PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 7, Pages 1903–1915
S 0002-9939(99)04671-7
Article electronically published on March 3, 1999

CONGRUENCE-PRESERVING EXTENSIONS OF FINITE LATTICES TO SECTIONALLY COMPLEMENTED LATTICES

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(Communicated by Lance W. Small)

Abstract. In 1962, the authors proved that every finite distributive lattice can be represented as the congruence lattice of a finite sectionally complemented lattice. In 1992, M. Tischendorf verified that every finite lattice has a congruence-preserving extension to an atomistic lattice.

In this paper, we bring these two results together. We prove that *every finite lattice has a congruence-preserving extension to a finite sectionally complemented lattice.*

1. Introduction

Early results on related structures (the automorphism group, the congruence lattice, and so on) of a lattice were characterization theorems, typified by the following result of R. P. Dilworth:

Theorem 1. *Let D be a finite distributive lattice. Then there exists a finite lattice L such that the congruence lattice of L is isomorphic to D.*

A stronger form of this result was published in G. Grätzer and E. T. Schmidt [6]:

Theorem 2. *Every finite distributive lattice D can be represented as the congruence lattice of a finite sectionally complemented lattice L.*

In the last decade, the emphasis has shifted from representation theorems to extension theorems, typified by the following important result of M. Tischendorf [12]:

Theorem 3. *Every finite lattice has a congruence-preserving embedding to a finite atomistic lattice.*

In this paper, in the spirit of Tischendorf's result, we shall prove the extension form of Theorem 2:

Received by the editors July 16, 1996 and, in revised form, September 22, 1997.
1991 *Mathematics Subject Classification.* Primary 06B10; Secondary 08A05.
Key words and phrases. Congruence lattice, congruence-preserving embedding, sectionally complemented lattice, finite.
The research of the first author was supported by the NSERC of Canada.
The research of the second author was supported by the Hungarian National Foundation for Scientific Research, under Grant No. T7442.

Theorem. *Every finite lattice K has a congruence-preserving embedding into a finite sectionally complemented lattice L.*

This result does not hold for infinite lattices: $F(\aleph_2)$ (the free lattice on $\aleph_2$ generators) does not have a congruence-preserving embedding into a sectionally complemented lattice according to M. Ploščica, J. Tůma, and F. Wehrung [10]. The problem is open for lattices of size $\aleph_0$ and $\aleph_1$.

See G. Grätzer and E. T. Schmidt [8] for a discussion of characterization theorems *vs.* extension theorems, and see G. Grätzer, H. Lakser, and E. T. Schmidt [2], [3], [4], and G. Grätzer and E. T. Schmidt [8] for further examples of extension theorems.

2. Preliminaries

We use the basic concepts and notations as in [1].

A finite lattice L is *atomistic* if every element is a join of atoms; L is *sectionally complemented* if, for every $a \leq b$ in L, there is an element c that is a complement of a in the interval $[0, b]$. Obviously, every finite sectionally complemented lattice is atomistic, but not conversely; the seven element lattice K of Figure 1 witnesses this.

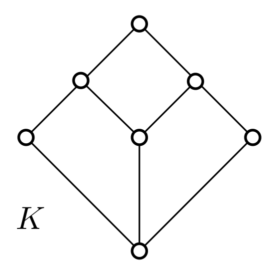

Figure 1

Let L be a lattice. If K is a sublattice of L, we call L an *extension* of K. If L is an extension of K, Θ is a congruence of K, and Φ is a congruence of L, then Φ is an *extension* of Θ to L iff the restriction of Φ to K equals Θ. We say that K in L has the *Congruence Extension Property* iff every congruence of K has an extension to L.

We call L a *congruence-preserving extension* of K iff every congruence of K has *exactly one* extension to L. In this case, the congruence lattice of K is isomorphic to the congruence lattice of L; in formula, $\operatorname{Con} K \cong \operatorname{Con} L$.

Let M be a finite poset such that $\inf\{a, b\}$ exists in M, for all a, $b \in M$. We define in M:

$$\begin{aligned} a \wedge b &= \inf\{a, b\}, && \text{for all } a,\ b \in M; \\ a \vee b &= \sup\{a, b\}, && \text{whenever } \sup\{a, b\} \text{ exists.} \end{aligned}$$

This makes M into a finite *chopped lattice*. Note that if L is a finite lattice with unit, 1, then $L - \{1\}$ is a finite chopped lattice; and conversely, if M is a finite chopped lattice, then $L = M \cup \{1\}$ is a finite lattice from which we obtain M by chopping off 1.

An equivalence relation Θ on the chopped lattice M is a *congruence relation* iff, for all a_0, a_1, b_0, $b_1 \in M$,

$$\begin{aligned} a_0 &\equiv b_0(\Theta), \\ a_1 &\equiv b_1(\Theta), \end{aligned}$$

imply that

$$\begin{aligned} a_0 \wedge a_1 &\equiv b_0 \wedge b_1(\Theta), \\ a_0 \vee a_1 &\equiv b_0 \vee b_1(\Theta), \quad \text{whenever } a_0 \vee a_1 \text{ and } b_0 \vee b_1 \text{ both exist.} \end{aligned}$$

The set $\operatorname{Con} M$ of all congruence relations of M is a lattice.

