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THE WIENER TRANSFORM ON THE BESICOVITCH SPACES

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Abstract. In his fundamental research on generalized harmonic analysis, Wiener proved that the integrated Fourier transform defined by $Wf(\gamma) = \int f(t)\,(e^{-2\pi i\gamma t} - \chi_{[-1,1]}(t))/(-2\pi i t)\,dt$ is an isometry from a nonlinear space of functions of bounded average quadratic power into a nonlinear space of functions of bounded quadratic variation. We consider this Wiener transform on the larger, linear, Besicovitch spaces $\mathcal{B}_{p,q}(\mathbf{R})$ defined by the norm $\|f\|_{\mathcal{B}_{p,q}} = \bigl(\int_0^\infty \bigl(\frac{1}{2T}\int_{-T}^T |f(t)|^p\,dt\bigr)^{q/p}\frac{dT}{T}\bigr)^{1/q}$. We prove that W maps $\mathcal{B}_{p,q}(\mathbf{R})$ continuously into the homogeneous Besov space $\dot{B}^{1/p'}_{p',q}(\mathbf{R})$ for $1 < p \le 2$ and $1 < q \le \infty$, and is a topological isomorphism when $p = 2$.

1. Introduction

In his fundamental research on generalized harmonic analysis, Wiener [W] proved that the integrated Fourier transform, or *Wiener transform*,

$$Wf(\gamma) \;=\; \int_{-\infty}^{\infty} f(t)\,\frac{e^{-2\pi i\gamma t} - \chi_{[-1,1]}(t)}{-2\pi i t}\,dt$$

satisfies the isometry relation

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|^2\,dt \;=\; \lim_{\lambda\to 0} \frac{2}{\lambda}\int_{-\infty}^{\infty} |\Delta_\lambda Wf(\gamma)|^2\,d\gamma, \tag{1}$$

where Δ_λ is the symmetric difference operator $\Delta_\lambda F(\gamma) = F(\gamma+\lambda) - F(\gamma-\lambda)$. Equation (1) is the *Wiener–Plancherel formula*.

The space of functions for which the limit of the quadratic averages on the left-hand side of (1) exists is nonlinear [HW], and therefore cannot be dealt with using the methods of ordinary functional analysis. It is therefore natural to consider the Wiener transform on larger linear spaces. The *Marcinkiewicz space* $M^p(\mathbf{R})$ is defined by the norm $\|f\|_{M^p} = \limsup_{T\to\infty}\bigl(\frac{1}{2T}\int_{-T}^T |f(t)|^p\,dt\bigr)^{1/p}$. Marcinkiewicz himself referred to this space as a Besicovitch space [M]. The usual Besicovitch space formed by completing the space of almost periodic functions is closely related.

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We shall deal with another collection of closely related spaces $\mathcal{B}_{p,q}(\mathbf{R})$, defined by the norm

$$\|f\|_{\mathcal{B}_{p,q}} \;=\; \begin{cases} \left(\int_0^\infty \left(\frac{1}{2T}\int_{-T}^{T} |f(t)|^p\,dt\right)^{q/p} \frac{dT}{T}\right)^{1/q}, & q<\infty, \\ \sup\limits_{T>0} \left(\frac{1}{2T}\int_{-T}^{T} |f(t)|^p\,dt\right)^{1/p}, & q=\infty. \end{cases}$$

By extension, we refer to $\mathcal{B}_{p,q}(\mathbf{R})$ as a Besicovitch space.

Chen and Lau [CL] considered the Wiener transform on $\mathcal{B}_{2,\infty}(\mathbf{R})$, and proved that W is a topological isomorphism of $\mathcal{B}_{2,\infty}(\mathbf{R})$ onto a space V defined by the norm $\|F\|_V = \sup_{\lambda>0} \left(\frac{2}{\lambda}\int_{-\infty}^{\infty} |\Delta_\lambda F(\gamma)|^2\,d\gamma\right)^{1/2}$. This space V coincides with the homogeneous Besov space $\dot{B}^{1/2}_{2,\infty}(\mathbf{R})$. It is our purpose in this paper to consider the Wiener transform on the Besicovitch space $\mathcal{B}_{p,q}(\mathbf{R})$ for $1<p\le 2$, $1<q\le\infty$, and to connect this study to Beurling's fundamental work on spectral synthesis [Beur].

