

CLASSIFICATION OF ACTIONS
OF DISCRETE AMENABLE GROUPS
ON STRONGLY AMENABLE SUBFACTORS OF TYPE III_λ

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ABSTRACT. Using the continuous decomposition, we classify strongly free actions of discrete amenable groups on strongly amenable subfactors of type III_λ , $0 < \lambda < 1$. Winsløw's fundamental homomorphism is a complete invariant. This removes the extra assumptions in the classification theorems of Loi and Winsløw and gives a complete classification up to cocycle conjugacy.

1. INTRODUCTION

In the theory of operator algebras, the study of automorphisms is one of the most important topics. Especially since Connes's work [3], much progress has been made on the classification of the actions of discrete amenable groups on injective factors.

In the subfactor theory, various studies of automorphisms have been done. In [13], Popa has introduced the notion of proper outeriness of automorphisms and proved that the properly outer actions of discrete amenable groups on strongly amenable subfactors of type II_1 are classified by the Loi invariant (see [9]). (In [1], Choda and Kosaki have introduced the same property independently and they call it strong outeriness.)

In the case of subfactors of type III_λ ($0 < \lambda < 1$), partial results on classification of group actions have been obtained by Winsløw and Loi ([10], [17], [18]). In [17] and [18], Winsløw has introduced the strong freeness and the fundamental homomorphism for actions. He has classified the strongly free actions of discrete amenable groups on subfactors of type III_λ for groups having the character lifting property. His fundamental homomorphism is a complete invariant. In [10], Loi gave a classification theorem when G is finite. Their idea of the proof is that they reduce the classification problem to the type II_∞ case using the discrete decomposition and apply Popa's classification result. The most difficult points of their proofs are to reduce the problem to the type II_∞ case. Because of this difficulty, they made extra assumptions such as the character lifting property or finiteness for groups. But it seems difficult to generalize their method to the arbitrary discrete amenable group case.

Our idea of a proof is using the continuous decomposition instead of the discrete decomposition based on the method in [14] and [15]. But in this case, we treat

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only factors of type III_λ , $0 < \lambda < 1$, with the trivial characteristic invariants, so the proof is less complicated than those in [14] and [15]. By using the continuous decomposition, we can more easily reduce the classification problem to the type II_∞ case than using the discrete decomposition and this method is valid for arbitrary discrete amenable groups.

After the submission of the first version of this paper, we have obtained a classification result also for group actions on subfactors of type III_0 in [11] in a more complicated method. But the simple argument in this paper is a key to a generalization in [11], and the author hopes that the short proof in this paper is of independent interest.

2. PRELIMINARIES

In this section, we recall several results about group actions on subfactors, and fix notations. The facts stated in this section are found in [1], [9], [10], [13], [17], [18], [19].

Let $N \subset M$ be an inclusion of factors with finite index and $N \subset M \subset M_1 \subset M_2 \subset \cdots$ the Jones tower. (Throughout this paper, we always assume that conditional expectations are minimal in the sense of [8] and inclusions of factors of type II are extremal.) For $\alpha \in \text{Aut}(M, N)$, we extend α to M_k such that $\alpha(e_k) = e_k$ inductively, where e_k denotes the Jones projection for $M_{k-1} \subset M_k$.

First we recall the Loi invariant and the strong outerness of group actions.

Definition 2.1 ([9, Section 5]). With above notations, Put

$$\Phi(\alpha) := \{\alpha|_{M' \cap M_k}\}_k.$$

We call Φ the Loi invariant for α .

Definition 2.2 ([1, Definition 1], [13, Definition 1.5.1]). An automorphism $\alpha \in \text{Aut}(M, N)$ is said to be properly outer or strongly outer if we have no non-zero $a \in \bigcup_k M_k$ satisfying $\alpha(x)a = ax$ for all $x \in M$. The action α of G on $N \subset M$ is said to be strongly outer if α_g is strongly outer except for $g = e$.

The most important result on classification of actions of groups on subfactors has been obtained by Popa.

Theorem 2.3 ([13, Theorem 3.1]). *Let $N \subset M$ be a strongly amenable inclusion of factors of type II_1 and G a countable discrete amenable group.*

If α and β are strongly outer actions of G on $N \subset M$, then α and β are cocycle conjugate if and only if $\Phi(\alpha) = \Phi(\beta)$.