An *ideal* I of a finite chopped lattice M is a non-empty subset $I \subseteq M$ with the following two properties:

(i) $i \wedge a \in I$, for $i \in I$ and $a \in M$;
(ii) $i \vee j \in I$, for i, $j \in I$, provided that $i \vee j$ exists in M.

The ideals of the finite chopped lattice M form the finite lattice $\operatorname{Id} M$.

The following lemma was published in G. Grätzer [1].

Lemma 4 (G. Grätzer and H. Lakser)**.** *Let M be a finite chopped lattice. Then for every congruence relation Θ of M, there exists exactly one congruence relation $\overline{\Theta}$ of $\operatorname{Id} M$ such that, for a, $b \in M$,*

$$(a] \equiv (b] \ (\overline{\Theta}) \quad \textit{iff} \quad a \equiv b \ (\Theta).$$

In particular, $\operatorname{Con} M \cong \operatorname{Con}(\operatorname{Id} M)$.

In the proof of the Theorem, we need a more detailed form of Theorem 2.

Theorem 2′. *For every finite distributive lattice D, there is a lattice L with the following properties:*

(i) *L is finite;*
(ii) *L is sectionally complemented;*
(iii) *$\operatorname{Con} L$ is isomorphic to D;*
(iv) *there is an independent set of atoms*

$$Q = \{\, q_\Phi \mid \Phi \in \mathrm{J}(\operatorname{Con} L) \,\}$$

of L such that $\Phi = \Theta(0, q_\Phi)$, for $\Phi \in \mathrm{J}(\operatorname{Con} L)$, and there is a one-to-one correspondence between congruences and certain subsets of Q; the subset that corresponds to the congruence Θ of L is

$$\{\, q_\Phi \mid \Phi \in \mathrm{J}(\operatorname{Con} L) \text{ and } \Phi \leq \Theta \,\}.$$

(Recall that a set of atoms is *independent*, if they generate a Boolean sublattice; $\mathrm{J}(\operatorname{Con} L)$ is the set of join-irreducible congruences of L.)

The two standards proofs of Theorem 2 (the original proof in [6] and the proof, due to the first author and H. Lakser, published in [1]) both exhibit the set Q satisfying (iv).

3. Sectionally complemented lattices

In this section, we present a new construction of finite sectionally complemented lattices.

Theorem 5. *Let A and B be (disjoint) finite sectionally complemented lattices with zero elements 0_A and 0_B, respectively. Let p_A be an atom of A and let p_B be an atom of B.*

Form the partial lattice M by identifying 0_A with 0_B and p_A with p_B. Then M is a finite chopped lattice and $\operatorname{Id} M$ is a finite sectionally complemented lattice.

Proof. Let $0 = 0_A = 0_B$ and $p = p_A = p_B$. Obviously, M is a finite chopped lattice.

An ideal I of M can be represented by a pair of elements $\langle i_A, i_B \rangle$ such that $i_A \in A$, $i_B \in B$, $I \cap A = (i_A]$, $I \cap B = (i_B]$, and $i_A \wedge p = i_B \wedge p$.

To show that $\operatorname{Id} M$ is sectionally complemented, let $I \subseteq J$ be two ideals of M, represented by $\langle i_A, i_B \rangle$ and $\langle j_A, j_B \rangle$, respectively. Let s_A be the sectional complement of i_A in j_A and let s_B be the sectional complement of i_B in j_B. If

$$p \wedge s_A = p \wedge s_B,$$

then $\langle s_A, s_B \rangle$ is an ideal S that is a sectional complement of I in $[(0], J]$. Otherwise, without loss of generality, we can assume that

$$p \wedge s_A = 0,$$
$$p \wedge s_B = p.$$

Let s'_B be a sectional complement of p in $[0, s_B]$. Then $\langle s_A, s'_B \rangle$ satisfies

$$p \wedge s_A = p \wedge s'_B$$

$(= 0)$, and so the pair represents an ideal S of M. Obviously, $I \wedge S = (0]$.

