence-preserving extension of  $L_0$  and that  $L_1$  also satisfies property (iv) of Theorem 2', in fact, with the same set of atoms as for  $L_0$ . Moreover,  $L_1$  is finite and sectionally complemented by Theorem 5. So we can apply Theorem 5 with  $A = L_1$ ,  $B = S(K/\Phi_2)$ , and the atoms  $q_{\Phi_2^\dagger}$  of  $L_1$  and  $p_{\Phi_2}$  of  $S(K/\Phi_2)$ , obtaining the chopped lattice  $M_2$  and  $L_2 = \text{Id } M_2$ . In  $n$  steps, we construct  $M_n$  and  $L = L_n = \text{Id } M_n$ .

It is clear that  $L$  is a finite sectionally complemented lattice. It is also evident that  $L$  is a congruence-preserving extension of  $L_0$ , hence,  $\text{Con } L \cong D$ . So to complete the proof of the Theorem, we have to show that  $K$  has an embedding into  $L$  with the Congruence Extension Property.

In the next step, we need the following statement:

**Lemma 11.** *Let  $A, B, C$  be pairwise disjoint finite lattices with zeroes  $0_A, 0_B, 0_C$ , and atoms  $p_A \in A, p_B, p'_B \in B, p_B \neq p'_B, p_C \in C$ . We construct some chopped lattices.*

- (i) *Let  $N_1$  be the chopped lattice obtained by forming the disjoint union of  $A$  and  $B$  and identifying  $0_A$  with  $0_B$  and  $p_A$  with  $p_B$ .*
- (ii) *Let  $N$  be the chopped lattice obtained by forming the disjoint union of  $A, B, C$  and identifying  $0_A$  with  $0_B$  and  $0_C, p_A$  with  $p_B$ , and  $p'_B$  with  $p_C$ .*
- (iii) *Form the ideal lattice  $\text{Id } N_1$ , which we consider an extension of  $N_1$ , so it has atoms  $p_A = p_B$  and  $p'_B$  and let  $N_2$  be the chopped lattice that is the disjoint union of  $\text{Id } N_1$  and  $C$  with the zeroes identified and also the atom  $p'_B \in \text{Id } N_1$  is identified with the atom  $p_C \in C$ .*

*Then  $\text{Id } N_2$  is isomorphic to  $\text{Id } N$ .*

*Proof.* The elements of  $\text{Id } N_1$  are pairs  $\langle a, b \rangle \in A \times B$  satisfying  $a \wedge p_a = b \wedge p_b$ . So the elements of  $\text{Id } N_2$  are pairs  $\langle \langle a, b \rangle, c \rangle$  such that  $\langle a, b \rangle \in \text{Id } N_1$  and  $\langle a, b \rangle \wedge p'_B = c \wedge p_C$ ; note that  $\langle a, b \rangle \in \text{Id } N_1$  iff  $a \wedge p_A = b \wedge p_B$  and note also that  $\langle a, b \rangle \wedge p'_B = b \wedge p'_B$ . On the other hand, the elements of  $\text{Id } N$  are triples  $\langle a, b, c \rangle \in A \times B \times C$  satisfying  $a \wedge p_A = b \wedge p_B$  and  $b \wedge p'_B = c \wedge p_C$ . It should now be obvious that  $\langle \langle a, b \rangle, c \rangle \mapsto \langle a, b, c \rangle$  is the required isomorphism  $\text{Id } N_2 \rightarrow \text{Id } N$ .  $\square$

Another view of  $L$  is the following. We form the chopped lattice  $M_1$ ; instead of proceeding to  $L_1 = \text{Id } M_1$ , let us now form the chopped lattice  $M'_2$  from  $M_1 = M'_1$  and  $S(K/\Phi_2)$  by identifying the zeros of  $M'_1$  and  $S(K/\Phi_2)$  and the atom  $q_{\Phi_2}$  of  $M'_1$  with the atom  $p_{\Phi_2}$  of  $S(K/\Phi_2)$ . Observe that  $M'_2$  is the union of  $L_0$ ,  $S(K/\Phi_1)$ , and  $S(K/\Phi_2)$  and  $S(K/\Phi_1) \cap S(K/\Phi_2) = \{0\}$ . By Lemma 11,  $\text{Id } M'_2 \cong L_2$ . Proceeding thus, we obtain the chopped lattice

