

HÖLDER ABSOLUTE VALUES ARE EQUIVALENT TO CLASSICAL ONES

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ABSTRACT. We study generalized absolute values on a field or a commutative ring with unit element satisfying an approximate triangle inequality and an approximate multiplicative property. We prove that they are always Hölder equivalent to an absolute value. This implies geometric rigidity results for Lipschitz and Hölder deformations of metric rings.

INTRODUCTION

We define in this article *Hölder absolute values*, which are a generalization of the classical notion of absolute value on a field. Hölder absolute values satisfy an approximate triangle inequality and an approximate multiplicative property.

More precisely, let $C_1 \geq 1$ and $C_2 \geq 1$. A (C_1, C_2) -Hölder absolute value on a ring R is a mapping $\|\cdot\| : R \rightarrow \mathbf{R}_+$ satisfying:

(HAV1) For $x \in R$, $\|x\| = 0 \Leftrightarrow x = 0$.

(HAV2) For $x, y \in R$, $\|x + y\| \leq C_2(\|x\| + \|y\|)$.

(HAV3) For $x, y \in R$, $C_1^{-1}\|x\| \|y\| \leq \|xy\| \leq C_1\|x\| \|y\|$.

Note that a ring which admits a Hölder absolute value is necessarily a ring without zero divisors. We note that a classical absolute value is a $(1, 1)$ -Hölder absolute value. Our theorem shows that any Hölder absolute value on a division ring or a commutative ring with unit element is Hölder equivalent to a classical one. We recall that two metric structures (X, d) and (X, δ) are (C_0, α) -Hölder equivalent, $C_0 \geq 1$, $\alpha > 0$, if for $x, y \in X$,

$$C_0^{-1}d(x, y)^{\alpha'} \leq \delta(x, y) \leq C_0d(x, y)^{\alpha'},$$

where $\alpha' = \alpha$ or $\alpha' = \alpha^{-1}$.

Theorem (Hölder Rigidity). *Let $\|\cdot\| : R \rightarrow \mathbf{R}_+$ be a (C_1, C_2) -Hölder absolute value on a division ring or a commutative ring R with unit element. There exists an absolute value on R , $|\cdot| : R \rightarrow \mathbf{R}_+$, which is (C_1^α, α) -Hölder equivalent to $\|\cdot\|$ with $\alpha = \log_2(2C_2)$, i.e. for $x \in R$,*

$$C_1^{-\alpha}|x|^\alpha \leq \|x\| \leq C_1^\alpha|x|^\alpha.$$

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Moreover $|\cdot|$ can be defined by

$$|x| = \lim_{n \rightarrow +\infty} \|x^n\|^{\frac{1}{n\alpha}}.$$

N. Bourbaki has studied $(1, C_2)$ -Hölder absolute values ([Bou]). Our Theorem generalizes Bourbaki's result (let $C_1 = 1$):

Corollary (Bourbaki). *If $\|\cdot\|$ is a $(1, C_2)$ -Hölder absolute value over a field, then there exists an absolute value $|\cdot|$ and an exponent α such that $\|x\| = |x|^\alpha$.*

Over \mathbf{Q} this result can be seen as a quantitative generalization of Ostrowski's Theorem for absolute values ($C_1 = 1$ and $C_2 = 1$ give Ostrowski's Theorem).

Corollary. *If $\|\cdot\|$ is a (C_1, C_2) -Hölder absolute value over \mathbf{Q} , normalized so that $\|1\| = 1$, then $\|\cdot\|$ is (C_1^α, α) -Hölder equivalent to a p -adic absolute value $|\cdot|_p$ or to $|\cdot|_\infty$, with $\alpha = \log_2(2C_2)$.*

Our main motivation is the study of geometric deformations of metric rings. The notion of Hölder absolute value appears naturally considering Hölder deformations of the metric space defined by a ring with an absolute value such that all translations are uniformly Lipschitz in the new metric. This explains the terminology used. In this geometric setting our Theorem can be formulated as a rigidity result for Lipschitz and Hölder deformations of metric rings.

The proof of the main theorem consists of two parts. First we prove that the limit defining $|\cdot|$ always exists. Second we show that $|\cdot|$ is an absolute value. The triangle inequality is the tricky part.

