

SOLVABILITY OF LINEAR SYSTEMS OF PDE'S WITH CONSTANT COEFFICIENTS

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ABSTRACT. In this paper we investigate the solvability of linear systems of partial differential equations with constant coefficients in a field of positive characteristic. In particular, we prove that consistence and compatibility are equivalent, which answers a question of Ehrenpreis and extends a result of Jia. The problem of uniqueness is also considered.

1. INTRODUCTION AND MAIN RESULTS

This paper deals with the solvability of linear systems of partial differential equations with constant coefficients in a field K of characteristic $p > 0$. In particular, we will show that consistence and compatibility for the rings of formal power series and of polynomials are equivalent.

Let $K[[X_1, \dots, X_s]]$ (resp. $K[X_1, \dots, X_s]$) be the ring of formal power series (resp. polynomials) in s indeterminates X_1, \dots, X_s over the field K .

Denote \mathbf{N} as the set of nonnegative integers. Given $\alpha := (\alpha_1, \dots, \alpha_s) \in \mathbf{N}^s$ and $\beta := (\beta_1, \dots, \beta_s) \in \mathbf{N}^s$, $\alpha \leq \beta$ means $\alpha_j \leq \beta_j$ for $1 \leq j \leq s$, while $\alpha < \beta$ means $\alpha \leq \beta$ and $\alpha \neq \beta$. For $\alpha \in \mathbf{N}^s$, we denote $\alpha! := \prod_{j=1}^s \alpha_j!$ and $X^\alpha := X_1^{\alpha_1} \cdots X_s^{\alpha_s}$. Moreover, the differential operator D^α on $K[[X_1, \dots, X_s]]$ or $K[X_1, \dots, X_s]$ is defined by the rule

$$(1.1) \quad D^\alpha \left(\sum_{\beta \in \mathbf{N}^s} a_\beta X^\beta \right) := \sum_{\beta \geq \alpha} a_\beta \frac{\beta!}{(\beta - \alpha)!} X^{\beta - \alpha}.$$

A polynomial

$$P(X) := \sum_{\alpha \in \mathbf{N}^s} c_\alpha X^\alpha \in K[X_1, \dots, X_s]$$

induces a differential operator $P(D) = \sum_{\alpha \in \mathbf{N}^s} c_\alpha D^\alpha$. Since the field K has **positive characteristic** p , we set

$$\mathcal{E} := \{\alpha \in \mathbf{N}^s : 0 \leq \alpha_j \leq p - 1 \text{ for } j = 1, \dots, s\}.$$

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Then $P(D)f = 0$ for all $f \in K[[X_1, \dots, X_s]]$ (or $K[X_1, \dots, X_s]$) if and only if $c_\alpha = 0$ for all $\alpha \in \mathcal{E}$. In this case we write $P(D) = 0$. This is essentially different from the case $p = 0$.

Now let $(P_{i,j})_{i \in I, j \in J}$ be a matrix, where I and J are finite sets, and each entry $P_{i,j}$ is an element of $K[[X_1, \dots, X_s]]$. The linear system of partial differential equations we want to consider has the following form:

$$(1.2) \quad \sum_{j \in J} P_{i,j}(D)u_j = f_i, \quad i \in I,$$

where f_i ($i \in I$) and u_j ($j \in J$) are both in $K[[X_1, \dots, X_s]]$ or $K[X_1, \dots, X_s]$.

The system (1.2) is said to be **consistent** if it has a solution for $(u_j)_{j \in J}$. It is said to be **compatible** if for any $q_i \in K[[X_1, \dots, X_s]]$ ($i \in I$),

$$(1.3) \quad \left(\sum_{i \in I} q_i P_{i,j} \right) (D) = 0, \quad j \in J,$$

implies

$$(1.4) \quad \sum_{i \in I} q_i(D)f_i = 0.$$

When K is the complex field \mathbf{C} , Ehrenpreis gave the well-known Ehrenpreis fundamental principle and showed in [3] for $\mathbf{C}[[X_1, \dots, X_s]]$ that the system (1.2) is consistent if and only if it is compatible. He raised the question whether his result could apply to an arbitrary field (see [3, p. 173]). In a recent paper [4], Jia confirmed the situation for $K[[X_1, \dots, X_s]]$ and fields of characteristic 0. Jia also characterized the uniqueness for $K = \mathbf{C}$ in [5]. For the case $\#J = 1$, Dahmen and Micchelli [1, 2] and Jia, Riemenschneider and Shen [6] considered a weak compatibility condition and showed that under some assumptions this weak compatibility is equivalent to the consistence. Oberst [7] gave some interesting examples and pointed out the essential differences between $p = 0$ and $p > 0$. His examples lead us to define the above compatibility condition which is equivalent to the usual definition when $p = 0$.

The purpose of this paper is to answer the question of Ehrenpreis and extend Jia's result to the following form.