Since $p \wedge s_B = p$, it follows that $p \leq s_B \leq j_B$. Since J is an ideal and $j_B \wedge p = p$, it follows that $j_A \wedge p = p$, that is,

$$p \leq j_A.$$

Obviously, $I \vee S \subseteq J$. So to show that $I \vee S = J$, it is sufficient to verify that j_A, $j_B \in I \vee S$. Evidently, $j_A = i_A \vee s_A \in I \vee S$. Note that $p \leq j_A = i_A \vee s_A \in I \vee S$. Thus, p, s'_B, $i_B \in I \vee S$, and therefore

$$p \vee s'_B \vee i_B = (p \vee s'_B) \vee i_B = s_B \vee i_B = j_B \in I \vee S,$$

which was to be proved. □

An easy consequence of this theorem is the following:

Corollary 6. *Let A, B, and M be as in Theorem 5. In addition, let us assume that B is a simple lattice. Then $\operatorname{Con} M \cong \operatorname{Con} A$ and $\operatorname{Id} M$ is a sectionally complemented congruence-preserving extension of A.*

Proof. It is clear that a congruence Θ of A has exactly one extension to a congruence Θ_M of M; indeed, if $p_A \equiv 0_A(\Theta)$, then let B be collapsed under Θ_M; otherwise, let Θ_M be ω_B on B. So this statement follows from Theorem 5 and Lemma 4. □

Lemma 7. *For every finite lattice L, there is a finite, simple, sectionally complemented extension $S(L)$.*

Proof. If $|L| \leq 2$, then we can take $S(L) = L$. So we can assume that $|L| > 2$.

Let 0 and 1 be the zero and unit element of L, respectively. We can assume that 1 is join-reducible; otherwise, adjoin a common complement u, that is, an element u satisfying

$$u \wedge x = 0,$$
$$u \vee x = 1,$$

for all $x \in L - \{0, 1\}$. So there are elements x_1, $x_2 \in L - \{1\}$ satisfying $x_1 \vee x_2 = 1$, and 1 is join-reducible.

Let

$$N(L) = L - \{\, x \mid x = 0 \text{ or } x \text{ is an atom} \,\}.$$

For every $a \in N(L)$, we adjoin an atom $p_a < a$ so that if $a \neq b$, then $p_a \neq p_b$. We make

$$S(L) = L \cup \{\, p_a \mid a \in N(L) \,\}$$

(a disjoint union) into a poset by the following rules:

(α) L is a subposet of $S(L)$;

(β) if $a \in N(L)$ and $x \in L$, then

$$\begin{aligned} x < p_a &\quad \text{iff } x = 0, \\ p_a < x &\quad \text{iff } a \leq x; \end{aligned}$$

(γ) if a and $b \in N(L)$, then

$$p_a \leq p_b \quad \text{iff} \quad a = b.$$

Then $S(L)$ is a lattice. The meet and the join in $S(L)$ are described by the following rules:

(i) L is a sublattice of $S(L)$;

(ii) if $a \in N(L)$ and $x \in L$, then

$$p_a \wedge x = \begin{cases} 0, & \text{if } a \nleq x, \\ p_a, & \text{if } a \leq x; \end{cases}$$

(iii) if $a \in N(L)$ and $x \in L$, then

$$p_a \vee x = \begin{cases} a \vee x, & \text{if } x \neq 0, \\ p_a, & \text{if } x = 0; \end{cases}$$

(iv) if a and $b \in N(L)$, $a \neq b$, then

$$\begin{aligned} p_a \wedge p_b &= 0, \\ p_a \vee p_b &= a \vee b. \end{aligned}$$

Obviously, $S(L)$ is a finite lattice; 0 and 1 are the zero and unit elements of $S(L)$, respectively. $S(L)$ an extension of L. To show that $S(L)$ is sectionally complemented, take $0 < u < v$ in $S(L)$. We consider two cases:

Case 1. $u \in L$. Then $v \in N(L)$ and p_v is a sectional complement of u in $[0, v]$ by (ii) and (iii).

Case 2. $u \notin L$. Then $u = p_a$, for some $a \in N(L)$, and $v \in L$ satisfies $a \leq v$. If $a = v$, then any x satisfying $0 < x < a$ is a sectional complement of u in $[0, v]$; and there is such an x because $a \in N(L)$. If $a < v$, then p_v is a sectional complement of u in $[0, v]$ by (iii) and (iv).

Finally, $S(L)$ is simple. Indeed, let $\Theta > \omega$ be a congruence of L. We verify that there is an $a > 0$, $a \in S(L)$, such that $a \equiv 0(\Theta)$. Indeed, since $\Theta > \omega$, there are u, $v \in L$ such that $u < v$ and $u \equiv v(\Theta)$. If $u = 0$, then $a = v$ satisfies the requirements. If $0 < u$, then $v \in N(L)$, so there is an element $p_v \in S(L)$. We have two cases to consider:

Case 1. $u \in L$. In this case, $u \equiv v(\Theta)$ implies that $p_v \equiv 0(\Theta)$, so we can take $a = p_v$.

Case 2. $u \in S(L) - L$. In this case, $u = p_x$, for some $x \in N(L)$. Obviously, $v \in L$ and $x \leq v$. Since $x \in N(L)$, there is an $a \in L$ satisfying $0 < a < x$. Now $u \equiv v(\Theta)$ implies that $p_x \equiv x(\Theta)$ and so $a \equiv 0(\Theta)$.

