ON THE RANGE AND THE KERNEL
OF THE OPERATOR $X \mapsto AXB - X$

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Abstract. Let $L(H)$ denote the algebra of (bounded linear) operators on the separable complex Hilbert space $H$, and let $(\mathcal{J}; \| . \|_3)$ denote a norm ideal in $L(H)$. For $A, B \in L(H)$, let the derivation $\delta_{A,B}: L(H) \to L(H)$ be defined by $\delta_{A,B}(X) = AX - XB$, and let $\Delta_{A,B}: L(H) \to L(H)$ be defined by $\Delta_{A,B}(X) = AXB - X$. The main result of this paper is to show that if $A, B$ are contractions, then for every operator $T \in \mathcal{J}$ such that $ATB = T$, then $\|AXB - X + T\|_3 \geq \|T\|_3$ for all $X \in \mathcal{J}$.

1. Introduction

Recently Du Hong Ke ([2]) proved that if $A, B$ are contractions, then for every operator $S$ such that $ASB = S$, $A^*SB^* = S$, then $\|AXB - X + S\| \geq \|S\|$ for all operators $X \in L(H)$.

Duggal ([4]) proved that if $A, B$ are contractions, then $S \in C^2$ and $ASB = S$ imply $\|AXB - X + S\|^2 = \|AXB - X\|^2 + \|S\|^2$, for all $X \in L(H)$, where $C^2$ denotes the (Hilbert) space of Hilbert-Schmidt operators on $H$. In this note, we shall prove the following theorem.

Theorem 1. If $A, B$ are contractions and $(\mathcal{J}; \| . \|_3)$ is a norm ideal in $L(H)$ and $T \in \mathcal{J}$ is such that $ATB = T$, then $\|AXB - X + T\|_3 \geq \|T\|_3$ for all operators $X \in L(H)$.

2. Some preliminaries

Definition 2.1 ([5]). A proper two-sided ideal $\mathcal{J}$ in $L(H)$ is said to be a norm ideal if there is a norm on $\mathcal{J}$ satisfying the following properties:

i) $(\mathcal{J}; \| . \|_3)$ is a Banach space;

ii) $\|AXB\|_3 \leq \|A\| \|X\| \|B\|$ for all $A, B \in L(H)$ and for all $X \in \mathcal{J}$;

iii) $\|X\|_3 = \|X\|$ for $X$ a rank one operator.

Remark. If $(\mathcal{J}; \| . \|_3)$ is a norm ideal, then the norm $\| . \|_3$ is unitarily invariant, in the sense that $\|UAV\|_3 = \|A\|_3$ for all $A$ in $\mathcal{J}$ and unitary $U, V$ in $L(H)$.
Example. Each proper ideal of $L(H)$ is contained in the ideal of compact operators. For any compact operator $A$, denote by $s_1(A) \geq s_2(A) \geq \cdots$ the singular values of $A$, i.e., the eigenvalues of $(A^*A)^{\frac{1}{2}}$.

Two special families of unitarily invariant norms satisfying conditions i), ii), and iii) of Definition 2.1 are the Schatten $p$-norms defined as

$$\|A\|_p = \left( \sum_{j=1}^{\infty} s_j(A)^p \right)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty,$$

where by convention

$$\|A\|_\infty = \max s_j(A) = s_1(A) = \|A\|,$$

and the Ky Fan norms defined as $\|A\|_k = \sum_{j=1}^{k} s_j(A)$, $k \geq 1$.

Theorem 2.2 ([3]). a) Let $A, B^* \in L(H)$ be contractions with $C_0$ completely non-unitary parts. If $\Delta_{A,B}(X) = 0$, then $\text{ran} X$ reduces $A$, $\ker X$ reduces $B^*$, and $A|_{\text{ran} X}$ and $B^*|_{\ker X}$ are unitarily equivalent operators.

b) Let $A$ and $B$ be contractions such that $\Delta_{A,B}(X) = 0$ and $X$ a compact operator. Then the conclusions of part a) hold.

3. Main results

Theorem 3.1. If $U$ is a unitary operator, $(\mathfrak{J}; \| \cdot \|_3)$ is a norm ideal in $L(H)$ and $T \in \mathfrak{J}$ is such that $TU = UT$, then

$$\|UX - XV + T\|_3 \geq \|T\|_3 \quad \text{for all operators } X \in \mathfrak{J}.$$

Proof. The proof is similar to J. Anderson’s ([1, Theorem 1.4, p. 136]).

