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A NOTE ON HOLOMORPHIC MAPS WITH UNIPOTENT JACOBIAN MATRICES

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Abstract. We prove that a holomorphic map $H : \mathbb{C}^2 \to \mathbb{C}^2$ is invertible if its Jacobian matrix JH is unipotent.

1. Introduction

Let $\mathbb{C}$ be the complex number field. Given a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ with $F = (f_1, f_2, \cdots, f_n)$, where $f_i \in \mathbb{C}[z_1, z_2, \cdots, z_n]$, a simple algebraic argument tells us that

$$J_F = \det[\frac{\partial f_i}{\partial z_j}] \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$$

whenever F is invertible. The Jacobian Conjecture asserts that the converse is true also.

The Jacobian Conjecture is false for holomorphic maps. An easy example is

$$F : \mathbb{C}^2 \to \mathbb{C}^2,$$
$$f_1 = e^{z_1},$$
$$f_2 = z_2 e^{-z_1}.$$

In [BCW] the Jacobian Conjecture has been reduced to the Unipotent Jacobian Conjecture, which states

The Unipotent Jacobian Conjecture. *If the Jacobian matrix JF of F is a unipotent matrix, then F is invertible.*

We suspect that this conjecture could also be true for holomorphic maps. In this note, we give a proof of the Unipotent Jacobian Conjecture for holomorphic maps with $n = 2$.

2. Main results

In this section, we first prove a theorem concerning holomorphic maps $F : \mathbb{C}^n \to \mathbb{C}^n$ with $JF^2 = 0$. The idea of the proof is obtained from [CSW]. We then apply the theorem to the case $n = 2$ and $F(z) = H(z) - z$, where $H(z)$ is an arbitrary holomorphic map $H : \mathbb{C}^2 \to \mathbb{C}^2$ with unipotent Jacobian matrix. This yields that H is invertible, i.e. Corollary 2.3.

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Theorem 2.1. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map. The following statements are equivalent:*

(1) $JF(z)^2 = 0$ *for all* $z \in \mathbb{C}^n$,
(2) $F(z + JF(z)z') = F(z)$ *for all* $z, z' \in \mathbb{C}^n$, *and*
(3) $JF(z + JF(z)z')JF(z) = 0$ *for all* $z, z' \in \mathbb{C}^n$.

Proof. (1)⇒(2). Using Taylor expansion

$$f(z+y) = f(z) + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_1,i_2,\cdots,i_k=1}^{n} \frac{\partial^k f(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \prod_{s=1}^{k} y_{i_s},$$

one has

$$\begin{aligned} &f_i(z + JF(z)z') = \\ &f_i(z) + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_1,i_2,\cdots,i_k=1}^{n} \frac{\partial^k f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \prod_{s=1}^{k} \sum_{j_s=1}^{n} \frac{\partial f_{i_s}(z)}{\partial z_{j_s}} z'_{j_s} = \\ &f_i(z) + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j_1,j_2,\cdots,j_k=1}^{n} \sum_{i_1,i_2,\cdots,i_k=1}^{n} \frac{\partial^k f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \prod_{s=1}^{k} \frac{\partial f_{i_s}(z)}{\partial z_{j_s}} \prod_{s=1}^{k} z'_{j_s}. \end{aligned}$$

Define

$$D^{[i]}_{j_1,j_2,\cdots,j_k} = \sum_{i_1,i_2,\cdots,i_k=1}^{n} \frac{\partial^k f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \prod_{s=1}^{k} \frac{\partial f_{i_s}}{\partial z_{j_s}}.$$

We now show that $D^{[i]}_{j_1,j_2,\cdots,j_k} = 0$ for all $k \geqslant 1$ and all $1 \leqslant j_1, j_2, \cdots, j_k, i \leqslant n$ by induction on k. When $k = 1$, the $D^{[i]}_{j_1}$'s are the entries of $JF(z)^2$, and therefore are equal to 0. Suppose $D^{[i]}_{j_1,j_2,\cdots,j_k} = 0$ for all $1 \leqslant j_1, j_2, \cdots, j_k, i \leqslant n$. Then