2. Notation

Feichtinger [F] has observed that the Besicovitch space $\mathcal{B}_{p,q}(\mathbf{R})$ coincides (in the sense of equivalent norms) with the "dyadic" amalgam space $(L^p(\mathbf{R}^\times), \ell^q)$, where $L^p(\mathbf{R}^\times)$ is the Lebesgue space with respect to the Haar measure $dt/|t|$ on the multiplicative group $\mathbf{R}^\times$ of nonzero real numbers. This amalgam space is defined by the norm

$$\|f\|_{(L^p(\mathbf{R}^\times),\ell^q)} \;=\; \left(\sum_{n\in\mathbf{Z}} \left(\int_{2^n\le|t|\le 2^{n+1}} |f(t)|^p\,\frac{dt}{|t|}\right)^{q/p}\right)^{1/q}.$$

As a consequence, $\mathcal{B}_{p,q}(\mathbf{R})$ is a Banach space and, for $1\le p,q<\infty$, its dual is $(\mathcal{B}_{p,q}(\mathbf{R}))' = \mathcal{B}_{p',q'}(\mathbf{R})$ when the duality is defined by

$$\langle f,g\rangle \;=\; \int_0^\infty \left(\frac{1}{2T}\int_{-T}^{T} f(t)\,\overline{g(t)}\,dt\right)\frac{dT}{T} \;=\; \frac{1}{2}\int f(t)\,\overline{g(t)}\,\frac{dt}{|t|}.$$

As usual, $\frac{1}{p}+\frac{1}{p'} = \frac{1}{q}+\frac{1}{q'} = 1$.

It is a standard fact [T] that, for the range of indices $1\le p,q<\infty$, and $0<s<1$, an equivalent norm for the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbf{R})$ is given by

$$\|F\|_{\dot{B}^s_{p,q}} \;=\; \left(\int_0^\infty \left(\frac{2}{\lambda^s}\left(\int_{-\infty}^{\infty} |\Delta_\lambda F(\gamma)|^p\,d\gamma\right)^{1/p}\right)^q \frac{d\lambda}{\lambda}\right)^{1/q}.$$

The usual adjustment is made when $q=\infty$.

The absolute value of the Dirichlet kernel will play an important role in our results. We use the notation

$$S(t) \;=\; \left|\frac{\sin 2\pi t}{\pi t}\right| \qquad\text{and}\qquad S_\lambda(t) \;=\; \lambda\, S(\lambda t) \;=\; \left|\frac{\sin 2\pi\lambda t}{\pi t}\right|.$$

Given an even, nonnegative function w on $\mathbf{R}$, we define its greatest decreasing minorant w_* and least decreasing majorant w^* by

$$w_*(t) \;=\; \inf_{0\le|u|\le|t|} w(u) \qquad\text{and}\qquad w^*(t) \;=\; \sup_{|t|\le|u|} w(u).$$

Note that $w_*(t) \le w(t) \le w^*(t)$ for all t, and that w_*, w^* are nonnegative, even functions that are decreasing on $[0,\infty)$. Also note in particular that $S^* \in L^p(\mathbf{R})$ if $p > 1$, and that $S_* = S \cdot \chi_{[-\pi/4,\pi/4]}$.

The Fourier transform is $\hat{f}(\gamma) = \int f(t)\, e^{-2\pi i \gamma t}\, dt$, and the inverse Fourier transform is $\check{f}(\gamma) = \int f(t)\, e^{2\pi i \gamma t}\, dt$.

3. The Wiener transform on $\mathcal{B}_{p,q}(\mathbf{R})$

We shall prove that the Wiener transform is a topological isomorphism of the Besicovitch space $\mathcal{B}_{2,q}(\mathbf{R})$ onto the homogeneous Besov space $\dot{B}^{1/2}_{2,q}(\mathbf{R})$ for $1 < q \le \infty$, and that W maps $\mathcal{B}_{p,q}(\mathbf{R})$ continuously into $\dot{B}^{1/p'}_{p',q}(\mathbf{R})$ for $1 < p \le 2$ and $1 < q \le \infty$.