Using the congruence $a \equiv 0(\Theta)$, we conclude that $p_1 \equiv 1(\Theta)$. Meeting with x_1, we obtain that $x_1 \equiv 0(\Theta)$, and similarly $x_2 \equiv 0(\Theta)$. Joining these two congruences, we obtain that $0 \equiv 1(\Theta)$, that is, $\Theta = \iota$. ☐

We should point out that since a finite partition lattice is simple and sectionally complemented by O. Ore [9], Lemma 7 follows from the following very deep result of P. Pudlák and J. Tůma [11]: *Every finite lattice can be embedded into a finite partition lattice.*

For the lattice K of Figure 1, we can choose for $S(K)$ the lattice depicted in Figure 2; note that this is much smaller than the lattice constructed in the proof of Lemma 7.

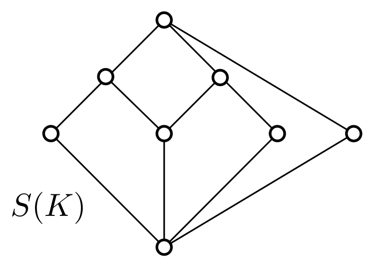

FIGURE 2

4. RECTANGULAR EXTENSIONS

The *rectangular extension* $\mathbb{R}(K)$ of a finite lattice K is defined as the direct product of *all* subdirect factors of K, that is,

$$\mathbb{R}(K) = \prod (\, K/\Phi \mid \Phi \in \mathrm{M}(\mathrm{Con}\, K)\,),$$

where $\mathrm{M}(\mathrm{Con}\, K)$ is the set of all meet-irreducible congruences of K.

K has a natural (diagonal) embedding into $\mathbb{R}(K)$ by

$$\psi \colon a \mapsto a^{\mathbb{R}} = \langle [a]\Phi \mid \Phi \in \mathrm{M}(\mathrm{Con}\, K)\rangle.$$

Let $K\psi = K^{\mathbb{R}}$ and for a congruence Θ of K, let $\Theta^{\mathbb{R}}$ denote the corresponding congruence of $K^{\mathbb{R}}$, that is, $\Theta^{\mathbb{R}} = \Theta\psi$.

Theorem 8. *Let K be a finite lattice. Then $K^{\mathbb{R}}$ has the Congruence Extension Property in $\mathbb{R}(K)$.*

Proof. Let $\Theta \in \mathrm{Con}\, K$ and $\Phi \in \mathrm{M}(\mathrm{Con}\, K)$; define the congruence Θ_Φ of K/Φ as follows:

$$\Theta_\Phi = \begin{cases} \omega, & \text{if } \Theta \leq \Phi, \\ \iota, & \text{if } \Theta \nleq \Phi; \end{cases}$$

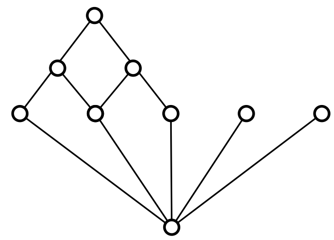

Figure 3

and define

$$\overline{\Theta}^{\mathbb{R}} = \prod(\, \Theta_\Phi \mid \Phi \in \mathrm{M}(\mathrm{Con}\, K)\,).$$

Obviously, $\overline{\Theta}^{\mathbb{R}}$ is a congruence of $\mathbb{R}(K)$. It remains to show the Congruence Extension Property, that is, that

$$u^{\mathbb{R}} \equiv v^{\mathbb{R}}\,(\Theta^{\mathbb{R}}) \text{ in } K^{\mathbb{R}} \quad \text{iff} \quad u^{\mathbb{R}} \equiv v^{\mathbb{R}}\,(\overline{\Theta}^{\mathbb{R}}) \text{ in } \mathbb{R}(K), \text{ for all } u,\ v \in K.$$

Indeed, $u^{\mathbb{R}} \equiv v^{\mathbb{R}}(\Theta^{\mathbb{R}})$ in $K^{\mathbb{R}}$ is equivalent to $uv(\Theta)$ in K, by the definition of $u^{\mathbb{R}}$, $v^{\mathbb{R}}$, $K^{\mathbb{R}}$, and $\Theta^{\mathbb{R}}$. The last congruence is equivalent to $uv(\Phi)$ (that is, $[u]\Phi = [v]\Phi$), for all $\Phi \in \mathrm{M}(\mathrm{Con}\, K)$ satisfying $\Theta \leq \Phi$, which, in turn, can be written as

$$[u]\Phi \equiv [v]\Phi\,(\Theta_\Phi), \text{ for all } \Phi \in \mathrm{M}(\mathrm{Con}\, K) \text{ satisfying } \Theta \leq \Phi$$

(since for $\Theta \leq \Phi$, by definition, $\Theta_\Phi = \omega$). Since $\Theta_\Phi = \iota$, for $\Theta \nleq \Phi$, the congruence $[u]\Phi \equiv [v]\Phi(\Theta_\Phi)$ always holds if $\Theta \nleq \Phi$. Therefore, the last displayed condition is equivalent to $[u]\Phi \equiv [v]\Phi(\Theta_\Phi)$, for all $\Phi \in \mathrm{M}(\mathrm{Con}\, K)$, which is the same as $u^{\mathbb{R}} \equiv v^{\mathbb{R}}(\Theta^{\mathbb{R}})$, as claimed. □