$$M'_n = L_0 \cup S(K/\Phi_1) \cup S(K/\Phi_2) \cup \dots \cup S(K/\Phi_n),$$

where, for all  $1 \leq i < j \leq n$ , we have  $S(K/\Phi_i) \cap S(K/\Phi_j) = \{0\}$ , and  $L \cong \text{Id } M'_n$ , again by Lemma 11. So the chopped sublattice  $S(K/\Phi_1) \cup S(K/\Phi_2) \cup \dots \cup S(K/\Phi_n)$  of  $M'_n$  is just the lattices  $S(K/\Phi_1), S(K/\Phi_2), \dots, S(K/\Phi_n)$  glued together at their zeros, and, therefore,

$$\text{Id}(S(K/\Phi_1) \cup S(K/\Phi_2) \cup \dots \cup S(K/\Phi_n)) \cong \widehat{\mathbb{R}}(K).$$

Thus we can view  $L$  as the ideal lattice of the chopped lattice obtained by gluing together  $L_0$  and  $\widehat{\mathbb{R}}(K)$  over the Boolean lattice of  $2^n$  elements generated in  $L_0$  and in  $\widehat{\mathbb{R}}(K)$  by  $P = Q$ . A congruence  $\Theta$  of  $L$  is then determined by a congruence  $\Theta_{L_0}$  on  $L_0$  and a congruence  $\Theta_{\widehat{\mathbb{R}}(K)}$  on  $\widehat{\mathbb{R}}(K)$  that agree on the Boolean lattice. Since  $\widehat{\mathbb{R}}(K)$  is a Congruence Preserving Extension of  $K \cong K^{\mathbb{R}}$ , it follows now that  $L$  is a Congruence Preserving Extension of  $K$ , concluding the proof of the Theorem.

Figure 5 shows the chopped lattice whose ideal lattice is the lattice  $L$  (more precisely, one candidate for the lattice  $L$ ) of the Theorem for the lattice  $K$  of Figure 1. As we pointed out in connection with Figure 3, the chopped lattice whose ideal lattice is  $\widehat{\mathbb{R}}(K)$  consists of  $S(K)$  and two more atoms, which are incorporated in  $L_0$ . So in this very easy illustration, the chopped lattice whose ideal lattice is  $L$ , in fact, consists of  $S(K)$  and  $L_0$  identified by their zeros and one atom each.  $L_0$  is also the smallest nontrivial example of the Theorem. It is easy to compute that  $L$  has 40 elements.

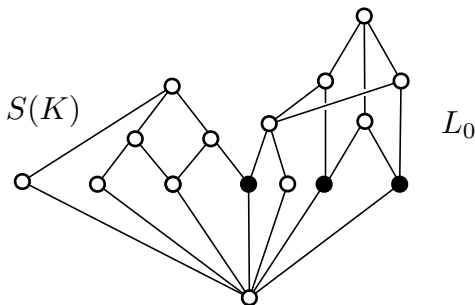


FIGURE 5

## ADDENDUM

While revising this paper (Aug. 1997), we were working on a problem proposed in our paper with H. Lakser [5] and we realized that generalizing the Theorem in this paper would yield a solution (see below).

Here is the generalization:

**Theorem'.** *Let  $K$  be a lattice and let  $\Gamma$  be a finite  $\{0, 1\}$ -meet subsemilattice of the congruence lattice of  $K$ . If  $\Gamma$  is a distributive lattice, then  $K$  has an extension  $L$  satisfying the following properties:*

- (i)  $L$  is sectionally complemented;
- (ii) the congruence lattice of  $L$  is isomorphic to  $\Gamma$ ;
- (iii) a congruence  $\Phi$  of  $K$  has an extension to  $L$  iff  $\Phi \in \Gamma$ ;
- (iv) each  $\Phi \in \Gamma$  has a unique extension to  $L$ .

We present two consequences of Theorem'.

**Corollary 12.** *Every lattice  $K$  with a finite congruence lattice has a congruence-preserving embedding into a sectionally complemented lattice  $L$ .*

Indeed, if  $\text{Con } K$  is finite, then set  $\Gamma = \text{Con } K$  and apply Theorem'.