Proof of the theorem. Let R be a division ring or a commutative ring with unit element endowed with a (C_1, C_2) -Hölder absolute value, denoted by $\|\cdot\|$, with $C_1 \geq 1$ and $C_2 \geq 1$.

Taking $\alpha = \frac{1}{\log_2(2C_2)}$, we have for any $x, y \in R$,

$$\begin{aligned} \|x + y\|^\alpha &\leq (2C_2)^\alpha (\max(\|x\|, \|y\|))^\alpha \\ &\leq 2 \max(\|x\|^\alpha, \|y\|^\alpha). \end{aligned}$$

□

Define the mapping $|||\cdot||| : R \rightarrow \mathbf{R}_+$ by $|||x||| = \|x\|^\alpha$. Obviously, we have:

Lemma 1. *The map $|||\cdot|||$ is a $(C_1^\alpha, 2)$ -Hölder absolute value on R .*

Consider $x \in R, x \neq 0$, and define the sequence of positive numbers $(a_n)_{n \geq 0}$ by $a_n = C_1^\alpha |||x^n|||$. The following Lemmata are straightforward:

Lemma 2. *The sequence $(a_n)_{n \geq 0}$ is sub-multiplicative, i.e. for $n, m \geq 1, a_{nm} \leq a_n a_m$.*

Lemma 3. *We have $C_1^{-\alpha} |||x||| \leq |||x^n|||^{1/n} \leq C_1^\alpha |||x|||$.*

It is a classical exercise that Lemma 2 implies $\liminf a_n^{1/n} = \limsup a_n^{1/n}$. Moreover Lemma 3 implies that the sequence $(a_n^{1/n})_{n \geq 0}$ is bounded above and bounded away from 0. So this sequence has a limit

$$|x| = \lim_{n \rightarrow +\infty} a_n^{1/n} = \lim_{n \rightarrow +\infty} |||x^n|||^{1/n}.$$

We also define $|0| = 0$. Using Lemma 3 we prove:

Proposition 1. For $x \in R$, $C_1^{-\alpha} |||x||| \leq |x| \leq C_1^\alpha |||x|||$.

So we only need to prove that $|\cdot|$ is indeed an absolute value. The only delicate part is proving the triangle inequality.

Lemma 4. (AV1) For $x \in R$, $|x| = 0 \Leftrightarrow x = 0$.

(AV3) For $x, y \in R$, $|xy| = |x||y|$.

Proof. (AV1) follows from Proposition 1.

For $x, y \in R$, we have

$$C_1^{-\alpha} |||x^n||| |||y^n||| \leq |||x^n y^n||| \leq C_1^\alpha |||x^n||| |||y^n|||.$$

Taking n roots and passing to the limit $n \rightarrow +\infty$ we get

$$|x| |y| \leq |xy| \leq |x| |y|,$$

so we have the equality and (AV3). □

So $|\cdot|$ has the expected multiplicative behavior. At first we only have an approximate version of the triangle inequality (AV2).

Lemma 5. For $x, y \in R$ we have

$$|x + y| \leq C_1^{2\alpha} (|x| + |y|) \leq 2C_1^{2\alpha} \max(|x|, |y|).$$

Proof. We use Proposition 1 twice and the triangle inequality for $|||\cdot|||$:

$$|x + y| \leq C_1^\alpha |||x + y||| \leq C_1^\alpha (|||x||| + |||y|||) \leq C_1^{2\alpha} (|x| + |y|).$$

□

Because of the previous approximate triangle inequality it is much more efficient to cut sums dyadically. The next lemma and its proof recalls the algorithmic problem of sorting a finite set of integers (the complexity of the naïve algorithm is $\mathcal{O}(n^2)$ but the dyadic optimal algorithm has complexity $\mathcal{O}(n \log n)$); see for example [We], p. 137).

Lemma 6 (Dyadic trick). Let $|\cdot|' : R \rightarrow \mathbf{R}_+$ such that for $x, y \in R$ we have

$$|x + y|' \leq M \max(|x|', |y|'),$$

for some positive constant M . Then for $x_1, x_2, \dots, x_n \in R$, we have

$$\left| \sum_{i=1}^n x_i \right|' \leq M^{\lceil \log_2 n \rceil + 1} \max_{1 \leq i \leq n} (|x_i|'),$$

where $[a]$ denotes the integer part of a .