For the lattice K of Figure 1, $\mathbb{R}(K)$ is $K \times (\mathfrak{C}_2)^2$ (where $\mathfrak{C}_2$ denotes the two-element lattice), which is representable as the ideal lattice of the chopped lattice of Figure 3. Indeed, K has three meet-irreducible congruences, ω, Φ_1, and Φ_2, so the three direct factors of $\mathbb{R}(K)$ are $K \cong K/\omega$, K/Φ_1, and K/Φ_2; and the latter two are isomorphic to $\mathfrak{C}_2$.

For each $\Phi \in \mathrm{M}(\mathrm{Con}\, K)$, we select $S(K/\Phi)$, a finite, simple, sectionally complemented extension of K/Φ, as provided by Lemma 7; let p_Φ be an atom of $S(K/\Phi)$. Then we form the extension

$$\widehat{\mathbb{R}}(K) = \prod(\, S(K/\Phi) \mid \Phi \in \mathrm{M}(\mathrm{Con}\, K)\,)$$

of $\mathbb{R}(K)$. We shall also denote by p_Φ the corresponding atom of $\widehat{\mathbb{R}}(K)$, that is, the element whose Φ component is p_Φ and all the other components are zero.

For the lattice K of Figure 1, we can choose $S(\mathfrak{C}_2) = \mathfrak{C}_2$ and then $\widehat{\mathbb{R}}(K)$ can be represented as the ideal lattice of the chopped lattice of Figure 4. The three atoms singled out in the previous paragraph are black-filled in the diagram. (The first of the black-filled atoms could be any one of the first four atoms.)

Lemma 9. *Let K be a finite lattice. Then the extension $\widehat{\mathbb{R}}(K)$ of $K \cong K^{\mathbb{R}}$ has the following properties:*

(i) *$\widehat{\mathbb{R}}(K)$ is sectionally complemented;*
(ii) *$\widehat{\mathbb{R}}(K)$ is finite;*

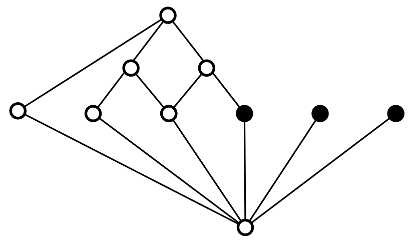

FIGURE 4

(iii) *let*

$$P = \{\, p_\Phi \mid \Phi \in \mathrm{M}(\mathrm{Con}\, K) \,\};$$

then there is a one-to-one correspondence between subsets of P *and congruences* Θ *of* $\widehat{\mathbb{R}}(K)$*; the subset of* P *corresponding to the congruence* Θ *of* $\widehat{\mathbb{R}}(K)$ *is*

$$\{\, p_\Phi \mid p_\Phi \equiv 0\,(\Theta) \,\};$$

hence, the congruence lattice of $\widehat{\mathbb{R}}(K)$ *is a finite Boolean lattice;*

(iv) *every congruence* Θ *of* K *has an extension* $\widehat{\Theta}$ *to a congruence of* $\widehat{\mathbb{R}}(K)$ *corresponding to the subset*

$$\{\, p_\Phi \mid \Phi \in \mathrm{M}(\mathrm{Con}\, K) \text{ and } \Theta \nleq \Phi \,\}$$

of P*.*

Proof. Properties (i) and (ii) follow from Lemma 7. Property (iii) is trivial since $\widehat{\mathbb{R}}(K)$ is a finite direct product of simple lattices.

To verify property (iv), observe that a congruence Θ of K extends to $\mathbb{R}(K)$ to a congruence that is a direct product of congruences of the form ω and ι of the factors; the corresponding direct product of congruences of the form ω and ι (on the factors $S(K/\Phi)$) extends Θ to a congruence $\widehat{\Theta}$ of $\widehat{\mathbb{R}}(K)$. The last statement is a rewrite of the definition of $\Theta^{\mathbb{R}}$ in the proof of Theorem 8. ☐

5. Proof of the Theorem

Let K be a finite lattice. In this section, we construct a finite, sectionally complemented, congruence-preserving extension L of K, as required by the Theorem.

Using Lemma 7, for every $\Phi \in \mathrm{M}(\mathrm{Con}\, K)$, we select a finite, sectionally complemented, simple extension $K_\Phi = S(K/\Phi)$ of K/Φ, with an atom, p_Φ, and zero, 0_Φ.