In the Introduction, we mention the problem whether every countable lattice has a congruence-preserving extension to a sectionally complemented lattice. In light of Theorem', we would like to rephrase this:

**Problem.** Let  $K$  be a lattice with the property that  $K$  has countably many compact congruence relations. Is it true that  $K$  has a congruence-preserving extension to a sectionally complemented lattice?

**Corollary 13.** *Let  $K$  be a lattice and let  $\Phi$  be a nontrivial congruence of  $K$ . Then there is an extension  $L$  of  $K$  and a standard ideal  $S$  of  $L$  such that the restriction of  $\Theta[S]$  to  $K$  is  $\Phi$ . Moreover,  $L$  is sectionally complemented and it has only one nontrivial congruence, namely,  $\Theta[S]$ .*

This is a much stronger form of a result in the paper [5] by H. Lakser and the authors, in which it is proved that there is an extension  $L$  of  $K$  and a standard ideal  $S$  of  $L$  such that the restriction of  $\Theta[S]$  (the smallest congruence under which  $S$  is in a single congruence class) to  $K$  is  $\Phi$  but the additional properties of  $L$  stated in the last sentence of Corollary 13 do not hold in the former construction.

We get Corollary 13 from Theorem', by setting  $\Gamma = \{\omega, \Phi, \iota\}$ .

We raised the problem in [5] whether the result in that paper can be generalized to finitely many congruences; the construction in [5] cannot be used to prove such a result; for instance, if we apply the construction twice, the standard ideal obtained in the first step is no longer standard after the second step. Of course, Theorem' gives a positive solution to this problem.

**Changes in the proof.** In conclusion, we describe the changes necessary in the proof of the Theorem, to obtain a proof of Theorem'.

The crucial step is the modification of Lemma 4; this was done in [7]. Let us say that a chopped lattice  $M$  satisfies (FG), if every finitely generated ideal is a finite union of principal ideals. If  $M$  satisfies (FG), then  $\text{Id}_{\text{fg}} M$  (the poset of finitely generated ideals of  $M$ ) forms a sublattice of  $\text{Id } M$ .

**Lemma 4'.** *Let  $M$  be a chopped lattice satisfying (FG). Then the lattice  $\text{Id}_{\text{fg}} M$  is a congruence-preserving extension of the chopped lattice  $M$ .*

This form of Lemma 4 should be used throughout the new proof. Note that all chopped lattices in the proof satisfy (FG).

To generalize Theorem 5 to arbitrary lattices, replace  $\text{Id } M$  by  $\text{Id}_{\text{fg}} M$ .

The rectangular extension  $\mathbb{R}(K)$  of a finite lattice  $K$  has to be changed to a  $\Gamma$ -rectangular extension of an arbitrary lattice  $K$  as follows:

$$\mathbb{R}_{\Gamma}(K) = \prod (K/\Phi \mid \Phi \in \mathbf{M}(\Gamma));$$

and we define

$$\widehat{\mathbb{R}}_{\Gamma}(K) = \prod (S(K/\Phi) \mid \Phi \in \mathbf{M}(\Gamma),$$

where  $S(K/\Phi)$  is a simple, sectionally complemented extension of  $K/\Phi$  with an atom  $p_{\Phi}$ .

Lemma 9 remains true, except that the finiteness statement has to be dropped.

In Section 5, we have to make very few changes. To start, define  $D = \Gamma$  (this step utilizes that  $\Gamma$  is distributive), and replace all references to  $\text{Con } K$  by  $\Gamma$ , so, for instance,

$$Q = \{q_{\Phi} \mid \Phi \in \mathbf{J}(\Gamma)\}.$$

Now we inductively construct the lattice  $L$ , except that

$$\Phi_1, \Phi_2, \dots, \Phi_n$$

lists the meet-irreducible congruences of  $\Gamma$ . Lemma 11 is valid with the only change that we form  $\text{Id}_{\text{fg}} N$ ,  $\text{Id}_{\text{fg}} N_1$ , and  $\text{Id}_{\text{fg}} N_2$  (rather than  $\text{Id } N$ ,  $\text{Id } N_1$ , and  $\text{Id } N_2$ ).

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