Corollary 3.2. If $U, V$ are unitary operators, $(\mathfrak{J}; \| \cdot \|_3)$ is a norm ideal in $L(H)$ and $T \in \mathfrak{J}$ is such that $UT = TV$, then

$$\|U_{UV}X + T\|_3 \geq \|T\|_3 \quad \text{for all operators } X \in \mathfrak{J}.$$

Proof. On $H \otimes H$, let

$$W = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, \quad \overline{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \overline{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$ 

Then $\overline{T} \in \mathfrak{J} \otimes \mathfrak{J}$ and $W \overline{X} - \overline{X}W + \overline{T} = \begin{pmatrix} 0 & UX - XV + T \\ 0 & 0 \end{pmatrix}$. Since $UT = TV$, it follows that $W\overline{T} = \overline{T}W$. By Theorem 3.1 we have $\|W\overline{X} - \overline{X}W + \overline{T}\|_{\mathfrak{J} \otimes \mathfrak{J}} \geq \|\overline{T}\|_3$ and so $\|WX - XV + T\|_3 \geq \|T\|_3$.

Corollary 3.3. If $U$ is a unitary operator, $(\mathfrak{J}; \| \cdot \|_3)$ is a norm ideal in $L(H)$, and $T \in \mathfrak{J}$ is such that $UTU = T$. Then

$$\|UXU - X + T\|_3 \geq \|T\|_3 \quad \text{for all operators } X \in \mathfrak{J}.$$

Proof. Let $T$ be such that $UTU = T$. Then $TU = U^*T$ and so

$$\|UXU - X + T\|_3 = \|U(XU - U^*X + U^*T)\|_3$$
$$= \|XU - U^*X + U^*T\|_3 \geq \|U^*T\|_3 \quad \text{(Corollary 3.2)}$$
$$\geq \|T\|_3.$$
Corollary 3.4. If $U, V$ are unitary operators, $(\mathfrak{I}; \| \cdot \|)$ is a norm ideal in $L(H)$, and $T \in \mathfrak{I}$ is such that $UTV = T$, then
\[
\|UXV - X + T\| \geq \|T\| \quad \text{for all operators } X \in \mathfrak{I}.
\]
Proof. On $H \oplus H$, let
\[
W = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, \quad \widetilde{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.
\]
Then $\widetilde{T} \in \mathfrak{I} \oplus \mathfrak{I}$ and $W\widetilde{X}W - \widetilde{X} + \widetilde{T} = \begin{pmatrix} U & X + T \end{pmatrix}$. Since $UTV = T$, it follows that $W\widetilde{X}W = \widetilde{T}$. By Corollary 3.3 we have $\|W\widetilde{X}W - \widetilde{X} + \widetilde{T}\| \geq \|\widetilde{T}\| \quad \text{and so} \quad \|UXV - X + T\| \geq \|T\|$.

Proof of Theorem 1. Let $T$ be an operator such that $\Delta(T) = 0$ and $T \in \mathfrak{I}$. By Theorem 2.2b $\operatorname{ran} X$ reduces $A$, $\ker X$ reduces $B^*$ and $A|\operatorname{ran} X$ and $B^*|\ker X$ are unitary operators. Put $H_1 = \operatorname{ran} T \oplus \ker \bigcap T$, $H_2 = \ker(T) \oplus \ker(T)$ so that we get decompositions of operators respectively:
\[
W = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B^* = \begin{pmatrix} B_1^* & 0 \\ 0 & B_2^* \end{pmatrix}.
\]
For linear operators $X, T$ from $H_2$ into $H_1$ we have:
\[
X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, \quad T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}.
\]
So
\[
\|\Delta_A,B(X) + T\| \geq \left\| \begin{pmatrix} T_1 + \Delta_{A_1,B_1}(X_1) & * \\ * & * \end{pmatrix} \right\| \geq \left\| \begin{pmatrix} T_1 + \Delta_{A_1,B_1}(X_1) & 0 \\ 0 & 0 \end{pmatrix} \right\| \geq \left\| \begin{pmatrix} T_1 + \Delta_{A_1,B_1}(X_1) \end{pmatrix} \right\|.
\]
Since $A_1, B_1$ are unitary operators, then Corollary 3.4 implies that
\[
\|\Delta_A,B(X) + T\| \geq \|T_1\| \geq \|T\|.
\]

References

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