$$\begin{aligned} D^{[i]}_{j_1,j_2,\cdots,j_{k+1}} &= \sum_{i_1,i_2,\cdots,i_{k+1}=1}^{n} \frac{\partial^{k+1} f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_{k+1}}} \prod_{s=1}^{k+1} \frac{\partial f_{i_s}(z)}{\partial z_{j_s}} \\ &= \sum_{i_{k+1}=1}^{n} \frac{\partial D^{[i]}_{j_1,j_2,\cdots,j_k}}{\partial z_{i_{k+1}}} \frac{\partial f_{i_{k+1}}}{\partial z_{j_{k+1}}} \\ &- \sum_{i_{k+1}=1}^{n} \sum_{i_1,i_2,\cdots,i_k=1}^{n} \frac{\partial^k f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \sum_{s=1}^{k} \frac{\partial^2 f_{i_s}(z)}{\partial z_{j_s} \partial z_{i_{k+1}}} \frac{\partial f_{i_{k+1}}(z)}{\partial z_{j_{k+1}}} \prod_{t=1, t\neq s}^{k} \frac{\partial f_{i_t}(z)}{\partial z_{j_t}}. \end{aligned}$$

Since

$$\sum_{i_{k+1}=1}^{n} \frac{\partial f_{i_s}(z)}{\partial z_{i_{k+1}}} \frac{\partial f_{i_{k+1}}(z)}{\partial z_{j_{k+1}}} = 0,$$

we have

$$\sum_{i_{k+1}=1}^{n} \frac{\partial^2 f_{i_s}(z)}{\partial z_{j_s} \partial z_{i_{k+1}}} \frac{\partial f_{i_{k+1}}(z)}{\partial z_{j_{k+1}}} = - \sum_{i_{k+1}=1}^{n} \frac{\partial f_{i_s}(z)}{\partial z_{i_{k+1}}} \frac{\partial^2 f_{i_{k+1}}(z)}{\partial z_{j_s} \partial z_{j_{k+1}}}.$$

Thus

$$\begin{aligned}
&D^{[i]}_{j_1,j_2,\cdots,j_{k+1}}\\
&=\sum_{s=1}^{k}\sum_{i_1,i_2,\cdots,i_k=1}^{n}\frac{\partial^k f_i(z)}{\partial z_{i_1}\partial z_{i_2}\cdots\partial z_{i_k}}\sum_{i_{k+1}=1}^{n}\frac{\partial f_{i_s}(z)}{\partial z_{i_{k+1}}}\frac{\partial^2 f_{i_{k+1}}(z)}{\partial z_{j_s}\partial z_{j_{k+1}}}\prod_{t=1,t\neq s}^{k}\frac{\partial f_{i_t}(z)}{\partial z_{j_t}}\\
&=\sum_{s=1}^{k}\sum_{i_{k+1}=1}^{n}D^{[i]}_{j_1,j_2,\cdots,j_{s-1},i_{k+1},j_{s+1},\cdots,j_{k-1},j_k}\frac{\partial^2 f_{i_{k+1}}}{\partial z_{j_s}\partial z_{j_{k+1}}}\\
&=0.
\end{aligned}$$

(2)⇒(3). We fix z and consider z' as variable and compute the Jacobian matrix with respect to z' in the left-hand side of (2). (3)⇒(1). Set $z'=0$ in (3). □

Theorem 2.2. *Let $F:\mathbb{C}^2\to\mathbb{C}^2$ be a holomorphic map. Then $JF(z)^2=0$ if and only if there exist an entire function $f:\mathbb{C}\to\mathbb{C}$ and constants a, b, c_1, and c_2 in $\mathbb{C}$, such that $F=(bf(az_1+bz_2)+c_1,-af(az_1+bz_2)+c_2)$.*

Proof. If $JF(z)=0$ for all $z\in\mathbb{C}^2$, then F is a constant map and the assertion is trivial. If $JF(z)\neq 0$, we define

$$\Omega=\{z\in\mathbb{C}^2|JF(z)\neq 0\}.$$

Note that Ω is an open subset of $\mathbb{C}^2$. For every $z\in\Omega$, define

$$L_z=\{z+JF(z)z'|z'\in\mathbb{C}^2\}.$$

Since $JF(z)\neq 0$ and $JF(z)^2=0$, L_z is a complex line in $\mathbb{C}^2$ passing through z. From Theorem 2.1(2), $F|_{L_z}$ is a constant map. We claim that all these lines are parallel to each other. Suppose to the contrary. Then there are two lines L_{z_1} and L_{z_2}, z_1,$z_2\in\Omega$, meeting at a point $z\in\mathbb{C}^2$. Thus, F is constant on $L_{z_1}\cup L_{z_2}$. Also for every $z\in\Omega$, L_z meets at least one of L_{z_1} and L_{z_2}. Therefore, F is a constant map on