The following fact will be useful: if w is an even, nonnegative function on $\mathbf{R}$ that is decreasing on $[0,\infty)$, then for each $\alpha > 1$ we have

$$\sum_{n\in\mathbf{Z}} \alpha^{n-1}\, w(\alpha^n) \;\le\; \frac{1}{\alpha-1}\int_0^\infty w(t)\, dt \;\le\; \sum_{n\in\mathbf{Z}} \alpha^n\, w(\alpha^n). \tag{2}$$

The following lemma establishes some basic facts about the Wiener transform.

Lemma 1. *Fix $1 < p \le 2$ and $1 \le q \le \infty$. If $f \in \mathcal{B}_{p,q}(\mathbf{R})$, then the following statements hold for each $\lambda > 0$.*

(a) $f \cdot S_\lambda \in L^p(\mathbf{R})$.

(b) $\Delta_\lambda(Wf) = \frac{1}{2}(f\cdot S_\lambda)^\wedge \in L^{p'}(\mathbf{R})$.

(c) $\|Wf\|_{\dot{B}^{1/p'}_{p',q}} \;\le\; \left(\int_0^\infty \left(\frac{\lambda}{2}\int_{-\infty}^\infty |f(t)|^p\, S(\lambda t)^p\, dt\right)^{q/p} \frac{d\lambda}{\lambda}\right)^{1/q}$, *with equality if* $p = 2$.

Proof. (a) Set $I_n = [-2^{n+1}, -2^n] \cup [2^n, 2^{n+1}]$. Then for each $\lambda > 0$,

$$\begin{aligned}
\int_{-\infty}^\infty |f(t)\, S_\lambda(t)|^p\, dt \;&\le\; \lambda^p \sum_{n\in\mathbf{Z}} \int_{I_n} |f(t)|^p\, S^*(\lambda t)^p\, \frac{2^{n+1}}{|t|}\, dt \\
&\le\; \lambda^p \left(\sum_{n\in\mathbf{Z}} 2^{n+1}\, S^*(2^n\lambda)^p\right)\left(\sup_{n\in\mathbf{Z}} \int_{I_n} |f(t)|^p\, \frac{dt}{|t|}\right) \\
&\le\; 4\lambda^p \left(\int_0^\infty S^*(\lambda t)^p\, dt\right) \|f\|_{(L^p(\mathbf{R}^\times),\ell^\infty)},
\end{aligned}$$

the last inequality following from (2). Since $S^* \in L^p(\mathbf{R})$ for $p > 1$ and since $\mathcal{B}_{p,q}(\mathbf{R}) = (L^p(\mathbf{R}^\times), \ell^q) \subset (L^p(\mathbf{R}^\times), \ell^\infty)$, the result follows.

(b) Follows from a direct computation and from part (a).

(c) By the Hausdorff–Young inequality,

$$\begin{aligned}
\|Wf\|_{\dot{B}^{1/p'}_{p',q}} \;&=\; \left(\int_0^\infty \left(\frac{2}{\lambda}\int_{-\infty}^\infty |\Delta_\lambda Wf(\gamma)|^{p'}\, d\gamma\right)^{q/p'} \frac{d\lambda}{\lambda}\right)^{1/q} \\
&\le\; \left(\int_0^\infty \left(\frac{2}{\lambda}\right)^{q/p'} \left(\int_{-\infty}^\infty |\tfrac{1}{2}\, f(t)\, S_\lambda(t)|^p\, dt\right)^{q/p} \frac{d\lambda}{\lambda}\right)^{1/q} \\
&=\; \left(\int_0^\infty \left(\frac{\lambda}{2}\int_{-\infty}^\infty |f(t)|^p\, S(\lambda t)^p\, dt\right)^{q/p} \frac{d\lambda}{\lambda}\right)^{1/q},
\end{aligned}$$

with equality if $p = 2$. □

The following two results estimate integrals of the form in Lemma 1(c).

