A NOTE ON HOLOMORPHIC MAPS WITH UNIPOTENT JACOBIAN MATRICES

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ABSTRACT. We prove that a holomorphic map $H : \mathbb{C}^2 \to \mathbb{C}^2$ is invertible if its Jacobian matrix $JH$ is unipotent.

1. INTRODUCTION

Let $\mathbb{C}$ be the complex number field. Given a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ with $F = (f_1, f_2, \cdots, f_n)$, where $f_i \in \mathbb{C}[z_1, z_2, \cdots, z_n]$, a simple algebraic argument tells us that

$$J_F = \det \left[ \frac{\partial f_i}{\partial z_j} \right] \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

whenever $F$ is invertible. The Jacobian Conjecture asserts that the converse is true also.

The Jacobian Conjecture is false for holomorphic maps. An easy example is

$$F : \mathbb{C}^2 \to \mathbb{C}^2, \quad f_1 = e^{z_1}, \quad f_2 = z_2 e^{-z_1}.$$

In [BCW] the Jacobian Conjecture has been reduced to the Unipotent Jacobian Conjecture, which states

The Unipotent Jacobian Conjecture. If the Jacobian matrix $JF$ of $F$ is a unipotent matrix, then $F$ is invertible.

We suspect that this conjecture could also be true for holomorphic maps. In this note, we give a proof of the Unipotent Jacobian Conjecture for holomorphic maps with $n = 2$.

2. MAIN RESULTS

In this section, we first prove a theorem concerning holomorphic maps $F : \mathbb{C}^n \to \mathbb{C}^n$ with $JF^2 = 0$. The idea of the proof is obtained from [CSW]. We then apply the theorem to the case $n = 2$ and $F(z) = H(z) - z$, where $H(z)$ is an arbitrary holomorphic map $H : \mathbb{C}^2 \to \mathbb{C}^2$ with unipotent Jacobian matrix. This yields that $H$ is invertible, i.e. Corollary 2.3.
**Theorem 2.1.** Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map. The following statements are equivalent:

1. $JF(z)^2 = 0$ for all $z \in \mathbb{C}^n$,
2. $F(z + JF(z)z') = F(z)$ for all $z, z' \in \mathbb{C}^n$, and
3. $JF(z + JF(z)z')JF(z) = 0$ for all $z, z' \in \mathbb{C}^n$.

**Proof.** $(1) \Rightarrow (2)$. Using Taylor expansion

$$f(z + y) = f(z) + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k=1}^{n} \frac{\partial^k f(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \prod_{s=1}^{k} y_{i_s},$$

one has

$$f_i(z + JF(z)z') =$$

$$f_i(z) + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j_1, j_2, \ldots, j_k=1}^{n} \frac{\partial^k f_i(z)}{\partial z_{j_1} \partial z_{j_2} \cdots \partial z_{j_k}} \prod_{s=1}^{k} \frac{\partial f_{i_s}(z)}{\partial z_{j_s}} z_{j_s} =$$

$$f_i(z) + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j_1, j_2, \ldots, j_k=1}^{n} \sum_{i=1}^{n} \frac{\partial^k f_i(z)}{\partial z_{j_1} \partial z_{j_2} \cdots \partial z_{j_k}} \prod_{s=1}^{k} \frac{\partial f_{i_s}(z)}{\partial z_{j_s}} z_{j_s}. $$

Define

$$D^{[i]}_{j_1, j_2, \ldots, j_k} = \sum_{i_1, i_2, \ldots, i_k=1}^{n} \frac{\partial^k f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \prod_{s=1}^{k} \frac{\partial f_{i_s}(z)}{\partial z_{j_s}}. $$