For type II_∞ inclusions, we have the following result due to Popa and Winsløw.

Theorem 2.4 ([13, Theorem 2.1], [17, Theorem 4.3]). *Let $N \subset M$ a strongly amenable inclusion of factors of type II_∞ . If α and β are actions of countable discrete amenable group G on $N \subset M$, then α and β are cocycle conjugate if and only if $\Phi(\alpha) = \Phi(\beta)$ and $\text{mod}(\alpha) = \text{mod}(\beta)$.*

Let $N \subset M$ be an arbitrary inclusion of factors with the common flow of weights. Fix a normal state of N and take a crossed product of $N \subset M$ by the modular automorphism. Put $\tilde{N} \subset \tilde{M} := N \rtimes_{\sigma^\phi} \mathbf{R} \subset M \rtimes_{\sigma^{\phi \circ E}} \mathbf{R}$, where E is the minimal conditional expectation from M onto N . Let $\tilde{\alpha}$ be the canonical extension of α to $\tilde{N} \subset \tilde{M}$ ([5], [7]), i.e.,

$$\tilde{\alpha}(x) := \alpha(x), \quad x \in M,$$

$$\tilde{\alpha}(\lambda(t)) := (D\phi \circ \alpha^{-1} : D\phi)_t \lambda(t),$$

where $\lambda(t)$ is the usual implementing unitary. The notions of strong freeness for automorphisms and the fundamental homomorphism are introduced by Winsløw in [17], [18].

Definition 2.5 ([17, Definition 3.2], [18, Definition 4.2]). An automorphism $\alpha \in \text{Aut}(M, N)$ is said to be strongly free if we have no non-zero $a \in \bigcup_k \tilde{M}_k$ satisfying $\tilde{\alpha}(x)a = ax$ for all $x \in \tilde{M}$. For an action α of G on $N \subset M$ is said to be strongly free if α_g is strongly free except for $g = e$.

According to [17] and [18], we set

$$\Upsilon(\alpha) := \{\tilde{\alpha}|_{\tilde{M}' \cap \tilde{M}_k}\}_k$$

and we call this the fundamental homomorphism.

3. CLASSIFICATION OF ACTIONS

Throughout this section, we assume that inclusions of factors of type III_λ are strongly amenable in the sense of Popa. (See [12] and [13].)

The following theorem is the main result of this paper.

Theorem 3.1. *Let $N \subset M$ be a strongly amenable inclusion of factors of type III_λ , $0 < \lambda < 1$, with the common flow of weights. Let G be a countable discrete amenable group, and α and β strongly free actions of G on $N \subset M$. Then α and β are cocycle conjugate if and only if $\Upsilon(\alpha) = \Upsilon(\beta)$.*

Our idea of proof is that we lift actions to inclusions of type II_∞ von Neumann algebras using continuous decomposition and apply Popa’s result.

The “only if” part is obvious, so we only prove the “if” part. Let (X, F_t) be the flow of weights of M . Since M is of type III_λ , (X, F_t) is of the form $([0, -\log \lambda], \text{translation})$. And we have an isomorphism

$$(\tilde{N} \subset \tilde{M} \subset \tilde{M}_1 \subset \dots) \cong (L^\infty(X) \otimes Q \subset L^\infty(X) \otimes P \subset L^\infty(X) \otimes P_1 \subset \dots),$$

where $Q \subset P \subset P_1 \subset \dots$ is a tower of factors of type II_∞ and $Q \subset P$ is strongly amenable by assumption.

Let θ_t be the usual trace scaling action of \mathbf{R} on $\tilde{N} \subset \tilde{M}$. Since $\tilde{\alpha}_g, g \in G$, commutes with θ_t , we can consider an action of $G \times \mathbf{R}$ by setting $(g, t) \rightarrow \tilde{\alpha}_g \theta_t$. If no confusion arises, we also denote this action of $G \times \mathbf{R}$ by $\tilde{\alpha}$. If we prove that two actions of $G \times \mathbf{R}$, $\tilde{\alpha}$ and $\tilde{\beta}$ are cocycle conjugate, the proof of [15, Proposition 1.1] also works in this case and we can deduce that the canonical extensions of $\tilde{\alpha}$ and $\tilde{\beta}$ on $N \rtimes_{\sigma^\phi} \mathbf{R} \rtimes_{\theta} \mathbf{R} \subset M \rtimes_{\sigma^{\phi \circ E}} \mathbf{R} \rtimes_{\theta} \mathbf{R} \cong N \otimes B(L^2(\mathbf{R})) \subset M \otimes B(L^2(\mathbf{R}))$ are also cocycle conjugate and get the conclusion that α and β are cocycle conjugate, since N and M are properly infinite.