$$\Omega\subset\bigcup_{z\in\Omega}L_z.$$

Since Ω is open, F is a constant map on $\mathbb{C}^2$, which contradicts $JF(z)\neq 0$. Hence there exist two complex numbers a and b, $|a|+|b|\neq 0$, such that

$$(a,b)JF(z)z'=0$$

for all $z,z'\in\mathbb{C}^2$. Therefore

$$(a,b)JF(z)=0$$

for all $z\in\mathbb{C}^2$. Let

$$\begin{aligned}
&G:\mathbb{C}^2\to\mathbb{C}^2,\\
&g_1=a'z_1+b'z_2,\\
&g_2=az_1+bz_2,
\end{aligned}$$

be a linear map such that G is invertible. Set $H=GFG^{-1}$. The Jacobian matrix

$$JH(z)=JG(z)JF(G^{-1}(z))JG(z)^{-1}$$

satisfies $JH(z)^2 = 0$ and its second row equals 0. This implies

$$H : \mathbb{C}^2 \to \mathbb{C}^2,$$
$$h_1 = h(z_2),$$
$$h_2 = \gamma,$$

where h is an entire function and γ is a constant. If we denote $\det JG(z)^{-1}$ by d, dh by f, $-db'\gamma$ by c_1 and $da'\gamma$ by c_2, then

$$F : \mathbb{C}^2 \to \mathbb{C}^2,$$
$$f_1 = bf(az_1 + bz_2) + c_1,$$
$$f_2 = -af(az_1 + bz_2) + c_2.$$

The converse is obvious and the proof is completed. □

Corollary 2.3. *If the Jacobian matrix JH of a holomorphic map $H : \mathbb{C}^2 \to \mathbb{C}^2$ is a unipotent matrix, then H is invertible.*

Proof. Let $H(z) = z+F(z)$, where F is a holomorphic map with nilpotent Jacobian matrix. By Theorem 2.2, it is easy to check that $G(z) = z - F(z - F(z))$ is the inverse of H. □

Remark 1. If F is a polynomial map, Theorem 2.2 can be derived from a result of [BCW].

Remark 2. Given a holomorphic function $f : \mathbb{C}^2 \to \mathbb{C}$, there is a holomorphic function $g : \mathbb{C}^2 \to \mathbb{C}$ such that the holomorphic map $F = (f, g)$ or $(g, f) : \mathbb{C}^2 \to \mathbb{C}^2$ has nilpotent Jacobian matrix JF if and only if f satisfies the partial differential equation

$$(*) \qquad (\frac{\partial f}{\partial z_1})^2 \frac{\partial^2 f}{\partial z_2^2} + (\frac{\partial f}{\partial z_2})^2 \frac{\partial^2 f}{\partial z_1^2} = 2 \frac{\partial f}{\partial z_1} \frac{\partial f}{\partial z_2} \frac{\partial^2 f}{\partial z_1 \partial z_2}.$$

This follows since $\det(JF) = 0$ and $\mathrm{Tr}(JF) = 0$ and we can then eliminate the function g by using the mixed second derivative of g. Thus Theorem 2.2 is equivalent to the following, which can also be proved in the same way as the combination of proofs of Theorem 2.1 and Theorem 2.2:

Theorem 2.4. *Let f be a holomorphic function on $\mathbb{C}^2$. Then f satisfies the differential equation (*) if and only if $f = h(az_1 + bz_2)$, where h is an analytic function on $\mathbb{C}$ and a and b are constants.*

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References

[BCW] H. Bass, E. H. Connell and D. Wright, *The Jacobian Conjecture: Reduction of Degree and Formal Expansion of the Inverse*, Bull. AMS **7** (1982), 287–330. MR **83k:**14028

[CSW] C. C. Cheng, T. Sakkalis and S. S. Wang, *A Case of the Jacobian Conjecture*, J. Pure Appl. Algebra **96** (1994), 15–18. MR **95i:**14018a

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