Let $D = \mathrm{Con}\, K$, and let L_0 be a lattice whose existence was stated in Theorem 2′ for the finite distributive lattice D. Since

$$\mathrm{J}(\mathrm{Con}\, L_0) \cong \mathrm{J}(D) \cong \mathrm{J}(\mathrm{Con}\, K),$$

the independent set of atoms $Q = \{\, q_\Phi \mid \Phi \in \mathrm{J}(\mathrm{Con}\, L_0) \,\}$ of L_0 (described in (iv) of Theorem 2′) can also be denoted as follows:

$$Q = \{\, q_\Phi \mid \Phi \in \mathrm{J}(\mathrm{Con}\, K) \,\},$$

and for Φ, $\Phi' \in \mathrm{J}(\mathrm{Con}\, K)$, $\Theta(0, q_\Phi) \leq \Theta(0, q_{\Phi'})$ iff $\Phi \leq \Phi'$ in $\mathrm{J}(\mathrm{Con}\, K)$.

For a congruence Θ of K, let Θ_0 denote the congruence of L_0 determined by the set $\{\, q_\Phi \mid \Phi \leq \Theta \,\}$, that is,

$$\Theta_0 = \bigvee (\, \Theta(q_\Phi, 0_{L_0}) \mid \Phi \leq \Theta \,);$$

obviously, $\Theta \mapsto \Theta_0$ sets up an isomorphism $\operatorname{Con} K \to \operatorname{Con} L_0$.

Let $\Phi \in \mathrm{M}(\operatorname{Con} K)$; then let $\Phi^\dagger$ be the smallest congruence of K not contained in Φ. Note that $\Phi^\dagger \in \mathrm{J}(\operatorname{Con} K)$. The map $\Phi \mapsto \Phi^\dagger$ is a natural order isomorphism between $\mathrm{M}(\operatorname{Con} K)$ and $\mathrm{J}(\operatorname{Con} K)$.

The following observation is trivial but crucial:

Lemma 10. *The map $p_\Phi \mapsto q_{\Phi^\dagger}$ sets up a bijection between the set*

$$\{\, p_\Phi \mid \Phi \in \mathrm{M}(\operatorname{Con} K) \text{ and } \Theta \nleq \Phi \,\}$$

that describes the congruence $\widehat{\Theta}$ of $\widehat{\mathbb{R}}(K)$ in Lemma 9(iv) *and the set*

$$\{\, q_\Phi \mid \Phi \in \mathrm{J}(\operatorname{Con} K) \text{ and } \Phi \leq \Theta \,\}$$

that describes the congruence Θ_0 in L_0.

In addition,

$$\{\, q_\Phi \mid \Phi \in \mathrm{J}(\operatorname{Con} K) \text{ and } \Phi \leq \Theta \,\} = \{\, q_{\Phi^\dagger} \mid \Phi \in \mathrm{M}(\operatorname{Con} K) \text{ and } \Theta \nleq \Phi \,\}.$$

Now we inductively construct the lattice L of the Theorem. Let

$$\Phi_1, \Phi_2, \ldots, \Phi_n$$

list the meet-irreducible congruences of K. We apply Theorem 5 with $A = L_0$, $B = S(K/\Phi_1)$, and the atoms $q_{\Phi_1^\dagger}$ of L_0 and p_{Φ_1} of $S(K/\Phi_1)$, to obtain the chopped lattice M_1 and $L_1 = \operatorname{Id} M_1$. From Corollary 6, it follows that L_1 is a congruence-preserving extension of L_0 and that L_1 also satisfies property (iv) of Theorem 2′, in fact, with the same set of atoms as for L_0. Moreover, L_1 is finite and sectionally complemented by Theorem 5. So we can apply Theorem 5 with $A = L_1$, $B = S(K/\Phi_2)$, and the atoms $q_{\Phi_2^\dagger}$ of L_1 and p_{Φ_2} of $S(K/\Phi_2)$, obtaining the chopped lattice M_2 and $L_2 = \operatorname{Id} M_2$. In n steps, we construct M_n and $L = L_n = \operatorname{Id} M_n$.

It is clear that L is a finite sectionally complemented lattice. It is also evident that L is a congruence-preserving extension of L_0, hence, $\operatorname{Con} L \cong D$. So to complete the proof of the Theorem, we have to show that K has an embedding into L with the Congruence Extension Property.

In the next step, we need the following statement:

Lemma 11. *Let A, B, C be pairwise disjoint finite lattices with zeroes 0_A, 0_B, 0_C, and atoms $p_A \in A$, p_B, $p'_B \in B$, $p_B \neq p'_B$, $p_C \in C$. We construct some chopped lattices.*

(i) *Let N_1 be the chopped lattice obtained by forming the disjoint union of A and B and identifying 0_A with 0_B and p_A with p_B.*

(ii) *Let N be the chopped lattice obtained by forming the disjoint union of A, B, C and identifying 0_A with 0_B and 0_C, p_A with p_B, and p'_B with p_C.*

(iii) *Form the ideal lattice* $\operatorname{Id} N_1$, *which we consider an extension of N_1, so it has atoms $p_A = p_B$ and p'_B and let N_2 be the chopped lattice that is the disjoint union of* $\operatorname{Id} N_1$ *and C with the zeroes identified and also the atom $p'_B \in \operatorname{Id} N_1$ is identified with the atom $p_C \in C$.*

Then $\operatorname{Id} N_2$ *is isomorphic to* $\operatorname{Id} N$.