Theorem 1. *Let K be a field of characteristic $p > 0$. Then the system of partial differential equations (1.2) is consistent if and only if it is compatible.*

Theorem 2. *Let K be a field of characteristic $p > 0$. Then the system of partial differential equations (1.2) is uniquely solvable if and only if it is compatible and the matrix $\{P_{i,j}(0)\}_{i \in I, j \in J}$ has rank $\#J$.*

Let us mention that the research on solvability of linear systems of partial differential equations also stems from multivariate splines and approximation theory, see [1, 2, 4, 5, 6].

2. PROOF OF THE THEOREMS

In this section we prove our theorems. The idea of the proofs is to split the formal power series and to take an algebraic approach to the reduced finite problem.

Proof of Theorem 1. The necessity is trivial.

To prove the sufficiency, we assume that the system (1.2) is compatible. Let

$$\begin{aligned} P_{i,j}(X) &:= \sum_{\alpha \in \mathbf{N}^s} c_{i,j}(\alpha) X^\alpha, \\ f_i(X) &:= \sum_{\alpha \in \mathbf{N}^s} \sum_{\beta \in \mathcal{E}} \frac{(p\alpha)!}{(p\alpha + \beta)!} \eta_i(\alpha, \beta) X^{p\alpha + \beta}, \\ u_j(X) &:= \sum_{\alpha \in \mathbf{N}^s} \sum_{\beta \in \mathcal{E}} \frac{(p\alpha)!}{(p\alpha + \beta)!} \xi_j(\alpha, \beta) X^{p\alpha + \beta}, \end{aligned}$$

where $i \in I, j \in J$ and all the coefficients are in K . Since $D^\alpha = 0$ for all $\alpha \in \mathbf{N}^s \setminus \mathcal{E}$, we may assume that

$$c_{i,j}(\alpha) = 0, \quad \alpha \in \mathbf{N}^s \setminus \mathcal{E}.$$

In terms of the above expressions,

$$\begin{aligned} & \sum_{j \in J} P_{i,j}(D) u_j \\ &= \sum_{j \in J} \sum_{\gamma \in \mathcal{E}} c_{i,j}(\gamma) \sum_{\alpha \in \mathbf{N}^s} \sum_{\beta \in \mathcal{E}} \frac{(p\alpha)!}{(p\alpha + \beta)!} \xi_j(\alpha, \beta) D^\gamma X^{p\alpha + \beta} \\ &= \sum_{j \in J} \sum_{\gamma \in \mathcal{E}} c_{i,j}(\gamma) \sum_{\alpha \in \mathbf{N}^s} \sum_{\gamma \leq \beta \in \mathcal{E}} \frac{(p\alpha)!}{(p\alpha + \beta - \gamma)!} \xi_j(\alpha, \beta) X^{p\alpha + \beta - \gamma} \\ (2.1) \quad &= \sum_{\alpha \in \mathbf{N}^s} \sum_{\beta \in \mathcal{E}} \frac{(p\alpha)!}{(p\alpha + \beta)!} \left\{ \sum_{j \in J} \sum_{\beta \leq \gamma \in \mathcal{E}} c_{i,j}(\gamma - \beta) \xi_j(\alpha, \gamma) \right\} X^{p\alpha + \beta}. \end{aligned}$$

Hence (1.2) is equivalent to

$$(2.2) \quad \sum_{j \in J} \sum_{\beta \leq \gamma \in \mathcal{E}} c_{i,j}(\gamma - \beta) \xi_j(\alpha, \gamma) = \eta_i(\alpha, \beta), \quad i \in I, \alpha \in \mathbf{N}^s, \beta \in \mathcal{E}.$$

We define a finite matrix $\{a_{(i,\beta),(j,\gamma)}\}_{(i,\beta) \in I \times \mathcal{E}, (j,\gamma) \in J \times \mathcal{E}}$ by

$$(2.3) \quad a_{(i,\beta),(j,\gamma)} = \begin{cases} c_{i,j}(\gamma - \beta), & \text{if } \beta \leq \gamma, i \in I, j \in J, \\ 0, & \text{otherwise.} \end{cases}$$

Then the system (1.2) is consistent if and only if for every $\alpha \in \mathbf{N}^s$, the following system of linear equations is solvable:

$$(2.4) \quad \sum_{j \in J} \sum_{\gamma \in \mathcal{E}} a_{(i,\beta),(j,\gamma)} x_{(j,\gamma)} = \eta_i(\alpha, \beta), \quad i \in I, \beta \in \mathcal{E}.$$

When $\eta_i(\alpha, \beta) = 0, i \in I, \beta \in \mathcal{E}$, for some $\alpha \in \mathbf{N}^s$, we choose $\xi_j(\alpha, \gamma) = 0, j \in J, \gamma \in \mathcal{E}$, to solve (2.2). Therefore, the system (1.2) is consistent for $K[[X_1, \dots, X_s]]$ as well as for $K[X_1, \dots, X_s]$ once we prove the solvability of the system (2.4) for every $\alpha \in \mathbf{N}^s$. To this end, we state that the vector $(\eta_i(\alpha, \beta) : i \in I, \beta \in \mathcal{E})$ lies in the linear span V of the vectors $\{(a_{(i,\beta),(j,\gamma)} : i \in I, \beta \in \mathcal{E}) : j \in J, \gamma \in \mathcal{E}\}$.