Proposition 2. *Let w be even and nonnegative on $\mathbf{R}$, and fix $1 \le p < \infty$ and $1 \le q \le \infty$. Then*

$$\left(\int_0^\infty \left(\frac{\lambda}{2}\int_{-\infty}^\infty |f(t)|^p\, w(\lambda t)\,dt\right)^{q/p} \frac{d\lambda}{\lambda}\right)^{1/q} \;\le\; C\,\|f\|_{\mathcal{B}_{p,q}},$$

with

$$C \;=\; \begin{cases} \left(\displaystyle\int_0^\infty w^*(t)\,dt\right)^{1/p}, & \text{if } p \le q,\\ 4^{1/q}\left(\displaystyle\sup_{0\le t\le 1} w^*(t)^{q/p} \;+\; \int_1^\infty w^*(t)^{q/p}\,dt\right)^{1/q}, & \text{if } q < p. \end{cases}$$

Proof. Fix $\alpha > 1$, and define $\omega_n = w^*(\alpha^n) - w^*(\alpha^{n+1})$. Then, since w^* is even and is decreasing on $[0,\infty)$,

$$\sum_{n\in\mathbf{Z}} \omega_n\,\chi_{[-\alpha^n,\alpha^n]}(t) \;\le\; w^*(t) \;\le\; \sum_{n\in\mathbf{Z}} \omega_n\,\chi_{[-\alpha^{n+1},\alpha^{n+1}]}(t). \tag{3}$$

Assume now that $1 \le p \le q < \infty$ (the case $q = \infty$ is similar). Then, using (3), the change of variables $T = \alpha^{n+1}/\lambda$, and the triangle inequality in the Banach space $L^{q/p}(\mathbf{R}^\times)$, we compute

$$\begin{aligned} &\left(\int_0^\infty \left(\frac{\lambda}{2}\int_{-\infty}^\infty |f(t)|^p\, w(\lambda t)\,dt\right)^{q/p} \frac{d\lambda}{\lambda}\right)^{p/q}\\ &\quad\le\; \left(\int_0^\infty \left(\frac{\lambda}{2}\sum_{n\in\mathbf{Z}} \omega_n \int_{-\alpha^{n+1}/\lambda}^{\alpha^{n+1}/\lambda} |f(t)|^p\,dt\right)^{q/p} \frac{d\lambda}{\lambda}\right)^{p/q}\\ &\quad=\; \left(\int_0^\infty \left(\sum_{n\in\mathbf{Z}} \frac{\alpha^{n+1}\omega_n}{2T} \int_{-T}^{T} |f(t)|^p\,dt\right)^{q/p} \frac{dT}{T}\right)^{p/q}\\ &\quad\le\; \left(\sum_{n\in\mathbf{Z}} \alpha^{n+1}\omega_n\right) \|f\|_{\mathcal{B}_{p,q}}^p\\ &\quad\le\; \left(\alpha\int_0^\infty w^*(t)\,dt\right) \|f\|_{\mathcal{B}_{p,q}}^p, \end{aligned}$$

the last inequality following by integrating (3) over $[0,\infty)$. Letting $\alpha \to 1$ therefore gives the result.

For the case $1 \le q < p < \infty$, we use instead the triangle inequality in the metric space $L^{q/p}(\mathbf{R}^\times)$ to derive the inequality

$$\int_0^\infty \left(\sum_{n\in\mathbf{Z}} \frac{\alpha^{n+1}\omega_n}{2T} \int_{-T}^{T} |f(t)|^p\,dt\right)^{q/p} \frac{dT}{T} \;\le\; \sum_{n\in\mathbf{Z}} (\alpha^{n+1}\omega_n)^{q/p}\,\|f\|_{\mathcal{B}_{p,q}}^q.$$

Then, taking $\alpha = 2^{p/q}$, we have the estimates

$$\begin{aligned} \sum_{n>0} (\alpha^{n+1}\omega_n)^{q/p} \;&\le\; \sum_{n>0} 2^{n+1}\,w^*(2^n)^{q/p} \;\le\; 4\int_1^\infty w^*(t)^{q/p}\,dt,\\ \sum_{n\le 0} (\alpha^{n+1}\omega_n)^{q/p} \;&\le\; \sum_{n\le 0} 2^{n+1} \sup_{0\le t\le 1} w^*(t)^{q/p} \;\le\; 4\sup_{0\le t\le 1} w^*(t)^{q/p}. \end{aligned}$$