So our purpose is the classification of actions of $G \times \mathbf{R}$ on $\tilde{N} \subset \tilde{M}$. Note that for $g \in G$, the equality $\text{tr}_{\tilde{M}} \tilde{\alpha}_g = \text{tr}_{\tilde{M}}$ holds.

Put $H := G \times \mathbf{R}$ and we consider the action $\tilde{\alpha}$ of the groupoid $H \rtimes X$ on $Q \subset P$ by the equality

$$\tilde{\alpha}_g(a) := \int_X^\oplus \tilde{\alpha}_{(g, g^{-1}x)}(a(g^{-1}x)) dx \quad (\text{see [15, Proposition 1.2]}).$$

Put $x_0 := 0 \in X$ and $H_0 := \{(g, x_0) \in H \rtimes X \mid gx_0 = x_0\}$. Then H_0 is a discrete amenable group acting on $Q \subset P$. For $x \in X = [0, -\log \lambda)$, we define

$h(x) : X \rightarrow X$ by $h(x)y := y + x$, where sum is taken modulo $-\log \lambda$. Especially $h(x)x_0 = x$.

Here we have the following proposition.

Proposition 3.2. *If the action of G is strongly free, then the action of H_0 is strongly outer.*

Proof. Assume that the action of H_0 is not strongly outer. Then there exists $g \in H \setminus \{e\}$ and a non-zero $a \in P_k$ for some k such that for every $b \in Q$, we have $\tilde{\alpha}_g(b)a = ab$. Since $\tilde{\alpha}_g$ is not strongly outer and $\text{mod } \tilde{\alpha}_g = 1$, we know that g is in G .

Set

$$\tilde{a} := \int_X^\oplus \tilde{\alpha}_{(h(x), x_0)}(a) dx.$$

Then an easy computation shows that the equality $\tilde{\alpha}_g(b)\tilde{a} = \tilde{a}b$ holds for every $b \in Q$ and this means that action α is not strongly free. \square

Proof of Theorem 3.1. Let α and β be strongly free actions of G on $N \subset M$ such that $\Upsilon(\alpha) = \Upsilon(\beta)$. Then we get two actions $\tilde{\alpha}$ and $\tilde{\beta}$ of the same groupoid $H \times X$. So we get two actions $\tilde{\alpha}$ and $\tilde{\beta}$ of a discrete amenable group H_0 .

By Proposition 3.2, both actions are strongly outer and by assumption both actions have the same Loi invariant and the same module. So there exists an automorphism $\theta \in \text{Aut}(P, Q)$ and $u_g \in Z_{\tilde{\beta}}(H_0, U(Q))$ such that

$$\theta \tilde{\alpha}_g \theta^{-1} = \text{Ad} u_g \tilde{\beta}_g, \quad g \in H_0.$$

Set

$$\theta_x := \tilde{\beta}_{(h(x), x_0)} \theta \tilde{\alpha}_{(h(x), x_0)}^{-1}, \quad x \in X, \quad \text{and}$$

$$u_{(g,x)} := \tilde{\beta}_{(h(gx)^{-1}gh(x), x_0)}(u_{h(gx)^{-1}gh(x)}), \quad (g,x) \in H \times X.$$

Then an easy computation shows that $u_{(g,x)} \in Z_{\tilde{\beta}}(H \times X, U(Q))$ and the equality

$$\theta_{gx} \tilde{\alpha}_{g,x} \theta_x^{-1} = \text{Ad} u_{(g,x)} \tilde{\beta}_{(g,x)}$$

holds.

Put $\tilde{\theta} := \int_X^\oplus \theta_x dx$ and $\tilde{u}_g := \int_X^\oplus u_{(g,x)} dx$. Then we get

$$\tilde{\theta} \tilde{\alpha}_g \tilde{\theta}^{-1} = \text{Ad } \tilde{u}_g \tilde{\beta}_g$$

and we get the conclusion. \square

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