Proof. The elements of $\operatorname{Id} N_1$ are pairs $\langle a, b\rangle \in A\times B$ satisfying $a\wedge p_a = b\wedge p_b$. So the elements of $\operatorname{Id} N_2$ are pairs $\langle\langle a, b\rangle, c\rangle$ such that $\langle a, b\rangle \in \operatorname{Id} N_1$ and $\langle a, b\rangle\wedge p'_B = c\wedge p_C$; note that $\langle a, b\rangle \in \operatorname{Id} N_1$ iff $a \wedge p_A = b \wedge p_B$ and note also that $\langle a, b\rangle \wedge p'_B = b \wedge p'_B$. On the other hand, the elements of $\operatorname{Id} N$ are triples $\langle a, b, c\rangle \in A \times B \times C$ satisfying $a\wedge p_A = b\wedge p_B$ and $b\wedge p'_B = c\wedge p_C$. It should now be obvious that $\langle\langle a, b\rangle, c\rangle \mapsto \langle a, b, c\rangle$ is the required isomorphism $\operatorname{Id} N_2 \to \operatorname{Id} N$. □

Another view of L is the following. We form the chopped lattice M_1; instead of proceeding to $L_1 = \operatorname{Id} M_1$, let us now form the chopped lattice M'_2 from $M_1 = M'_1$ and $S(K/\Phi_2)$ by identifying the zeros of M'_1 and $S(K/\Phi_2)$ and the atom $q_{\Phi_2^\dagger}$ of M'_1 with the atom p_{Φ_2} of $S(K/\Phi_2)$. Observe that M'_2 is the union of L_0, $S(K/\Phi_1)$, and $S(K/\Phi_2)$ and $S(K/\Phi_1) \cap S(K/\Phi_2) = \{0\}$. By Lemma 11, $\operatorname{Id} M'_2 \cong L_2$. Proceeding thus, we obtain the chopped lattice

$$M'_n = L_0 \cup S(K/\Phi_1) \cup S(K/\Phi_2) \cup \cdots \cup S(K/\Phi_n),$$

where, for all $1 \le i < j \le n$, we have $S(K/\Phi_i) \cap S(K/\Phi_j) = \{0\}$, and $L \cong \operatorname{Id} M'_n$, again by Lemma 11. So the chopped sublattice $S(K/\Phi_1)\cup S(K/\Phi_2)\cup\cdots\cup S(K/\Phi_n)$ of M'_n is just the lattices $S(K/\Phi_1)$, $S(K/\Phi_2)$, … , $S(K/\Phi_n)$ glued together at their zeros, and, therefore,

$$\operatorname{Id}(S(K/\Phi_1) \cup S(K/\Phi_2) \cup \cdots \cup S(K/\Phi_n)) \cong \widehat{\mathbb{R}}(K).$$

Thus we can view L as the ideal lattice of the chopped lattice obtained by gluing together L_0 and $\widehat{\mathbb{R}}(K)$ over the Boolean lattice of 2^n elements generated in L_0 and in $\widehat{\mathbb{R}}(K)$ by $P = Q$. A congruence Θ of L is then determined by a congruence Θ_{L_0} on L_0 and a congruence $\Theta_{\widehat{\mathbb{R}}(K)}$ on $\widehat{\mathbb{R}}(K)$ that agree on the Boolean lattice. Since $\widehat{\mathbb{R}}(K)$ is a Congruence Preserving Extension of $K \cong K^{\mathbb{R}}$, it follows now that L is a Congruence Preserving Extension of K, concluding the proof of the Theorem.

Figure 5 shows the chopped lattice whose ideal lattice is the lattice L (more precisely, one candidate for the lattice L) of the Theorem for the lattice K of Figure 1. As we pointed out in connection with Figure 3, the chopped lattice whose ideal lattice is $\widehat{\mathbb{R}}(K)$ consists of $S(K)$ and two more atoms, which are incorporated in L_0. So in this very easy illustration, the chopped lattice whose ideal lattice is L, in fact, consists of $S(K)$ and L_0 identified by their zeros and one atom each. L_0 is also the smallest nontrivial example of the Theorem. It is easy to compute that L has 40 elements.

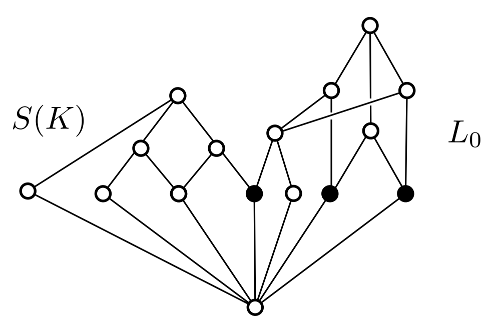

FIGURE 5

Addendum

While revising this paper (Aug. 1997), we were working on a problem proposed in our paper with H. Lakser [5] and we realized that generalizing the Theorem in this paper would yield a solution (see below).