□

Proposition 3. *Let w be even and nonnegative on $\mathbf{R}$, and fix $1 \le p, q \le \infty$. Then*

$$\left(\int_0^\infty \left(\frac{\lambda}{2}\int_{-\infty}^{\infty} |f(t)|^p\, w(\lambda t)\, dt\right)^{q/p} \frac{d\lambda}{\lambda}\right)^{1/q} \;\ge\; \left(\sup_{b\in\mathbf{R}} b\, w_*(b)\right)^{1/p} \|f\|_{\mathcal{B}_{p,q}}.$$

Proof. Assume that $1 \le p, q < \infty$ (the cases $p = \infty$ or $q = \infty$ are similar). Fix $b \in \mathbf{R}$. Then, using the change of variables $\lambda = b/T$, we compute

$$\begin{aligned}
\int_0^\infty \left(\frac{\lambda}{2}\int_{-\infty}^{\infty} |f(t)|^p\, w(\lambda t)\, dt\right)^{q/p} \frac{d\lambda}{\lambda} \;&\ge\; \int_0^\infty \left(\frac{b}{2T}\int_{-\infty}^{\infty} |f(t)|^p\, w_*(bt/T)\, dt\right)^{q/p} \frac{dT}{T}\\
&\ge\; \int_0^\infty \left(\frac{b}{2T}\int_{-T}^{T} |f(t)|^p\, w_*(b)\, dt\right)^{q/p} \frac{dT}{T}\\
&=\; \big(b\, w_*(b)\big)^{q/p}\, \|f\|^q_{\mathcal{B}_{p,q}}.
\end{aligned}$$

□

Theorem 4. (a) *If $1 < p \le 2$ and $1 < q \le \infty$, then W maps $\mathcal{B}_{p,q}(\mathbf{R})$ continuously into $\dot{B}^{1/p'}_{p',q}(\mathbf{R})$.*

(b) *W is a topological isomorphism of $\mathcal{B}_{2,q}(\mathbf{R})$ onto $\dot{B}^{1/2}_{2,q}(\mathbf{R})$ for $1 < q \le \infty$.*

Proof. (a) In light of Lemma 1, apply Proposition 2 using $w(t) = S(t)^p$. The result follows for $p \le q \le \infty$ since $(S^*)^p$ is integrable, and for $1 < q \le p$ since $(S^*)^{p\cdot q/p} = (S^*)^q$ is integrable and bounded.

(b) Since $0 < \sup\,(b\, S_*(b)^2) < \infty$, it follows from Proposition 3 that W^{-1} : Range(W) $\to \mathcal{B}_{2,q}(\mathbf{R})$ is continuous. It therefore remains only to show that W is surjective. Fix $G \in \dot{B}^{1/2}_{2,q}(\mathbf{R})$. Then $\Delta_\lambda G \in L^2(\mathbf{R})$ for almost every λ. Set $\beta_\lambda(t) = -i \sin 2\pi\lambda t$, and define $f_\lambda = (\Delta_\lambda G)^\vee/\beta_\lambda$. Then $\beta_\mu\, \beta_\lambda\, f_\lambda = (\Delta_\mu \Delta_\lambda G)^\vee = (\Delta_\lambda \Delta_\mu G)^\vee = \beta_\lambda\, \beta_\mu\, f_\mu$ and $\beta_\lambda\, \beta_\mu$ is nonzero a.e., so $f = f_\lambda$ is independent of λ. Define $g(t) = -2\pi i t f(t)$. Then, since $(\Delta_\lambda G)^\vee(t) = -i\,(\sin 2\pi\lambda t)\, f(t) = \frac{\lambda}{2}\, S(\lambda t)\, g(t)$, we have by the Plancherel formula and Proposition 3 that

$$\begin{aligned}
\|G\|^q_{\dot{B}^{1/2}_{2,q}} \;&=\; \int_0^\infty \left(\frac{2}{\lambda}\int_{-\infty}^{\infty} |\Delta_\lambda G(\gamma)|^2\, d\gamma\right)^{q/2} \frac{d\lambda}{\lambda}\\
&=\; \int_0^\infty \left(\frac{\lambda}{2}\int_{-\infty}^{\infty} |g(t)|^2\, S(\lambda t)^2\, dt\right)^{q/2} \frac{d\lambda}{\lambda} \;\ge\; C\, \|g\|^q_{\mathcal{B}_{2,q}}.
\end{aligned}$$