To prove the statement, fix $\alpha \in \mathbf{N}^s$. Suppose that $\{\lambda_{(i,\beta)}\} \in K^{I \times \mathcal{E}}$ lies in V^\perp , i.e.,

$$(2.5) \quad \sum_{i \in I} \sum_{\beta \in \mathcal{E}} \lambda_{(i,\beta)} a_{(i,\beta),(j,\gamma)} = 0, \quad j \in J, \gamma \in \mathcal{E},$$

we show that

$$(2.6) \quad \sum_{i \in I} \sum_{\beta \in \mathcal{E}} \lambda_{(i,\beta)} \eta_i(\alpha, \beta) = 0.$$

To prove (2.6), define a set of polynomials $\{q_i\}_{i \in I}$ by

$$(2.7) \quad q_i(X) := \sum_{\beta \in \mathcal{E}} \lambda_{(i,\beta)} X^\beta.$$

Then, by (2.5), for $j \in J$,

$$\begin{aligned} \left(\sum_{i \in I} q_i P_{i,j} \right)(D) &= \sum_{i \in I} \sum_{\beta \in \mathcal{E}} \sum_{\gamma \in \mathcal{E}} \lambda_{(i,\beta)} c_{i,j}(\gamma) D^{\beta+\gamma} \\ &= \sum_{i \in I} \sum_{\beta \in \mathcal{E}} \sum_{\beta \leq \gamma \in \mathcal{E}} \lambda_{(i,\beta)} c_{i,j}(\gamma - \beta) D^\gamma \\ &= \sum_{\gamma \in \mathcal{E}} \left\{ \sum_{i \in I} \sum_{\beta \in \mathcal{E}} \lambda_{(i,\beta)} a_{(i,\beta),(j,\gamma)} \right\} D^\gamma = 0. \end{aligned}$$

This means that for our chosen $\{q_i\}_{i \in I}$ (1.3) holds. By the compatibility condition of (1.2), we know that (1.4) holds, i.e.,

$$\begin{aligned} 0 &= \sum_{i \in I} q_i(D) f_i \\ &= \sum_{\delta \in \mathbf{N}^s} \sum_{\beta \in \mathcal{E}} \frac{(p\delta)!}{(p\delta + \beta)!} \left\{ \sum_{i \in I} \sum_{\beta \leq \gamma \in \mathcal{E}} \lambda_{(i,\gamma-\beta)} \eta_i(\delta, \gamma) \right\} X^{p\delta + \beta}. \end{aligned}$$

Choose $\beta = 0, \delta = \alpha$, (2.6) follows. Thus the statement is true, which implies the solvability of the system (2.4).

The proof of Theorem 1 is complete. \square

Proof of Theorem 2. We use the same notations as in the proof of Theorem 1.

Necessity. Suppose that the matrix $\{P_{i,j}(0)\}_{i \in I, j \in J}$ has rank less than $\#J$, then there is some $0 \neq \{d_j\}_{j \in J} \in K^J$ such that

$$\sum_{j \in J} c_{i,j}(0) d_j = \sum_{j \in J} P_{i,j}(0) d_j = 0, \quad i \in I.$$

Define $u_j \in K[X_1, \dots, X_s]$ by

$$u_j(X) = d_j, \quad j \in J.$$

Then by (2.1)

$$\sum_{j \in J} P_{i,j}(D) u_j = 0,$$

which is a contradiction to the uniqueness.

Sufficiency. Suppose that the uniqueness does not hold. Then there are $u_j \in K[[X_1, \dots, X_s]], j \in J$, not all trivial, such that

$$\sum_{j \in J} P_{i,j}(D) u_j = 0, \quad i \in I.$$

By (2.2), we have

$$(2.8) \quad \sum_{j \in J} \sum_{\beta \leq \gamma \in \mathcal{E}} c_{i,j}(\gamma - \beta) \xi_j(\alpha, \gamma) = 0, \quad i \in I, \alpha \in \mathbf{N}^s, \beta \in \mathcal{E}.$$

Notice that $u_j, j \in J$, are not all trivial. Hence there is some $\gamma_0 \in \mathcal{E}$ such that $\xi_j(\alpha, \gamma) = 0$ for all $\alpha \in \mathbf{N}^s, j \in J$ and all $\gamma \in \mathcal{E}$ with $\gamma_0 < \gamma$, while $\xi_{j_0}(\alpha_0, \gamma_0) \neq 0$ for some $j_0 \in J$ and $\alpha_0 \in \mathbf{N}^s$.

Thus, by (2.8) for $\beta = \gamma_0$ and $\alpha = \alpha_0$, we obtain

$$\sum_{j \in J} c_{i,j}(0) \xi_j(\alpha_0, \gamma_0) = \sum_{j \in J} P_{i,j}(0) \xi_j(\alpha_0, \gamma_0) = 0, \quad i \in I.$$

This implies that the matrix $\{P_{i,j}(0)\}_{i \in I, j \in J}$ has rank less than $\#J$, which is again a contradiction.

The proof of Theorem 2 is complete. \square

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