Here is the generalization:

Theorem′. *Let K be a lattice and let Γ be a finite $\{0, 1\}$-meet subsemilattice of the congruence lattice of K. If Γ is a distributive lattice, then K has an extension L satisfying the following properties:*

(i) *L is sectionally complemented;*
(ii) *the congruence lattice of L is isomorphic to Γ;*
(iii) *a congruence Φ of K has an extension to L iff $\Phi \in \Gamma$;*
(iv) *each $\Phi \in \Gamma$ has a unique extension to L.*

We present two consequences of Theorem′.

Corollary 12. *Every lattice K with a finite congruence lattice has a congruence-preserving embedding into a sectionally complemented lattice L.*

Indeed, if $\operatorname{Con} K$ is finite, then set $\Gamma = \operatorname{Con} K$ and apply Theorem′.

In the Introduction, we mention the problem whether every countable lattice has a congruence-preserving extension to a sectionally complemented lattice. In light of Theorem′, we would like to rephrase this:

Problem. Let K be a lattice with the property that K has countably many compact congruence relations. Is it true that K has a congruence-preserving extension to a sectionally complemented lattice?

Corollary 13. *Let K be a lattice and let Φ be a nontrivial congruence of K. Then there is an extension L of K and a standard ideal S of L such that the restriction of $\Theta[S]$ to K is Φ. Moreover, L is sectionally complemented and it has only one nontrivial congruence, namely, $\Theta[S]$.*

This is a much stronger form of a result in the paper [5] by H. Lakser and the authors, in which it is proved that there is an extension L of K and a standard ideal S of L such that the restriction of $\Theta[S]$ (the smallest congruence under which S is in a single congruence class) to K is Φ but the additional properties of L stated in the last sentence of Corollary 13 do not hold in the former construction.

We get Corollary 13 from Theorem′, by setting $\Gamma = \{\omega, \Phi, \iota\}$.

We raised the problem in [5] whether the result in that paper can be generalized to finitely many congruences; the construction in [5] cannot be used to prove such a result; for instance, if we apply the construction twice, the standard ideal obtained in the first step is no longer standard after the second step. Of course, Theorem′ gives a positive solution to this problem.

Changes in the proof. In conclusion, we describe the changes necessary in the proof of the Theorem, to obtain a proof of Theorem′.

The crucial step is the modification of Lemma 4; this was done in [7]. Let us say that a chopped lattice M satisfies (FG), if every finitely generated ideal is a finite union of principal ideals. If M satisfies (FG), then $\operatorname{Id}_{\mathrm{fg}} M$ (the poset of finitely generated ideals of M) forms a sublattice of $\operatorname{Id} M$.

Lemma 4′. *Let M be a chopped lattice satisfying* (FG). *Then the lattice* $\operatorname{Id}_{\mathrm{fg}} M$ *is a congruence-preserving extension of the chopped lattice* M.

This form of Lemma 4 should be used throughout the new proof. Note that all chopped lattices in the proof satisfy (FG).

To generalize Theorem 5 to arbitrary lattices, replace $\operatorname{Id} M$ by $\operatorname{Id}_{\mathrm{fg}} M$.

The rectangular extension $\mathbb{R}(K)$ of a finite lattice K has to be changed to a Γ-*rectangular extension* of an arbitrary lattice K as follows:

$$\mathbb{R}_\Gamma(K) = \prod(\, K/\Phi \mid \Phi \in \mathrm{M}(\Gamma\,)\,);$$

and we define

$$\widehat{\mathbb{R}}_\Gamma(K) = \prod(\, S(K/\Phi) \mid \Phi \in \mathrm{M}(\Gamma\,),$$

where $S(K/\Phi)$ is a simple, sectionally complemented extension of K/Φ with an atom p_Φ.

Lemma 9 remains true, except that the finiteness statement has to be dropped.

In Section 5, we have to make very few changes. To start, define $D = \Gamma$ (this step utilizes that Γ is distributive), and replace all references to $\operatorname{Con} K$ by Γ, so, for instance,

$$Q = \{\, q_\Phi \mid \Phi \in \mathrm{J}(\Gamma)\,\}.$$

Now we inductively construct the lattice L, except that

$$\Phi_1, \Phi_2, \dots, \Phi_n$$

lists the meet-irreducible congruences of Γ. Lemma 11 is valid with the only change that we form $\operatorname{Id}_{\mathrm{fg}} N$, $\operatorname{Id}_{\mathrm{fg}} N_1$, and $\operatorname{Id}_{\mathrm{fg}} N_2$ (rather than $\operatorname{Id} N$, $\operatorname{Id} N_1$, and $\operatorname{Id} N_2$)).

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