Hence $g \in \mathcal{B}_{2,q}(\mathbf{R})$, and therefore $Wg \in \dot{B}^{1/2}_{2,q}(\mathbf{R})$. Finally, $\Delta_\lambda W g = \frac{1}{2}(g \cdot S_\lambda)^\wedge = (\beta_\lambda \cdot f)^\wedge = \Delta_\lambda G$, so $Wg = G$ in $\dot{B}^{1/2}_{2,q}(\mathbf{R})$. □

4. Connection to Beurling's A^p, B^p spaces

In one of his deep investigations into spectral synthesis, Beurling [Beur] introduced the spaces

$$A^p \;=\; \bigcup_{w\in\Omega} L^p_{w^{1-p}}(\mathbf{R}) \qquad \text{and} \qquad B^p \;=\; \bigcap_{w\in\Omega} L^p_w(\mathbf{R}).$$

Here Ω is the class of all even, positive, integrable functions on $\mathbf{R}$, and the weighted L^p space $L^p_w(\mathbf{R})$ is defined by the norm $\|f\|_{L^p_w} = \left(\int |f(t)|^p\, w(t)\, dt\right)^{1/p}$. A^p is a convolution algebra contained in $L^1(\mathbf{R})$. The spaces A^p and $B^{p'}$ are duals when the duality is defined by $\langle f,g\rangle = \int f(t)\,\overline{g(t)}\,dt$. Moreover, B^p coincides with the Besicovitch space $\mathcal{B}_{p,\infty}(\mathbf{R})$, with equality of norms [Beur].

By using clever estimates on the weights w, Beurling proved that the Fourier transform is a topological isomorphism of A^2 onto a space U defined by the norm $\|F\|_U = \int_0^\infty \bigl(\frac{2}{\lambda}\int_{-\infty}^\infty |\Delta_\lambda F(\gamma)|^2\,d\gamma\bigr)^{1/2}\,\frac{d\lambda}{\lambda}$. This space U is the homogeneous Besov space $\dot{B}^{1/2}_{2,1}(\mathbf{R})$. We will show in this section how Beurling's isomorphism result for the Fourier transform connects to our isomorphism result for the Wiener transform.

The following lemma establishes the relationship between A^p and $\mathcal{B}_{p,1}(\mathbf{R})$.

Lemma 5. *$A^p = \{tf(t) : f \in \mathcal{B}_{p,1}(\mathbf{R})\}$, with $\|f\|_{A^p} = 2\,\|tf(t)\|_{\mathcal{B}_{p,1}}$.*

Proof. This is a consequence of the following facts:

(a) $(A^p)' = B^{p'}$ under the duality $\langle f,g\rangle = \int f(t)\,\overline{g(t)}\,dt$,

(b) $(\mathcal{B}_{p,1}(\mathbf{R}))' = \mathcal{B}_{p',\infty}(\mathbf{R})$ under the duality $\langle f,g\rangle = \frac{1}{2}\int f(t)\,\overline{g(t)}\,\frac{dt}{|t|}$,

(c) $B^{p'} = \mathcal{B}_{p',\infty}(\mathbf{R})$, with equality of norms. □

Theorem 6. *W is a topological isomorphism of $\mathcal{B}_{2,1}(\mathbf{R})$ onto $\dot{B}^{1/2}_{2,1}(\mathbf{R})$.*

Proof. The fact that the Fourier transform is a topological isomorphism of A^2 onto $\dot{B}^{1/2}_{2,1}(\mathbf{R})$ implies that $C_1\,\|g\|_{A^2} \le \|\hat{g}\|_{\dot{B}^{1/2}_{2,1}} \le C_2\,\|g\|_{A^2}$ for all $g \in A^2$.