We now show that $D^{[i]}_{j_1, j_2, \ldots, j_k} = 0$ for all $k \geq 1$ and all $1 \leq j_1, j_2, \ldots, j_k, i \leq n$ by induction on $k$. When $k = 1$, the $D^{[i]}_{j_1}$'s are the entries of $JF(z)^2$, and therefore are equal to 0. Suppose $D^{[i]}_{j_1, j_2, \ldots, j_k} = 0$ for all $1 \leq j_1, j_2, \ldots, j_k, i \leq n$. Then

$$D^{[i]}_{j_1, j_2, \ldots, j_k+1} = \sum_{i_1, i_2, \ldots, i_{k+1}=1}^{n} \frac{\partial^{k+1} f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_{k+1}}} \prod_{s=1}^{k+1} \frac{\partial f_{i_s}(z)}{\partial z_{j_s}} =$$

$$= \sum_{i_{k+1}=1}^{n} \frac{\partial D^{[i]}_{j_1, j_2, \ldots, j_k}}{\partial z_{i_{k+1}}} \frac{\partial f_{i_{k+1}}}{\partial z_{j_{k+1}}} - \sum_{i_{k+1}=1}^{n} \sum_{i_1, i_2, \ldots, i_{k}=1}^{n} \frac{\partial^k f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \sum_{s=1}^{k} \frac{\partial^2 f_{i_s}(z)}{\partial z_{j_s} \partial z_{i_{k+1}}} \frac{\partial f_{i_{k+1}}(z)}{\partial z_{j_{k+1}}} \prod_{t=1}^{k} \frac{\partial f_{i_t}(z)}{\partial z_{j_t}}. $$

Since

$$\sum_{i_{k+1}=1}^{n} \frac{\partial f_{i_s}(z)}{\partial z_{i_{k+1}}} \frac{\partial f_{i_{k+1}}(z)}{\partial z_{j_{k+1}}} = 0,$$

we have

$$\sum_{i_{k+1}=1}^{n} \frac{\partial^2 f_{i_s}(z)}{\partial z_{j_s} \partial z_{i_{k+1}}} \frac{\partial f_{i_{k+1}}(z)}{\partial z_{j_{k+1}}} = - \sum_{i_{k+1}=1}^{n} \frac{\partial f_{i_s}(z)}{\partial z_{i_{k+1}}} \frac{\partial^2 f_{i_{k+1}}(z)}{\partial z_{j_{k+1}}}.$$
Thus
\[
D^{[i]}_{j_1,j_2,\ldots,j_{k+1}} = \sum_{s=1}^{k} \sum_{i_1,i_2,\ldots,i_k=1}^{n} \frac{\partial^k f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \sum_{k+1=1}^{n} \frac{\partial f_{i_k}(z)}{\partial z_{j_{k+1}}} \frac{\partial^2 f_{i_{k+1}}(z)}{\partial z_{j_{k+1}} \partial z_{j_{k+1}}} \prod_{t=1, t \not= s}^{k} \frac{\partial f_{i_t}(z)}{\partial z_{j_t}}
\]
= \sum_{s=1}^{k} D_{j_1,j_2,\ldots,j_{k-1},j_{k+1}}^{[i]} \frac{\partial^2 f_{i_{k+1}}}{\partial z_{j_{k+1}} \partial z_{j_{k+1}}}
= 0.
\]

(2) $\Rightarrow$ (3). We fix $z$ and consider $z'$ as variable and compute the Jacobian matrix with respect to $z'$ in the left-hand side of (2). (3) $\Rightarrow$ (1). Set $z' = 0$ in (3). 

**Theorem 2.2.** Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic map. Then $JF(z)^2 = 0$ if and only if there exist an entire function $f : \mathbb{C} \to \mathbb{C}$ and constants $a, b, c_1,$ and $c_2$ in $\mathbb{C}$, such that $F = (bf(a z_1 + b z_2) + c_1, -af(a z_1 + b z_2) + c_2)$.

**Proof.** If $JF(z) = 0$ for all $z \in \mathbb{C}^2$, then $F$ is a constant map and the assertion is trivial. If $JF(z) \neq 0$, we define
\[
\Omega = \{ z \in \mathbb{C}^2 | JF(z) \neq 0 \}.
\]

Note that $\Omega$ is an open subset of $\mathbb{C}^2$. For every $z \in \Omega$, define
\[
L_z = \{ z + JF(z) z' | z' \in \mathbb{C}^2 \}.
\]