Proof. Let $m = \lceil \log_2 n \rceil + 1$ and complete the sequence $(x_i)_{1 \leq i \leq n}$ into $(x_i)_{1 \leq i \leq 2^m}$ adjoining 0 elements. Then using the approximate triangle inequality:

$$\begin{aligned} \left| \sum_{i=1}^{2^m} x_i \right|' &\leq M \max \left(\left| \sum_{i=1}^{2^{m-1}} x_i \right|', \left| \sum_{i=2^{m-1}+1}^{2^m} x_i \right|' \right) \\ &\leq M^2 \max \left(\left| \sum_{i=1}^{2^{m-2}} x_i \right|', \left| \sum_{i=2^{m-2}+1}^{2^{m-1}} x_i \right|', \left| \sum_{i=2^{m-1}+1}^{3 \cdot 2^{m-2}} x_i \right|', \left| \sum_{i=3 \cdot 2^{m-2}+1}^{2^m} x_i \right|' \right) \\ &\leq \dots \\ &\leq M^m \max_{1 \leq i \leq 2^m} |x_i|'. \quad \square \end{aligned}$$

Lemma 7. Let $\bar{\mathbf{Z}}$ be the image of \mathbf{Z} in R .

For $n \in \mathbf{N}$,

$$\|\bar{n}\| \leq 2n\|1\|.$$

Proof. We use the dyadic trick (Lemma 6) with $|\cdot|' = \|\cdot\|$ and $M = 2$. Take $m = \lfloor \log_2 n \rfloor + 1$, $n \leq 2^m \leq 2n$, and $x_i = 1$ for $1 \leq i \leq n$. We have

$$\|n\| = \left\| \sum_{i=1}^n x_i \right\| \leq 2^m \|1\| \leq 2n\|1\|.$$

□

Lemma 8. Let $\bar{\mathbf{Z}}$ be the image of \mathbf{Z} in R .

For $n \in \mathbf{N}$,

$$|\bar{n}| \leq n.$$

Proof. Using Lemma 7,

$$\|\bar{n}^k\| = \|\bar{n}^k\| \leq 2n^k\|1\|,$$

and $\|\bar{n}^k\|^{1/k} \leq 2^{1/k}n\|1\|^{1/k}$. Taking $k \rightarrow +\infty$ we have $|\bar{n}| \leq n$. □

Proposition 2. For $x, y \in R$, $|x + y| \leq |x| + |y|$.

Proof. We consider first the case when R is a commutative ring with unit element. Let $x, y \in R$ and $n \geq 1$. Let $m = \lfloor \log_2 n \rfloor + 1$. Using Lemma 5 and Lemma 7 we have

$$\begin{aligned} |(x + y)^n| &= \left| \sum_{i=0}^n \binom{n}{i} x^i y^{n-i} \right| \\ &\leq (2C_1^{2\alpha})^m \max_{0 \leq i \leq n} \left| \binom{n}{i} x^i y^{n-i} \right|. \end{aligned}$$

Now using Lemma 4 and Lemma 8,

$$\begin{aligned} |(x + y)^n| &\leq (2C_1^{2\alpha})^m \max_{0 \leq i \leq n} \binom{n}{i} |x|^i |y|^{n-i} \\ &\leq (2C_1^{2\alpha})^m \max_{0 \leq i \leq n} \binom{n}{i} |x|^i |y|^{n-i} \\ &\leq (2C_1^{2\alpha})^m \sum_{i=0}^n \binom{n}{i} |x|^i |y|^{n-i} \\ &\leq (2C_1^{2\alpha})^m (|x| + |y|)^n. \end{aligned}$$

Finally exploiting the multiplicative property for $|\cdot|$,

$$|x + y| = |(x + y)^n|^{1/n} \leq (2C_1^{2\alpha})^{\frac{1}{n}(\lfloor \log_2 n \rfloor + 1)} (|x| + |y|),$$

and passing to the limit $n \rightarrow +\infty$ we get the sharp triangle inequality $|x + y| \leq |x| + |y|$. We observe that the better bound obtained using the “dyadic trick” is crucial in this last step.

In the case where R is a division ring we can assume that $x \neq 0$ and we obtain in the same way

$$|(1 + x^{-1}y)^n| \leq (2C_1^{2\alpha})^m (1 + |x|^{-1}|y|)^n.$$

So multiplying by $|x|^n$ and using the multiplicative property we prove the same inequality as before. The end of the proof is identical. \square

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