Fix now any $f \in \mathcal{B}_{2,1}(\mathbf{R})$. Then, by Lemma 5, $g(t) = f(t)/t \in A^2 \subset L^1(\mathbf{R})$, with $\|g\|_{A^2} = 2\,\|f\|_{\mathcal{B}_{2,1}}$. Since $f(t)/t$ is integrable, we have that

$$\Delta_\lambda Wf(\gamma) \;=\; \int_{-\infty}^\infty \frac{f(t)}{-2\pi it}\,\bigl(e^{-2\pi i(\gamma+\lambda)t} - e^{-2\pi i(\gamma-\lambda)t}\bigr)\,dt \;=\; -\frac{1}{2\pi i}\,\Delta_\lambda\hat{g}(\gamma).$$

Hence, $\|Wf\|_{\dot{B}^{1/2}_{2,1}} = \frac{1}{2\pi}\,\|\hat{g}\|_{\dot{B}^{1/2}_{2,1}}$.

As a consequence of these remarks, it follows that W maps $\mathcal{B}_{2,1}(\mathbf{R})$ continuously into $\dot{B}^{1/2}_{2,1}(\mathbf{R})$ and is invertible on its range. The same argument as in Theorem 4(b) shows that W is surjective and completes the proof. □

5. Extensions to higher dimensions

The extension of the Wiener–Plancherel formula to higher dimensions is non-trivial. Several extensions, based on differing geometries for $\mathbf{R}^d$, have recently been derived [BBE], [Bene], [Benk]. For example, the "rectangular" Wiener transform of [BBE] is

$$Wf(\gamma) \;=\; \int_{\mathbf{R}^d} f(t)\,\mathcal{E}(t,\gamma)\,dt, \qquad \text{where } \mathcal{E}(t,\gamma) \;=\; \prod_{i=1}^d \frac{e^{-2\pi it_j\gamma_j} - \chi_{[-1,1]}(t_j)}{-2\pi it_j}.$$

The Wiener–Plancherel formula is then

$$\lim_{T\to\infty} \frac{1}{|R_T|}\int_{R_T} |f(t)|^2\,dt \;=\; \lim_{\lambda\to 0} \frac{2^d}{\lambda_1\cdots\lambda_d}\int_{\mathbf{R}^d} |\Delta_\lambda Wf(\gamma)|^2\,d\gamma,$$

where $R_T = [-T,T]^d$, Δ_λ is the rectangular symmetric difference operator

$$\Delta_\lambda F(\gamma) \;=\; \frac{1}{2^d}\sum_{\omega\in\{-1,1\}^d} (-1)^{|\omega|}\,F(\gamma+\lambda\omega),$$

and the limits are taken according to a particular natural convergence criterion.

Our results all extend to the setting of [BBE]. The Besicovitch space $\mathcal{B}_{p,q}(\mathbf{R}^d)$ is defined by the norm

$$\|f\|_{\mathcal{B}_{p,q}} \;=\; \left(\int_{\mathbf{R}_+^d} \left(\frac{1}{|R_T|}\int_{R_T} |f(t)|^p\,dt\right)^{q/p} \frac{dT}{T_1\cdots T_d}\right)^{1/q},$$

and a "rectangular" Besov space norm is defined by

$$\|F\|_{\dot{B}^s_{p,q}} \;=\; \left(\int_{\mathbf{R}_+^d} \left(\frac{2^d}{(\lambda_1\cdots\lambda_d)^s}\left(\int_{\mathbf{R}^d} |\Delta_\lambda F(\gamma)|^p\,d\gamma\right)^{1/p}\right)^q \frac{d\lambda}{\lambda_1\cdots\lambda_d}\right)^{1/q}.$$

Easy extensions of our results show that this Wiener transform is a topological isomorphism of $\mathcal{B}_{2,q}(\mathbf{R}^d)$ onto $\dot{B}^{1/2}_{2,q}(\mathbf{R}^d)$ for $1 < q \le \infty$, etc. We expect that similar results should also hold true using the spherical Wiener transform of [Bene] or [Benk].

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