Since $JF(z) \neq 0$ and $JF(z)^2 = 0$, $L_z$ is a complex line in $\mathbb{C}^2$ passing through $z$. From Theorem 2.1(2), $F|_{L_z}$ is a constant map. We claim that all these lines are parallel to each other. Suppose to the contrary. Then there are two lines $L_{z_1}$ and $L_{z_2}$, $z_1, z_2 \in \Omega$, meeting at a point $z \in \mathbb{C}^2$. Thus, $F$ is constant on $L_{z_1} \cup L_{z_2}$. Also for every $z \in \Omega$, $L_z$ meets at least one of $L_{z_1}$ and $L_{z_2}$. Therefore, $F$ is a constant map on
\[
\Omega \subset \bigcup_{z \in \Omega} L_z.
\]

Since $\Omega$ is open, $F$ is a constant map on $\mathbb{C}^2$, which contradicts $JF(z) \neq 0$. Hence there exist two complex numbers $a$ and $b$, $|a| + |b| \neq 0$, such that
\[
(a,b) JF(z) z' = 0
\]
for all $z, z' \in \mathbb{C}^2$. Therefore
\[
(a,b) JF(z) = 0
\]
for all $z \in \mathbb{C}^2$. Let
\[
G : \mathbb{C}^2 \to \mathbb{C}^2,
\]
\[
g_1 = a' z_1 + b' z_2,
\]
\[
g_2 = a z_1 + b z_2,
\]
be a linear map such that $G$ is invertible. Set $H = G F G^{-1}$. The Jacobian matrix
\[
JH(z) = JG(z) JF(G^{-1}(z)) JG(z)^{-1}
\]
satisfies $JH(z)^2 = 0$ and its second row equals 0. This implies
\[
H : \mathbb{C}^2 \rightarrow \mathbb{C}^2, \\
h_1 = h(z_2), \\
h_2 = \gamma,
\]
where $h$ is an entire function and $\gamma$ is a constant. If we denote $\det JH(z)^{-1}$ by $d$, $dh$ by $f$, $-db^T\gamma$ by $c_1$ and $da^T\gamma$ by $c_2$, then
\[
F : \mathbb{C}^2 \rightarrow \mathbb{C}^2, \\
f_1 = bf(a z_1 + b z_2) + c_1, \\
f_2 = -af(a z_1 + b z_2) + c_2.
\]
The converse is obvious and the proof is completed. \hfill \Box

**Corollary 2.3.** If the Jacobian matrix $JH$ of a holomorphic map $H : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a unipotent matrix, then $H$ is invertible.

**Proof.** Let $H(z) = z + F(z)$, where $F$ is a holomorphic map with nilpotent Jacobian matrix. By Theorem 2.2, it is easy to check that $G(z) = z - F(z - F(z))$ is the inverse of $H$. \hfill \Box

**Remark 1.** If $F$ is a polynomial map, Theorem 2.2 can be derived from a result of [BCW].

**Remark 2.** Given a holomorphic function $f : \mathbb{C}^2 \rightarrow \mathbb{C}$, there is a holomorphic function $g : \mathbb{C}^2 \rightarrow \mathbb{C}$ such that the holomorphic map $F = (f, g)$ or $(g, f) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ has nilpotent Jacobian matrix $JF$ if and only if $f$ satisfies the partial differential equation
\[
(*) \quad \left( \frac{\partial f}{\partial z_1} \right)^2 \frac{\partial^2 f}{\partial z_2^2} + \left( \frac{\partial f}{\partial z_2} \right)^2 \frac{\partial^2 f}{\partial z_1^2} = 2 \frac{\partial f}{\partial z_1} \frac{\partial f}{\partial z_2} \frac{\partial^2 f}{\partial z_1 \partial z_2}.
\]
This follows since $\det(JF) = 0$ and $\text{Tr}(JF) = 0$ and we can then eliminate the function $g$ by using the mixed second derivative of $g$. Thus Theorem 2.2 is equivalent to the following, which can also be proved in the same way as the combination of proofs of Theorem 2.1 and Theorem 2.2:

**Theorem 2.4.** Let $f$ be a holomorphic function on $\mathbb{C}^2$. Then $f$ satisfies the differential equation $(*)$ if and only if $f = h(a z_1 + b z_2)$, where $h$ is an analytic function on $\mathbb{C}$ and $a$ and $b$ are constants.

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**References**


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